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On energy functionals, Kähler–Einstein metrics, and the Moser–Trudinger–Onofri neighborhood

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Abstract

We prove that the existence of a Kähler–Einstein metric on a Fano manifold is equivalent to the properness of the energy functionals defined by Bando, Chen, Ding, Mabuchi and Tian on the set of Kähler metrics with positive Ricci curvature. We also prove that these energy functionals are bounded from below on this set if and only if one of them is. This answers two questions raised by X.-X. Chen. As an application, we obtain a new proof of the classical Moser–Trudinger–Onofri inequality on the two-sphere, as well as describe a canonical enlargement of the space of Kähler potentials on which this inequality holds on higher-dimensional Fano Kähler–Einstein manifolds.

Keywords: Energy functionals; Kähler–Einstein manifolds; Moser–Trudinger–Onofri inequality

1. Introduction

Our main purpose in this article is to give a new analytic characterization of Kähler–Einstein manifolds in terms of certain functionals defined on the infinite-dimensional space of Kähler forms. As a corollary of our approach we also obtain a new proof of the classical Moser–Trudinger–Onofri inequality on the two-sphere as well as an optimal extension of it to higher-dimensional Fano Kähler–Einstein manifolds.

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A necessary condition for a manifold to admit a Kähler–Einstein metric is that its first Chern class be either positive, negative or zero. Aubin and Yau proved that this condition is also sufficient in the second case and Yau proved that the same is true also in the third case.

Yet additional geometric assumptions are necessary in the first case (in this case the manifold is called Fano): Matsushima proved that the group of automorphisms must be reductive, Futaki proved that a certain character on the algebra of holomorphic vector fields must be trivial, and Kobayashi and Lübke proved that the tangent bundle must be stable. Since then much work has been done on the subject (see for example the recent expositions [9,22,41,42]).

In this article we will restrict attention to two closely related analytic criteria relating the existence of Kähler–Einstein metrics to properties of certain energy functionals (see the end of this section and Section 2 for notation and definitions) on the space of Kähler forms $\mathcal{H}_{c_1}$. The first, introduced by Tian, can be thought of as a "stability" criterion [45]. It expresses the existence of a Kähler–Einstein metric as equivalent to the properness of an energy functional.

Theorem 1.1. *(See [45,46,48].) Let* (M, J) *be a Fano manifold and assume that* $\mathrm{Aut}(M, \mathrm{J})$ *is finite. Then the following are equivalent*:

(i) (M, J) *admits a Kähler–Einstein metric,*
(ii) E_0 *is proper on* $\mathcal{H}_{c_1}^+$,
(iii) F *is proper on* $\mathcal{H}_{c_1}^+$.

The finiteness assumption[2] covers, for example, all Kähler–Einstein Fano surfaces except the product of two Riemann spheres, the projective plane $\mathbb{P}^2$, and $\mathbb{P}^2$ blown up at 3 non-collinear points [39,44,47]. However, there is a slightly more technically involved version of Theorem 1.1, also due to Tian, which applies to all Kähler–Einstein Fano manifolds, that will be stated in Section 4 (Theorem 4.1).

The second analytic criterion, introduced by Bando and Mabuchi, can be thought of as a "semi-stability" condition [6]. Two related formulations appeared subsequently [5,17]. It expresses the existence of "almost" Kähler–Einstein metrics as a consequence of the lower boundedness of an energy functional:

Theorem 1.2. *(See [6,17].) Let* (M, J) *be a Fano manifold. Assume that either* F *or* E_0 *is bounded from below on* $\mathcal{H}_{c_1}^+$ *and let* $\epsilon > 0$. *Then* (M, J) *admits a Kähler metric* $\omega_\epsilon \in \mathcal{H}_{c_1}$ *satisfying* $\mathrm{Ric}\,\omega_\epsilon > (1-\epsilon)\omega_\epsilon$.

It is worth mentioning that a precise characterization of Fano manifolds for which these functionals are bounded from below is still lacking. Also, examples of such manifolds which are not Kähler–Einstein are yet to be given.

We point out that Theorem 1.1 and the version of Theorem 1.2 for the functional F were originally stated with the assumptions on properness and boundedness made on the whole space of Kähler forms $\mathcal{H}_{c_1}$ rather than on the subspace of forms of positive Ricci curvature $\mathcal{H}_{c_1}^+$. However, the respective existence proofs only make use of those assumptions on $\mathcal{H}_{c_1}^+$. Thus Theorem 1.1 implies that the properness of the functionals on $\mathcal{H}_{c_1}^+$ implies their properness on $\mathcal{H}_{c_1}$. In ad-

[2] Since automorphisms of the complex structure preserve the first Chern class this assumption is equivalent to the triviality of $\mathrm{aut}(M, \mathrm{J})$ [19, Theorem 4.8].

dition, in Remark 4.4 we prove that for any Fano manifold also the lower boundedness of the functionals on $\mathcal{H}_{c_1}^+$ implies their lower boundedness on $\mathcal{H}_{c_1}$. Therefore it seems more natural to state Theorems 1.1 and 1.2 in the equivalent manner above. This will also be justified by the results of Section 5 (in particular Corollary 5.5).

Chen and Tian constructed a family of energy functionals $E_1, \ldots, E_n$, analogues of the 'K-energy' ('Kähler energy') E_0 corresponding to higher degree elementary symmetric polynomial expressions of the eigenvalues of the Ricci tensor [15]. As with E_0 and F, Kähler–Einstein metrics are critical points of these functionals and it is therefore a natural idea to seek to extend Theorems 1.1 and 1.2 to $k = 1, \ldots, n$. In this direction, an analogue of Theorem 1.1 for $k = 1$ was proved recently by Song and Weinkove [40]. The main purpose of the present article is to prove the following two statements.

Theorem 1.3. *Let (M, J) be a Fano manifold and assume that* $\mathrm{Aut}(M, \mathrm{J})$ *is finite. Let $k \in \{0, \ldots, n\}$. Then the following are equivalent*:

(i) (M, J) *admits a Kähler–Einstein metric,*
(ii) *E_k is proper on $\mathcal{H}_{c_1}^+$,*
(iii) *F is proper on $\mathcal{H}_{c_1}^+$.*

Theorem 1.4. *Let (M, J) be a Fano manifold and let $k \in \{0, \ldots, n\}$. Assume that either F or E_k is bounded from below on $\mathcal{H}_{c_1}^+$ and let $\epsilon > 0$. Then (M, J) admits a Kähler metric $\omega_\epsilon \in \mathcal{H}_{c_1}$ satisfying* $\mathrm{Ric}\, \omega_\epsilon > (1 - \epsilon)\omega_\epsilon$.

Our proofs carry over to Kähler–Einstein manifolds admitting holomorphic vector fields (for the more general statements the reader is referred to Sections 3 and 4). We remark that while Theorem 1.3 generalizes the work of Song and Weinkove, our methods provide a considerable simplification over the ones used there.

These theorems show that the functionals E_k are, on the one hand, closely related to geometric stability, and, on the other hand, all equivalent in a suitable sense.[3]

To prove these theorems we first observe that a certain formula of Bando and Mabuchi for the 'Ricci energy' E_n extends naturally to all of the functionals E_k. The merit of this new formula (Proposition 2.6) is that it succinctly captures the relation between the different functionals. This shows in particular that the lower boundedness of E_k implies the lower boundedness of E_{k+1}. We then interpret another observation of Bando and Mabuchi in order to close the loop and prove that the lower boundedness of E_n on $\mathcal{H}_{c_1}^+$ implies that of F on $\mathcal{H}_{c_1}$. This step is crucial in proving Theorem 1.4. In fact it proves more, namely, that the lower boundedness of any one of the functionals implies that of the rest (Corollary 4.2). Special cases of this fact have been observed previously [14,17,27,35] (see Remark 4.4).

To prove Theorem 1.3 we consider the continuity method path (18) introduced by Aubin [3]. As before, we show that the properness of E_k implies the properness of E_{k+1}. Next, assuming E_n is proper and using Theorem 1.4 we conclude that this path exists for all $t \in [0, 1)$. We show that on a fixed interval $[t_0, 1)$ each of the functionals E_k is uniformly bounded from above with t_0 depending only on n, and then conclude.

[3] In particular, after posting the first version of this article I became aware of the fact that Theorems 1.3 and 1.4 answer questions posed recently by Chen [14].

In Section 5 we observe that Proposition 2.6 allows to obtain without additional effort a strengthened version of the second main result of Song and Weinkove, the one concerning the non-negativeness of the energy functionals with respect to a Kähler–Einstein base metric. We also observe that our results, when combined with previous ones [6,17], provide for a new and entirely Kähler geometric proof of the Moser–Trudinger–Onofri inequality on the Riemann sphere. As a corollary of this approach we also characterize the functions for which this inequality continues to hold on higher-dimensional Kähler–Einstein Fano manifolds, thus extending the work of Ding and Tian. We call the set of all such functions the Moser–Trudinger–Onofri neighborhood of the space of Kähler potentials. It is a canonically defined set that strictly contains the space of Kähler potentials and lies within $\mathcal{C}^\infty(M)$. This provides a higher dimensional analogue of the original Moser–Trudinger–Onofri inequality that is optimal in a certain sense and brings Ricci curvature into the picture (Theorem 5.4). Finally, we are able to show that the energy functionals $E_2, E_3, \ldots$ are not bounded from below on $\mathcal{H}_{c_1}$ (Corollary 5.5).

The results herein have applications also to the study of the Kähler–Ricci flow and geometric stability [37] that will appear in a subsequent article.

The article is organized as follows. In Section 2 we review the relevant background concerning energy functionals and present the formula for the functionals E_k (Proposition 2.6) whose proof appears in Appendix A. In Section 3 we review results concerning the continuity method approach. The proofs of our main results are contained in Section 4. Section 5 concludes with our results on the lower boundedness of the functionals E_k and on the generalized Moser–Trudinger–Onofri inequality.

Setup and notation. Let (M, J) be a connected compact closed Kähler manifold of complex dimension n and let $\Omega \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ be a Kähler class with $d = \partial + \bar\partial$. Define the Laplacian $\Delta = -\bar\partial \circ \bar\partial^\star - \bar\partial^\star \circ \bar\partial$ with respect to a Riemannian metric g on M and assume that J is compatible with g and parallel with respect to its Levi-Civita connection. Let $g_{\mathrm{Herm}} = 1/\pi \cdot g_{i\bar j}(z)\, dz^i \otimes \overline{dz^j}$ be the associated Kähler metric, that is the induced Hermitian metric on $(T^{1,0}M, \mathrm{J})$, and let $\omega := \omega_g = \sqrt{-1}/2\pi \cdot g_{i\bar j}(z)\, dz^i \wedge d\bar z^j$ denote its corresponding Kähler form, a closed positive $(1,1)$-form on (M, J) such that $g_{\mathrm{Herm}} = \frac{1}{2} g - \frac{\sqrt{-1}}{2}\omega$. Similarly denote by g_ω the Riemannian metric induced from ω by $g_\omega(\cdot,\cdot) = \omega(\cdot, \mathrm{J}\,\cdot)$. For any Kähler form we let $\mathrm{Ric}(\omega) = -\sqrt{-1}/2\pi \cdot \partial\bar\partial \log\det(g_{i\bar j})$ denote the Ricci form of ω. It is well-defined globally and represents the first Chern class $c_1 := c_1(T^{1,0}M, \mathrm{J}) \in H^2(M, \mathbb{Z}) \cap H^{1,1}(M, \mathbb{C})$. One calls ω Kähler–Einstein if $\mathrm{Ric}\,\omega = a\omega$ for some real a.

Denote by $\mathcal{D}_\Omega$ the space of all closed $(1,1)$-forms whose cohomology class is Ω. For a Kähler form ω with $[\omega] = \Omega$ we will consider the space of strictly ω-plurisubharmonic functions (also called Kähler potentials)

$$\mathcal{H}_\omega = \left\{\varphi \in \mathcal{C}^\infty(M)\colon\ \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi > 0\right\},$$

and the subspace $\mathcal{H}_\Omega \subseteq \mathcal{D}_\Omega$ of Kähler forms cohomologous to Ω. We denote by $\mathcal{H}_\Omega^+ \subseteq \mathcal{H}_\Omega$ the subspace of those Kähler forms whose Ricci curvature is positive. Let $\mathrm{Aut}(M, \mathrm{J})$ denote the complex Lie group of automorphisms (biholomorphisms) of (M, J) and denote by $\mathrm{aut}(M, \mathrm{J})$ its Lie algebra of infinitesimal automorphisms consisting of real vector fields X satisfying $\mathcal{L}_X \mathrm{J} = 0$. Let G be any compact real Lie subgroup of $\mathrm{Aut}(M, \mathrm{J})$, and let $\mathrm{Aut}(M, \mathrm{J})_0$ denote the identity component of $\mathrm{Aut}(M, \mathrm{J})$. We denote by $\mathcal{H}_\Omega(G) \subseteq \mathcal{H}_\Omega$ and $\mathcal{H}_\Omega^+(G) \subseteq \mathcal{H}_\Omega^+$ the corresponding subspaces of G-invariant forms.

2. Certain energy functionals on the space of Kähler forms

We call a real-valued function A defined on a subset $\mathrm{Dom}(A)$ of $\mathcal{D}_\Omega \times \mathcal{D}_\Omega$ an energy functional if it is zero on the diagonal restricted to $\mathrm{Dom}(A)$. By a Donaldson-type functional, or exact energy functional, we will mean an energy functional that satisfies the cocycle condition $A(\omega_1,\omega_2)+A(\omega_2,\omega_3)=A(\omega_1,\omega_3)$ with each of the pairs appearing in the formula belonging to $\mathrm{Dom}(A)$ [18,29,46]. We will occasionally refer to both of these simply as functionals and exact functionals, respectively. Note that if an exact functional is defined on $U \times W$ with $U \subseteq W$ then there exists a unique exact functional defined on $W \times W$ extending it.

Let $V := \int_M \omega^n = [\omega]^n([M])$. The energy functionals I, J, introduced by Aubin [3], are defined for each pair $(\omega, \omega_\varphi := \omega + \sqrt{-1}\partial\bar\partial\varphi) \in \mathcal{D}_\Omega \times \mathcal{D}_\Omega$ by

$$I(\omega,\omega_\varphi) = V^{-1}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l}\wedge\omega_\varphi^l = V^{-1}\int_M \varphi(\omega^n - \omega_\varphi^n), \tag{1}$$

$$J(\omega,\omega_\varphi) = \frac{V^{-1}}{n+1}\int_M \sqrt{-1}\partial\varphi \wedge \bar\partial\varphi \wedge \sum_{l=0}^{n-1}(n-l)\omega^{n-l-1}\wedge\omega_\varphi^l. \tag{2}$$

One may also define them via a variational formula. Connect each pair $(\omega, \omega_{\varphi_1} := \omega + \sqrt{-1}\partial\bar\partial\varphi_1)$ with a piecewise smooth path $\{\omega_{\varphi_t}\}_{t\in[0,1]}$ (we regard this path as a function on $M\times[0,1]$ and occasionally suppress the subscript t). Then we have for any such path

$$(I-J)(\omega,\omega_{\varphi_1}) = -\frac{1}{V}\int_{M\times[0,1]} \varphi_t n\sqrt{-1}\partial\bar\partial\dot\varphi_t \wedge \omega_{\varphi_t}^{n-1}\wedge dt, \tag{3}$$

$$J(\omega,\omega_{\varphi_1}) = \frac{1}{V}\int_{M\times[0,1]} \dot\varphi_t\big(\omega^n - \omega_{\varphi_t}^n\big)\wedge dt. \tag{4}$$

On $\mathcal{H}_\Omega \times \mathcal{H}_\Omega$ I, J and $I-J$ are all nonnegative (and hence non-exact) and equivalent, namely,

$$\frac{1}{n^2}(I-J) \leqslant \frac{1}{n(n+1)}I \leqslant \frac{1}{n}J \leqslant I-J \leqslant \frac{n}{n+1}I \leqslant nJ. \tag{5}$$

Note that pulling-back both arguments of these functionals by an automorphism of (M,J) does not change their value. It is important to understand the behavior of these functionals also outside the subspace $\mathcal{H}_\Omega$:

Lemma 2.1. *Let $\omega \in \mathcal{H}_\Omega$. Then $I(\omega,\cdot)$ is unbounded from above on $\mathcal{H}_\Omega$ and, when $n>1$, unbounded on $\mathcal{D}_\Omega$.*

Proof. Fix a holomorphic coordinate patch

$$\psi : U \to \mathbb{C}^n, \quad \psi(q) = \mathbf{z}(q) := \big(z^1(q),\dots,z^n(q)\big),\ \forall q \in U \subseteq M.$$

Let $a>0$ be such that $\psi^{-1}(\{v\in\mathbb{C}^n\colon |v|<3a\}) \subseteq U$. For the first statement, define $\tilde\varphi_b$ by letting $\tilde\varphi_b = b|\mathbf{z}|^2$ on $\psi^{-1}(\{v\in\mathbb{C}^n\colon a<|v|<2a\})$ and constant elsewhere on U in such a way that it is continuous. Approximate $\tilde{\varphi}_b$ by smooth functions $\varphi_{b,m}$ that agree with it outside the set $\psi^{-1}(\{v \in \mathbb{C}^n\colon |v| \in (a - \frac{1}{m}, a + \frac{1}{m}) \cup (2a - \frac{1}{m}, 2a + \frac{1}{m})\})$ and that satisfy $|\varphi_b - \varphi_{b,m}| < \frac{1}{m}$ on U. Given $a_2 > 0$ there exists b and a corresponding m such that $\varphi_{b,m} \in \mathcal{H}_\omega$ and $I(\omega, \omega_{\varphi_{b,m}}) > a_2$.

For the second statement, construct similarly functions, as above, now setting $\tilde{\varphi}_b = -b(|z_1|^2 + |z_2|^2)$ on $\psi^{-1}(\{v \in \mathbb{C}^n\colon a < |v| < 2a\})$. Again one may approximate using functions $\varphi_{b,m}$. Expanding $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_{b,m})^l$ using the binomial formula it then follows that up to a term that is uniformly bounded for m sufficiently large, $I(\omega, \omega_{\varphi_{b,m}})$ equals $V^{-1}\int_M \sqrt{-1}\partial\varphi_{b,m} \wedge \bar{\partial}\varphi_{b,m} \wedge \omega^{n-2} \wedge (a_2\omega + a_3\sqrt{-1}\partial\bar{\partial}\varphi_{b,m})$ for some $a_2, a_3 > 0$. We then see that given any $a_4 > 0$ there exists b and a corresponding m such that $I(\omega, \omega_{\varphi_{b,m}}) < -a_4$. □

We say that an exact functional A is bounded from below on $U \subseteq \mathcal{H}_\Omega$ if for every ω such that $(\omega, \omega_\varphi) \in \mathrm{Dom}(A)$ and $\omega_\varphi \in U$ holds $A(\omega, \omega_\varphi) \geqslant C_\omega$ with C_ω independent of ω_φ. We say it is proper (in the sense of Tian) on a set $U \subseteq \mathcal{H}_\Omega$ if for each $\omega \in \mathcal{H}_\Omega$ there exists a smooth function $\tau_\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{s\to\infty} \tau_\omega(s) = \infty$ such that $A(\omega, \omega_\varphi) \geqslant \tau_\omega((I - J)(\omega, \omega_\varphi))$ for every $\omega_\varphi \in U$. This is well-defined, in other words depends only on $[\omega]$ since the failure of $I - J$ to satisfy the cocycle condition is under control with respect to the two base metrics, $\omega, \omega_{\varphi_1}$ say, to wit,

$$(I - J)(\omega, \omega_{\varphi_2}) - (I - J)(\omega_{\varphi_1}, \omega_{\varphi_2}) = (I - J)(\omega, \omega_{\varphi_1}) - \frac{1}{V}\int_M \varphi_1\big(\omega_{\varphi_2}^n - \omega_{\varphi_1}^n\big),$$

with the last term controlled by the oscillation of φ_1. Properness of a functional implies it has a lower bound.

We introduce the following collection of energy functionals for each $k \in \{0, \dots, n\}$:

$$\begin{aligned} I_k(\omega, \omega_\varphi) &= \frac{1}{V}\int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \sum_{l=0}^{k-1} \frac{k-l}{k+1}\omega^{n-1-l} \wedge \omega_\varphi^l \\ &= \frac{V^{-1}}{k+1}\int_M \varphi\left(k\omega^n - \sum_{l=1}^{k} \omega^{n-l} \wedge \omega_\varphi^l\right). \end{aligned} \tag{6}$$

Note that $I_n = J$, $I_{n-1} = ((n+1)J - I)/n$.

Chen and Tian [15] defined another such family:

$$J_k(\omega, \omega_{\varphi_1}) = V^{-1}\int_{M\times[0,1]} \dot{\varphi}_t\big(\omega_{\varphi_t}^k \wedge \omega^{n-k} - \omega_{\varphi_t}^n\big) \wedge dt, \quad k = 0, \dots, n. \tag{7}$$

(Note that $J_{n-k-1}/(k+1)$ in their article corresponds to J_k in this article.) The following computation relates these two families of functionals.

Lemma 2.2. *The following relation holds on $\mathcal{H}_\Omega \times \mathcal{H}_\Omega$:*

$$I_k(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - J_k(\omega, \omega_\varphi).$$

Proof. Given a path $\{\omega_{\varphi_t}\}_{t\in[0,1]}$ we compute the variational equation for I_k.

$$\begin{aligned}(k+1)\frac{d}{dt}I_k(\omega,\omega_{\varphi_t}) &= -\frac{1}{V}\int_M\sum_{l=0}^{k-1}\big(2\dot\varphi\sqrt{-1}\partial\bar\partial\varphi\wedge(k-l)\omega^{n-1-l}\wedge\omega_\varphi^l\\
&\quad+\varphi\sqrt{-1}\partial\bar\partial\varphi\wedge\sqrt{-1}\partial\bar\partial\dot\varphi\wedge l(k-l)\omega^{n-1-l}\wedge\omega_\varphi^{l-1}\big)\\
&= -\frac{1}{V}\int_M\dot\varphi\sqrt{-1}\partial\bar\partial\varphi\wedge\sum_{l=0}^{k-1}\big(2(k-l)\omega^{n-1-l}\wedge\omega_\varphi^l\\
&\quad+(\omega_\varphi-\omega)\wedge l(k-l)\omega^{n-1-l}\wedge\omega_\varphi^{l-1}\big)\\
&= -\frac{1}{V}\int_M\dot\varphi\sqrt{-1}\partial\bar\partial\varphi\wedge\Bigg(\sum_{l=0}^{k-1}2(k-l)\omega^{n-1-l}\wedge\omega_\varphi^l\\
&\quad+\sum_{l=1}^{k-1}l(k-l)\omega^{n-1-l}\wedge\omega_\varphi^l\\
&\quad-\sum_{l=0}^{k-2}(k-l-1)(l+1)\omega^{n-1-l}\wedge\omega_\varphi^l\Bigg)\\
&= -(k+1)\frac{1}{V}\int_M\dot\varphi\sqrt{-1}\partial\bar\partial\varphi\wedge\sum_{l=0}^{k-1}\omega^{n-1-l}\wedge\omega_\varphi^l,\end{aligned}$$

and putting $\sqrt{-1}\partial\bar\partial\varphi=\omega_\varphi-\omega$ we have

$$\frac{d}{dt}I_k(\omega,\omega_{\varphi_t}) = V^{-1}\int_M\dot\varphi_t\big(\omega^n-\omega^{n-k}\wedge\omega_{\varphi_t}^k\big). \tag{8}$$

Combining with (7) and (4) we conclude. □

Note that from the definitions it follows that

$$0\leqslant I_k(\omega,\omega_\varphi)\leqslant J(\omega,\omega_\varphi),\quad \text{on } \mathcal{H}_\Omega\times\mathcal{H}_\Omega. \tag{9}$$

As a corollary of Lemma 2.2 we have therefore $0\leqslant J_k(\omega,\omega_\varphi)\leqslant J(\omega,\omega_\varphi)$ on $\mathcal{H}_\Omega\times\mathcal{H}_\Omega$. We point out that this upper bound improves [15, Corollary 4.5] while the lower bound appears to be new. Also from (6)

$$\frac{k+2}{k+1}I_{k+1}\geqslant\frac{k+1}{k}I_k,\quad \text{on } \mathcal{H}_\Omega\times\mathcal{H}_\Omega. \tag{10}$$

Note that in particular $I_{k+1}\geqslant I_k$ and so by Lemma 2.2 $J_k\geqslant J_{k+1}$. We note in passing that this lemma also yields the following formula:

$$J_k(\omega,\omega_\varphi)=\frac{V^{-1}}{n+1}\int_M \sqrt{-1}\partial\varphi\wedge\bar\partial\varphi\wedge\left(\frac{n-k}{k+1}\sum_{l=0}^{k-1}(l+1)\omega^{n-1-l}\wedge\omega_\varphi^l + \sum_{l=k}^{n-1}(n-l)\omega^{n-1-l}\wedge\omega_\varphi^l\right). \tag{11}$$

The Chen–Tian energy functionals E_k, $k=0,\dots,n$, are defined by

$$E_k(\omega,\omega_{\varphi_1})=V^{-1}\int_{M\times[0,1]}\Delta_{\varphi_t}\dot\varphi_t\,\mathrm{Ric}(\omega_{\varphi_t})^k\wedge\omega_{\varphi_t}^{n-k}\wedge dt - \frac{n-k}{k+1}V^{-1}\int_{M\times[0,1]}\dot\varphi_t\big(\mathrm{Ric}(\omega_{\varphi_t})^{k+1}-\mu_k\omega_{\varphi_t}^{k+1}\big)\wedge\omega_{\varphi_t}^{n-1-k}\wedge dt, \tag{12}$$

where $\mu_k := \frac{c_1^{k+1}\cup[\omega]^{n-k-1}([M])}{[\omega]^n([M])}$. This gives rise to well-defined exact energy functionals [15] (note that $E_k/(k+1)$ in the aforementioned article corresponds to E_k in this article). The K-energy, E_0, was introduced by Mabuchi [29], while E_n, that we refer to as the 'Ricci energy,' was introduced by Bando and Mabuchi[4] [6].

For each $\omega\in\mathcal{H}_\Omega$ these functionals (being exact) induce a (real) Lie group homomorphism $\mathrm{Aut}(M,\mathrm{J})_0\to\mathbb{R}$ given by $h\mapsto E_k(\omega,h^\star\omega)$. The corresponding Lie algebra homomorphism $\mathrm{aut}(M,\mathrm{J})\to\mathbb{R}$ is given by $X\mapsto \frac{d}{dt}|_0 E_k(\omega,(\exp tX)^\star\omega)$. This naturally extends to a complex Lie algebra homomorphism

$$X\mapsto\mathcal{F}_k(X;\omega):=\frac{d}{dt}\bigg|_0 E_k\big(\omega,(\exp tX)^\star\omega\big)-\sqrt{-1}\frac{d}{dt}\bigg|_0 E_k\big(\omega,(\exp tJX)^\star\omega\big). \tag{13}$$

Changing ω within a fixed cohomology class does not change the homomorphism [15,30]. This is an extension of the Bando–Calabi–Futaki theorem, the case $k=0$ [8,10,20] (the construction was further generalized by Futaki [21]). One calls these homomorphisms Futaki characters (or invariants). When (M,J,ω) is Fano Kähler–Einstein it follows from (12) that $\mathcal{F}_k$ is trivial and hence $E_k(\omega,\omega_\varphi)=0$ if ω_φ is Kähler–Einstein, since the set of Kähler–Einstein metrics is equal to an $\mathrm{Aut}(M,\mathrm{J})_0$-orbit of ω [6].

[4] Kähler–Einstein forms are the only critical points of these two functionals when $\Omega=\mu c_1$, $\mu\in\{\pm1\}$: For E_0 see [46, p. 19] while for E_n the critical forms satisfy $(\mu\,\mathrm{Ric}\,\omega)^n=\omega^n$ and writing $\mu\,\mathrm{Ric}\,\omega=\omega+\sqrt{-1}\partial\bar\partial f$ we see that $\mu\,\mathrm{Ric}\,\omega>0$ at the minimum of f. Since the smallest eigenvalue of a Hölder continuous matrix-valued function is also Hölder continuous [1, p. 438] we conclude that $\mu\,\mathrm{Ric}\,\omega>0$ implying that f is constant by the uniqueness argument of Calabi (for a different proof see [30, Section 8]). However when $c_1=0$ there are nontrivial solutions of $(\mathrm{Ric}\,\omega)^n=0$ if the manifold is a product. For $\mu=1$ critical points of E_k with nonnegative Ricci curvature are necessarily Kähler–Einstein [49].

Unless otherwise stated, from now and on we will assume that (M, J) is Fano and let $\omega_\varphi \in \mathcal{H}_{c_1}$. Let $f_{\omega_\varphi} \in \mathcal{C}^\infty(M)$ denote the unique function satisfying $\sqrt{-1}\partial\bar\partial f_{\omega_\varphi} = \operatorname{Ric}\omega_\varphi - \omega_\varphi$ and $V^{-1}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n = 1$. Following Ding [16], define an exact functional on $\mathcal{H}_{c_1} \times \mathcal{D}_{c_1}$ by

$$F(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - \frac{1}{V}\int_M \varphi\omega^n - \log\frac{1}{V}\int_M e^{f_\omega - \varphi}\omega^n.$$

The critical points of this functional are the Kähler–Einstein metrics. We state the following relation between the functionals E_0 and F.

Lemma 2.3. *(See [17].) Let $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$. Then*

$$F(\omega, \omega_\varphi) = E_0(\omega, \omega_\varphi) + \frac{1}{V}\int_M f_{\omega_\varphi}\omega_\varphi^n - \frac{1}{V}\int_M f_\omega\omega^n.$$

Note that

$$\frac{1}{V}\int_M f_{\omega_\varphi}\omega_\varphi^n \leqslant \frac{1}{V}\int_M e^{f_{\omega_\varphi}}\omega_\varphi^n - 1 = 0. \tag{14}$$

Note also that one may define a Lie algebra homomorphism corresponding to F similarly to the construction for E_k in (13). Lemma 2.3 implies that this homomorphism will coincide with $\mathcal{F}_0$.

An equivalent form of the following was stated by Bando and Mabuchi [6, (1.5)].

Lemma 2.4. *For every $(\omega, \omega_\varphi) \in \mathcal{H}^+_{c_1} \times \mathcal{H}_{c_1}$ one has*

$$E_n(\omega, \omega_\varphi) = F(\operatorname{Ric}\omega, \operatorname{Ric}\omega_\varphi).$$

Note that by exactness this formula completely determines E_n on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$, as remarked earlier.

Proof. Let $\{\varphi_t\}$ denote a smooth family of functions such that $\omega_{\varphi_0} = \omega$, $\omega_{\varphi_1} = \omega_\varphi$. Write $\operatorname{Ric}\omega_{\varphi_t} = \operatorname{Ric}\omega + \sqrt{-1}\partial\bar\partial\log\frac{\omega^n}{\omega_{\varphi_t}^n}$. Then $f_{\operatorname{Ric}\omega} = \log\frac{\omega^n}{(\operatorname{Ric}\omega)^n}$. Thus for each $t \in [0, 1]$,

$$F(\operatorname{Ric}\omega, \operatorname{Ric}\omega_{\varphi_t}) = J(\operatorname{Ric}\omega, \operatorname{Ric}\omega_{\varphi_t}) - \frac{1}{V}\int_M \log\frac{\omega^n}{\omega_{\varphi_t}^n}(\operatorname{Ric}\omega)^n.$$

Hence,

$$\frac{d}{dt}F(\operatorname{Ric}\omega, \operatorname{Ric}\omega_{\varphi_t}) = -V^{-1}\int_M (-\Delta_t\dot\varphi_t)(\operatorname{Ric}\omega_{\varphi_t})^n = \frac{d}{dt}E_n(\omega, \omega_{\varphi_t}),$$

from which we conclude by integration. □

Bando and Mabuchi derived the following elegant formula.

Proposition 2.5. *(See [6, (1.8.1)].) For every* $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$,

$$E_n(\omega, \omega_\varphi) = E_0(\omega, \omega_\varphi) + J(\omega_\varphi, \operatorname{Ric}\omega_\varphi) - J(\omega, \operatorname{Ric}\omega).$$

We now show that Proposition 2.5 can be generalized as follows.

Proposition 2.6. *Let* $k \in \{0, \dots, n\}$. *For every* $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$,

$$\begin{aligned} E_k(\omega, \omega_\varphi) &= E_n(\omega, \omega_\varphi) - J_k(\omega_\varphi, \operatorname{Ric}\omega_\varphi) + J_k(\omega, \operatorname{Ric}\omega) && (15)\\ &= E_0(\omega, \omega_\varphi) + I_k(\omega_\varphi, \operatorname{Ric}\omega_\varphi) - I_k(\omega, \operatorname{Ric}\omega) && (16)\\ &= \left(\left(1 - \frac{l}{k+1}\right)E_0 + \frac{l}{k+1}E_n\right)(\omega, \omega_\varphi) + \left(I_k - \frac{l}{k+1}J\right)(\omega_\varphi, \operatorname{Ric}\omega_\varphi) && (17)\\ &\quad - \left(I_k - \frac{l}{k+1}J\right)(\omega, \operatorname{Ric}\omega), \quad \forall l \in \{0, \dots, k+1\}. \end{aligned}$$

The proof appears in Appendix A. The functionals E_k are thus seen to be described as 'Kähler–Ricci' energies, "interpolating" between the Kähler energy E_0 and the Ricci energy E_n. We note that there exist counterparts of the formulas presented in this section for some other Kähler classes [37].

One particularly visible consequence of Proposition 2.6 is the fact that the homomorphisms $\mathcal{F}_k$ all coincide, a result first proved by Maschler [30, (17)] using an equivariant formulation and later by Liu [28, Section 3] by a direct computation (see also [26]). For other explicit expressions for the functionals E_k see [15,26,35,40].

3. Continuity method approach

Consider the path $\{\omega_{\varphi_t}\} \subseteq \mathcal{H}_{c_1}$ given implicitly by

$$\begin{aligned} \omega_{\varphi_t}^n &= e^{(t+1)f_\omega + c_t}\omega^n, \quad t \in [-1, 0],\\ \omega_{\varphi_t}^n &= e^{f_\omega - t\varphi_t}\omega^n, \quad t \in [0, 1], \end{aligned} \tag{18}$$

with the normalizations $\int_M e^{(t+1)f_\omega + c_t}\omega^n = V$ for $t \in [-1, 0]$ and $\int_M e^{f_\omega - t\varphi_t}\omega^n = V$ for $t \in [0, 1]$. Note that the first segment always exists by the proof of the Calabi–Yau theorem [51] while the second, when it exists, deforms the metric to a Kähler–Einstein metric [3]:

$$\operatorname{Ric}\omega_{\varphi_t} - \omega_{\varphi_t} = -(1-t)\sqrt{-1}\partial\bar\partial\varphi_t, \quad t \in [0, 1]. \tag{19}$$

We will make use of the following proposition:

Proposition 3.1. *(See [6, Theorem 5.7].) Assume that* (M, J) *is Fano and let* G *be a compact subgroup of* $\operatorname{Aut}(M, \mathrm{J})$. *Assume that* E_0 *is bounded from below on* $\mathcal{H}_{c_1}^+(G)$ *and let* $\omega \in \mathcal{H}_{c_1}(G)$. *Then* (18) *has a unique smooth solution for each* $t \in [0, 1)$.

Note that by Lemma 2.3 and (14) the same conclusion holds with E_0 replaced by F. In particular, Theorem 1.2 is a direct corollary of Proposition 3.1 combined with this observation (one obtains a version of Theorem 1.2 with the free choice of a subgroup G, although this, as opposed to the refinement of Theorem 1.1 that will be given in the next section, should not be considered as a gain in generality). We also note that one of the important ingredients in the proof of Proposition 3.1 is the fact that $(I - J)(\omega, \cdot)$ is nondecreasing along the continuity path (18) [6, Theorem 5.1], [43, p. 232], [46, Lemma 6.25].

It is worth noting that Bando has shown that if $\omega \in \mathcal{H}_{c_1}(G)$ satisfies $\operatorname{Ric}\omega > (1-\epsilon)\omega$, $\epsilon > 0$, then "flowing" it along the Ricci flow will produce another metric in $\mathcal{H}_{c_1}(G)$ whose scalar curvature differs from n by at most a fixed constant times ϵ [5]. Therefore, the existence of a lower bound for E_0 or for F implies the existence of Kähler metrics in $\mathcal{H}_{c_1}(G)$ whose scalar curvature is as close to a constant as desired (the original result of Bando extends to the G-invariant setting since its proof makes use of a Kähler–Ricci flow which, like the continuity method, preserves $\mathcal{H}_{c_1}(G)$). These can be thought of as "almost Kähler–Einstein" metrics since a Kähler metric of constant scalar curvature in $\mathcal{H}_{c_1}$ is necessary Kähler–Einstein.

4. Boundedness and properness properties of energy functionals

By Matsushima's theorem, when a Kähler–Einstein form ω exists the Lie algebra of Killing vector fields is a real form of $\operatorname{aut}(M, \mathrm{J})$ [8,31,38]. In other words, when a Kähler–Einstein metric exists we may take G to be the isometry group $\operatorname{Iso}(M, g_\omega)$. Also, $\operatorname{aut}(M, \mathrm{J})$ is then isomorphic to an eigenspace of the Laplacian, namely,

$$\operatorname{aut}(M, \mathrm{J}) \cong \Lambda_1 := \left\{\psi \in \mathcal{C}^\infty(M)\colon -\Delta_\omega \psi = \psi\right\}.$$

Set

$$\mathcal{H}_{c_1}^{+}(\Lambda_1) := \left\{\omega_\varphi \in \mathcal{H}_{c_1}^{+}\colon \int_M \varphi\psi\omega^n = 0,\ \forall\psi \in \Lambda_1\right\}.$$

Similarly, define $\mathcal{H}_{c_1}(\Lambda_1)$. We may now state the following theorem of Tian which is a refined version of Theorem 1.1.[5]

Theorem 4.1. *(See [45,46,48].) Let (M, J) be a Fano manifold and G be a compact subgroup of* $\operatorname{Aut}(M, \mathrm{J})$. *If F or E_0 is proper on $\mathcal{H}_{c_1}^{+}(G)$ then (M, J) admits a G-invariant Kähler–Einstein metric. Conversely, if (M, J) admits a G-invariant Kähler–Einstein metric then F and E_0 are proper on $\mathcal{H}_{c_1}^{+}(\Lambda_1)$.*

We remark that when $\operatorname{aut}(M, \mathrm{J})$ is semisimple then $\mathcal{H}_{c_1}(G) \subseteq \mathcal{H}_{c_1}(\Lambda_1)$ [36].

Let us turn to the proof of our main theorems and begin with Theorem 1.3. Assume that a Kähler–Einstein form ω exists. Then F is proper on $\mathcal{H}_{c_1}^{+}$ by Theorem 1.1. By Lemma 2.3 and (14) so is E_0. From Proposition 2.6 we have

$$E_{k+1}(\omega, \omega_\varphi) = E_k(\omega, \omega_\varphi) + (I_{k+1} - I_k)(\omega_\varphi, \operatorname{Ric}\omega_\varphi) - (I_{k+1} - I_k)(\omega, \operatorname{Ric}\omega),$$

[5] A detailed exposition of this theorem will be found in a forthcoming article of Tian and Zhu.

with $I_{k+1} \geqslant I_k$ on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$ as noted after (10). It follows that if E_k is proper on $\mathcal{H}^+_{c_1}$ so is E_{k+1}. We conclude that E_n is proper on $\mathcal{H}^+_{c_1}$.

Assume that E_n is proper on $\mathcal{H}^+_{c_1}$. Then from Lemma 2.4 and the Calabi–Yau theorem we see that F is bounded from below on $\mathcal{H}_{c_1}$ and from Lemma 2.3 and (14) it follows that so is E_0. Therefore from Proposition 3.1, given $\omega \in \mathcal{H}_{c_1}$, the continuity path (18) extends for all $t < 1$.

From the properness and exactness of E_n there exists a function τ_ω as in Section 2 satisfying $E_n(\omega_{\varphi_0}, \omega_{\varphi_t}) \geqslant \tau_\omega(I(\omega, \omega_{\varphi_t})) - E_n(\omega, \omega_{\varphi_0})$. Hence it suffices now to show that $E_n(\omega_{\varphi_0}, \omega_{\varphi_t})$ is uniformly bounded from above for all $t > t_0$ with t_0 depending only on (M, J, ω). We will then have that $I(\omega, \omega_{\varphi_t})$ is uniformly bounded independently of $t \in [0, 1)$. This will entail a uniform bound on $\|\varphi_t\|_{L^\infty}$ [4, Proposition 7.35], [46, Lemma 6.19] and hence a uniform bound on $\|\varphi_t\|_{\mathcal{C}^{2,\beta}(M, g_\omega)}$ for some $\beta \in (0, 1)$ [2,51]. By the continuity method arguments therein one then concludes that a unique smooth solution exists at $t = 1$ that is a Kähler potential for a Kähler–Einstein form.

In fact we will find such a t_0 depending only on n for each E_k. The computation that follows involves expressions similar to those that figure in the work of Song and Weinkove; using Proposition 2.6 considerably simplifies our calculations compared to the ones there.

Fix $\tau \in [0, 1]$. First, from (19) and the definition of E_0 we have

$$\begin{aligned} E_0(\omega_{\varphi_0}, \omega_{\varphi_\tau}) &= \int_{[0,\tau]} \frac{d}{dt} E_0(\omega_{\varphi_0}, \omega_{\varphi_t})\, dt \\ &= \frac{1}{V} \int_{M \times [0,\tau]} (1 - t) n \dot{\varphi}_t \sqrt{-1}\partial\bar{\partial}\varphi_t \wedge \omega_{\varphi_t}^{n-1} \wedge dt \\ &= -\int_{[0,\tau]} (1 - t) \frac{d}{dt} (I - J)(\omega, \omega_{\varphi_t})\, dt \\ &= -(1 - \tau)(I - J)(\omega, \omega_{\varphi_\tau}) \\ &\quad + (I - J)(\omega, \omega_{\varphi_0}) - \int_{[0,\tau]} (I - J)(\omega, \omega_{\varphi_t})\, dt. \end{aligned} \tag{20}$$

From Proposition 2.6, (5) and (9) we therefore conclude that there exists a constant c_ω depending only on (M, J, ω) for which

$$(n + 1) E_k(\omega_{\varphi_0}, \omega_{\varphi_\tau}) \leqslant -(1 - \tau) I(\omega, \omega_{\varphi_\tau}) + n I(\omega_{\varphi_\tau}, \operatorname{Ric} \omega_{\varphi_\tau}) + c_\omega. \tag{21}$$

From (19),

$$\begin{aligned} &I(\omega_{\varphi_\tau}, \operatorname{Ric} \omega_{\varphi_\tau}) \\ &\quad = (1 - \tau)^2 \frac{1}{V} \int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{l=0}^{n-1} \omega_{\varphi_\tau}^{n-l-1} \wedge \big(\tau\omega_{\varphi_\tau} + (1 - \tau)\omega\big)^l \\ &\quad = (1 - \tau)^2 \frac{1}{V} \int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{l=0}^{n-1} \sum_{j=0}^{l} \binom{l}{j} \tau^{l-j} (1 - \tau)^j \omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j \end{aligned}$$

$$= (1-\tau)^2 \frac{1}{V} \int_M \sqrt{-1}\partial\varphi_\tau \wedge \bar{\partial}\varphi_\tau \wedge \sum_{j=0}^{n-1} (1-\tau)^j \sum_{l=j}^{n-1} \binom{l}{j} \tau^{l-j} \omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j.$$

Note that

$$(1-\tau)^j \sum_{l=j}^{n-1} \binom{l}{j} \tau^{l-j} \leqslant (1-\tau)^j (n-1) \binom{n-1}{j}. \tag{22}$$

We may choose $t_1 \in [0,1)$ depending only on n in such a way that for all $\tau \in [t_1, 1]$ the expression on the right-hand side of (22) is smaller than n for each $j = 0, \ldots, n-1$. We conclude that

$$I(\omega_{\varphi_\tau}, \operatorname{Ric}\omega_{\varphi_\tau}) \leqslant n(1-\tau)^2 I(\omega, \omega_{\varphi_\tau}), \quad \forall \tau \in [t_1, 1). \tag{23}$$

Returning to (21) we then see that $E_k(\omega_{\varphi_0}, \omega_{\varphi_\tau}) \leqslant c_\omega/(n+1)$ whenever $\tau \in [\max\{t_1, 1-\frac{1}{n^2}\}, 1)$. This concludes the proof of Theorem 1.3. □

As a corollary of the proof we record the following fact.

Corollary 4.2. *Let (M, J) be a Fano manifold. If one of the functionals F, $E_0, \ldots, E_n$ is bounded from below on $\mathcal{H}_{c_1}^+$ so are the rest.*

Combining Corollary 4.2 with Theorem 1.2 concludes the proof of Theorem 1.4. □

We end this section with several remarks.

Remark 4.3. Our methods imply that the refined version of Theorem 1.1 (Theorem 4.1) also extends to each of the functionals E_k.

Remark 4.4. Note that one may state Corollary 4.2 with $\mathcal{H}_{c_1}^+$ replaced by $\mathcal{H}_{c_1}$ for F, E_0 and E_1. Indeed, recall that once F is bounded from below on $\mathcal{H}_{c_1}^+$ so are each of the E_k while a lower bound for E_n on $\mathcal{H}_{c_1}^+$ implies a lower bound for F on $\mathcal{H}_{c_1}$ (by Lemma 2.4) which, in turn, implies the same for E_0 (using Lemma 2.3) and for E_1 (using Proposition 2.6). Some special cases of Corollary 4.2 appeared previously, namely the fact that when F is bounded from below so is E_0 [17] and vice versa [27], and the fact that when E_0 is bounded from below so is E_1 [35] and vice versa [14].

Remark 4.5. Assume that the functionals F and E_k, $k \in \{0, \ldots, n\}$ are bounded from below on $\mathcal{H}_{c_1}^+$ and for each $\omega \in \mathcal{H}_{c_1}$ set $l(\omega) = \inf_{\omega_\varphi \in H_{c_1}} F(\omega, \omega_\varphi)$ and

$$l_k(\omega) = \begin{cases} \inf_{\omega_\varphi \in H_{c_1}} E_k(\omega, \omega_\varphi), & \text{for } k = 0, 1, \\ \inf_{\omega_\varphi \in H_{c_1}^+} E_k(\omega, \omega_\varphi), & \text{for } k = 2, \ldots, n. \end{cases}$$

Then the following relations hold between the various lower bounds:

$$l(\omega) + \frac{1}{V} \int_M f_\omega \omega^n = l_0(\omega) = l_k(\omega) + I_k(\omega, \operatorname{Ric}\omega). \tag{24}$$

This generalizes the relation between l and l_0 [27] and between l and l_1 [14] that appeared recently; our proof, given below, appears considerably simpler.

Proof. By Lemma 2.3, (14) and Proposition 2.6

$$l(\omega) + \frac{1}{V}\int\limits_M f_\omega \omega^n \leqslant l_0(\omega) \leqslant l_k(\omega) + I_k(\omega, \operatorname{Ric}\omega). \tag{25}$$

(For the second inequality we used (16) and the fact that $I_k(\omega_\varphi, \operatorname{Ric}\omega_\varphi) \geqslant 0$ for $\omega_\varphi \in \mathcal{H}^+_{c_1}$.) On the other hand, note first that from (20) it follows that $\int_{[0,1]}(I-J)(\omega, \omega_{\varphi_t})\,dt$ is bounded. As remarked in Section 3 the function $(I-J)(\omega, \omega_{\varphi_t})$ is nondecreasing in t. Hence

$$(1-\tau)(I-J)(\omega, \omega_{\varphi_\tau}) \leqslant \int\limits_{[\tau,1]} (I-J)(\omega, \omega_{\varphi_t})\,dt,$$

and therefore [17, p. 67]

$$\lim_{\tau\to 1^-} (1-\tau)(I-J)(\omega, \omega_{\varphi_\tau}) = 0. \tag{26}$$

Going back to (20) and using the identity $E_0(\omega, \omega_{\varphi_0}) + (I-J)(\omega, \omega_{\varphi_0}) = V^{-1}\int_M f_\omega \omega^n$ we have

$$\lim_{\tau\to 1^-} E_0(\omega, \omega_{\varphi_t}) = \frac{1}{V}\int\limits_M f_\omega \omega^n - \int\limits_{[0,1]} (I-J)(\omega, \omega_{\varphi_t})\,dt.$$

By a theorem of Ding and Tian we have [17, Theorem 1.2]

$$l(\omega) = \lim_{t\to 1^-} F(\omega, \omega_{\varphi_t}) = -\int\limits_{[0,1]} (I-J)(\omega, \omega_{\varphi_t})\,dt. \tag{27}$$

Combining with (25) we conclude that

$$l_0(\omega) = \lim_{t\to 1^-} E_0(\omega, \omega_{\varphi_t}) = l(\omega) + \frac{1}{V}\int\limits_M f_\omega \omega^n.$$

Finally, using (9), (5), (23) and (26) it follows that $\lim_{t\to 1^-} I_k(\omega_{\varphi_t}, \operatorname{Ric}\omega_{\varphi_t}) = 0$. Therefore, using Proposition 2.6 (16) again we have $l_0(\omega) \geqslant l_k(\omega) + I_k(\omega, \operatorname{Ric}\omega)$. $\square$

Remark 4.6. Note that from Proposition 2.6 it follows that if F is proper on $\mathcal{H}_{c_1}$ (equivalently on $\mathcal{H}^+_{c_1}$) with $F(\omega, \omega_\varphi) \geqslant \tau_\omega((I-J)(\omega, \omega_\varphi))$ then we have the inequality $E_k(\omega, \omega_\varphi) \geqslant \tau_\omega((I-J)(\omega, \omega_\varphi)) - I_k(\omega, \operatorname{Ric}\omega)$ on $\mathcal{H}^+_{c_1}$ (and for $k = 0, 1$ on $\mathcal{H}_{c_1}$). On the determination of explicit functions τ_ω we refer to [36,45,46].

5. Boundedness of energy functionals and the Moser–Trudinger–Onofri inequality

In this section we suppose that a Kähler–Einstein metric ω exists. First, we state the following fundamental theorem.

Theorem 5.1. *(See [6, Theorem A, Corollary 8.3], [5, Theorem 1].) Let (M, J, ω) be a Kähler–Einstein Fano manifold. Then $E_0(\omega, \omega_\varphi) \geqslant 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}$ and $E_n(\omega, \omega_\varphi) \geqslant 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}^+$ with equality if and only if $\omega_\varphi = h^\star \omega$ with $h \in \mathrm{Aut}(M, \mathrm{J})_0$.*

Building on these results, Song and Weinkove proved: (i) the first statement holds with E_0 replaced by E_1 (see also [35]), and (ii) the second statement holds with E_n replaced by E_k for each $k \in \{2, \ldots, n-1\}$. Proposition 2.6 provides a much simplified proof of these two facts. Moreover, it allows to improve on (ii). Let

$$\mathcal{A}_k := \{\omega_\varphi \in \mathcal{H}_{c_1} \colon\ E_k(\omega, \omega_\varphi) \geqslant 0\}. \tag{28}$$

Then we have shown that

$$\mathcal{A}_k \supseteq \mathcal{B}_k := \bigl\{\omega_\varphi \in \mathcal{H}_{c_1} \colon\ I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi) \geqslant 0\bigr\}. \tag{29}$$

For example, for $k = 1$ this gives $\mathcal{A}_1 = \mathcal{H}_{c_1}$, when $k = 2$ we have

$$\mathcal{A}_2 \supseteq \mathcal{B}_2 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} \colon\ \mathrm{Ric}\,\omega_\varphi + 2\omega_\varphi \geqslant 0\},$$

for $k = 3$

$$\mathcal{A}_3 \supseteq \mathcal{B}_3 \supseteq \{\omega_\varphi \in \mathcal{H}_{c_1} \colon\ \mathrm{Ric}\,\omega_\varphi + \omega_\varphi \geqslant 0\},$$

and for arbitrary k one may readily obtain an explicit bound (depending on k) on the set $\mathcal{B}_k$, and hence on $\mathcal{A}_k$, in terms of a lower bound on the Ricci curvature, using the definition (6).

Let $\omega_{\mathrm{FS},c}$ denotes the Fubini–Study form of constant Ricci curvature c on (S^2, J), the Riemann sphere, given locally by

$$\omega_{\mathrm{FS},c} = \frac{\sqrt{-1}}{c\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Here $V = \int_{S^2} \omega_{\mathrm{FS},c} = c_1([M])/c = 2/c$. For $c = 1/2\pi$ it is induced from restricting the Euclidean metric on $\mathbb{R}^3$ to the radius 1 sphere. Denote by $W^{1,2}(S^2)$ the space of functions on S^2 that are square-summable and so is their gradient (with respect to some Riemannian metric). The Moser–Trudinger–Onofri inequality states:

Theorem 5.2. *(See [32,33,50].) For $\omega = \omega_{\mathrm{FS},2/V}$ and any function φ on S^2 in $W^{1,2}(S^2)$ one has*

$$\frac{1}{V} \int\limits_{S^2} e^{-\varphi + \frac{1}{V}\int_{S^2} \varphi\omega} \omega \leqslant e^{\frac{1}{V}\int_{S^2} \frac{1}{2}\sqrt{-1}\partial\varphi\wedge\bar{\partial}\varphi}. \tag{30}$$

Equality holds if and only if ω_φ is the pull-back of ω by a Möbius transformation.

An alternative proof of this inequality has been given by Ding and Tian for those functions φ that belong to the subspace $\mathcal{H}_\omega \subseteq W^{1,2}(S^2)$. The proof uses the properties of F. We now note that our work provides a new and succinct proof of the original Moser–Trudinger–Onofri inequality entirely within the framework of exact energy functionals. This is the first proof that does not use symmetrization/rearrangement arguments. Other proofs of this inequality have been given by Onofri [33], Hong [25], Osgood, Phillips and Sarnak [34], Beckner [7], Carlen and Loss [11,12], Ghigi [23] (for more background we refer to Chang [13]).

Proof. By Theorem 5.1 and Proposition 2.6 $E_1(\omega,\cdot) \geqslant 0$ on $\mathcal{H}_\Omega$. Given $\varphi \in \mathcal{C}^\infty(S^2)$ there exists $\psi \in \mathcal{H}_\omega$ such that $\operatorname{Ric}\omega_\psi = \omega_\varphi$ by solving the Poisson equation on S^2. Thus by Lemma 2.4 $F(\omega,\cdot) \geqslant 0$ on $\mathcal{D}_\Omega$. Using the definition of F, for any smooth function φ we obtain (30). Since $\mathcal{C}^\infty(S^2)$ is dense in $W^{1,2}(S^2)$ we conclude. □

Ding and Tian showed that a restricted analogue of this inequality holds also for higher-dimensional manifolds:

Theorem 5.3. *(See [17].) Let (M, J) be a Fano manifold and let $\omega \in \mathcal{H}_{c_1}$. Assume that F is bounded from below on $\mathcal{H}_{c_1}$ and let $a = -\inf_{\mathcal{H}_{c_1}} F(\omega,\cdot)$. Then for each $\varphi \in \mathcal{H}_\omega$ holds*

$$\frac{1}{V}\int_M e^{-\varphi+\frac{1}{V}\int_M \varphi\omega^n}\omega^n \leqslant e^{J(\omega,\omega_\varphi)+a}. \tag{31}$$

If (M, J, ω) is Kähler–Einstein then $a = 0$.

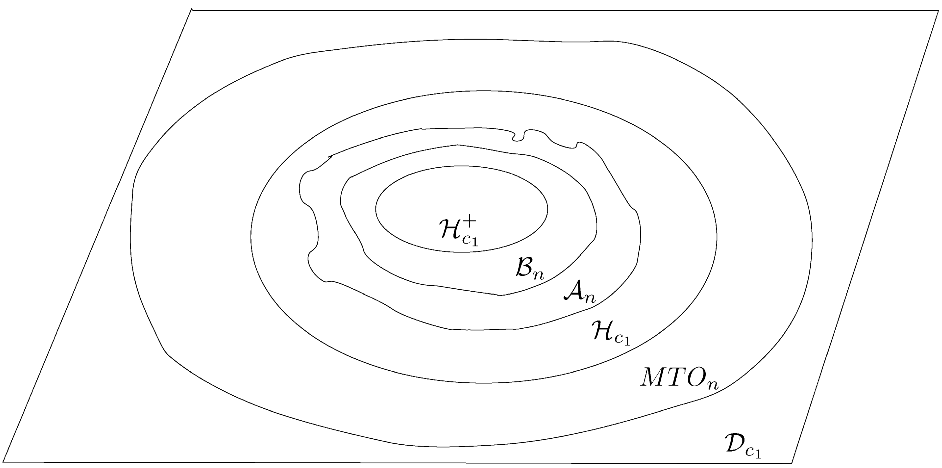

Fig. 1. Subspaces inside the space of closed forms representing the first Chern class of a Fano manifold.

Recall Jensen's inequality $\frac{1}{V}\int_M e^{-\varphi+\frac{1}{V}\int_M \varphi\omega^n}\omega^n \geqslant 1$ [24]. Now observe that in higher dimensions, due to Lemma 2.1, inequality (31) cannot be extended to all of $\mathcal{C}^\infty(M)$. A natural question is therefore: on a Kähler–Einstein manifold, what is the largest neighborhood of $\mathcal{H}_\omega$

inside $\mathcal{C}^\infty(M)$ on which (31) does hold? Naturally, we call such a neighborhood the Moser–Trudinger–Onofri neighborhood of $\mathcal{H}_\omega$, and put

$$MTO_n = \left\{\varphi \in \mathcal{C}^\infty(M)\colon \varphi \text{ satisfies (31) on the Fano manifold } (M, \mathrm{J}), \dim_{\mathbb{C}} M = n\right\}. \quad (32)$$

Using Lemma 2.4, we have the following characterization of the Moser–Trudinger–Onofri neighborhood. By abuse of notation we do not distinguish here between the set $\mathrm{Ric}(\mathcal{A}_n)$ in $\mathcal{H}_\Omega$ and the corresponding set in $\mathcal{H}_\omega$.

Theorem 5.4. *Let (M, J, ω) be a Kähler–Einstein Fano manifold. Then $\varphi \in \mathcal{C}^\infty(M)$ satisfies the generalized Moser–Trudinger–Onofri inequality* (31) *if and only if there exists a function $\psi \in \mathcal{C}^\infty(M)$ such that $\mathrm{Ric}\,\omega_\psi = \omega_\varphi$ and $\omega_\psi \in \mathcal{A}_n$. That is, $MTO_n = \mathrm{Ric}(\mathcal{A}_n) \supset \mathcal{H}_\omega$.*

Recall that $\mathcal{B}_n \subseteq \mathcal{A}_n$ and that we have bounds on $\mathcal{B}_n$ in terms of the Ricci curvature. Therefore, Theorem 5.4 shows that in higher dimensions the Moser–Trudinger–Onofri inequality is related to Ricci curvature and holds on a set strictly larger than the space of Kähler potentials. It would be interesting to improve the bounds both on $\mathcal{A}_n$ and on $\mathcal{B}_n$.

We now state another corollary of our arguments.

Corollary 5.5. *Let (M, J, ω) be a Fano manifold. Then the Ricci energy E_n is unbounded from below on $\mathcal{H}_{c_1}$ if and only if $n > 1$.*

Before concluding, we remark that Theorem 5.3 can be strengthened using the results obtained here. The same applies to later extensions of this inequality [36,46] and will figure in a subsequent article.

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Appendix A

In this appendix we prove Proposition 2.6. First, in order to establish formula (16) we show that the variations of both sides of the equation agree.

$$-(k+1)V\frac{d}{dt}I_k(\omega_\varphi, \mathrm{Ric}\,\omega_\varphi)$$
$$= \frac{d}{dt}\int\limits_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial f_{\omega_\varphi} \wedge \sum_{l=0}^{k-1}(k-l)\omega_\varphi^{n-1-l} \wedge (\mathrm{Ric}\,\omega_\varphi)^l$$

$$= \frac{d}{dt}\int_M f_{\omega_\varphi}(\operatorname{Ric}\omega_\varphi - \omega_\varphi)\wedge\sum_{l=0}^{k-1}(k-l)\omega_\varphi^{n-1-l}\wedge(\operatorname{Ric}\omega_\varphi)^l$$

$$= \frac{d}{dt}\int_M f_{\omega_\varphi}\left(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l\right)$$

$$= \int_M \dot f_{\omega_\varphi}\left(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l\right) \tag{A.1}$$

$$+ \int_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial\dot\varphi\wedge\left(\sum_{l=1}^{k}(n-l)\omega_\varphi^{n-l-1}\wedge(\operatorname{Ric}\omega_\varphi)^l - kn\omega_\varphi^{n-1}\right) \tag{A.2}$$

$$- \int_M f_{\omega_\varphi}\sqrt{-1}\partial\bar\partial\Delta_\varphi\dot\varphi\sum_{l=1}^{k}l\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^{l-1}. \tag{A.3}$$

First, we write (A.1) as

$$\int_M \dot f_{\omega_\varphi}\left(-k\omega_\varphi^n + \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l\right) =: \iota_1 + \mu_1.$$

We will evaluate (A.2) and (A.3) by substituting once again $\sqrt{-1}\partial\bar\partial f_{\omega_\varphi} = \operatorname{Ric}\omega_\varphi - \omega_\varphi$. For (A.2) we get

$$\int_M \dot\varphi(\operatorname{Ric}\omega_\varphi - \omega_\varphi)\wedge\left(-kn\omega_\varphi^{n-1} + \sum_{l=1}^{k}(n-l)\omega_\varphi^{n-l-1}\wedge(\operatorname{Ric}\omega_\varphi)^l\right)$$

$$= \int_M \dot\varphi\bigg(kn\omega_\varphi^n - kn\omega_\varphi^{n-1}\wedge\operatorname{Ric}\omega_\varphi - (n-1)\omega_\varphi^{n-1}\wedge\operatorname{Ric}\omega_\varphi$$

$$+ \sum_{l=2}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l + (n-k)\omega_\varphi^{n-k-1}\wedge(\operatorname{Ric}\omega_\varphi)^{k+1}\bigg)$$

$$= \int_M \dot\varphi\bigg(\big[-(n-k)+(k+1)n-k\big]\omega_\varphi^n - (k+1)n\omega_\varphi^{n-1}\wedge\operatorname{Ric}\omega_\varphi$$

$$+ \sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l + (n-k)\omega_\varphi^{n-k-1}\wedge(\operatorname{Ric}\omega_\varphi)^{k+1}\bigg)$$

$$=: (\kappa_1 + \lambda_1 + \iota_2) + \lambda_2 + \mu_2 + \kappa_2.$$

For (A.3) we get

$$\begin{aligned}
&\int_M \Delta_\varphi\dot\varphi(\omega_\varphi-\operatorname{Ric}\omega_\varphi)\wedge\sum_{l=1}^{k} l\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^{l-1}\\
&=\int_M \Delta_\varphi\dot\varphi\left(\omega_\varphi^n+\sum_{l=1}^{k-1}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l-k\omega_\varphi^{n-k}\wedge(\operatorname{Ric}\omega_\varphi)^k\right)\\
&=\int_M \Delta_\varphi\dot\varphi\left(\sum_{l=1}^{k}\omega_\varphi^{n-l}\wedge(\operatorname{Ric}\omega_\varphi)^l-(k+1)\omega_\varphi^{n-k}\wedge(\operatorname{Ric}\omega_\varphi)^k\right)\\
&=: \mu_3+\kappa_3.
\end{aligned}$$

Noting that $\dot f_{\omega_\varphi}=-\Delta_\varphi\dot\varphi-\dot\varphi+c$ with c a constant yields $\iota_1+\iota_2=-kcV$ and $\mu_1+\mu_2+\mu_3=kcV$. Note that $\kappa_1+\kappa_2+\kappa_3=-(k+1)V\frac{d}{dt}E_k(\omega,\omega_\varphi)$ and $\lambda_1+\lambda_2=(k+1)V\frac{d}{dt}E_0(\omega,\omega_\varphi)$. This completes the proof of (16).

Formulas (15) and (17) now follow: first use (16) with $k=n$ to express E_0 in terms of E_n and J, and then substitute this expression back into (16) and apply Lemma 2.2.

Let us note that one way one could arrive at these formulas would be to use the expression for $(k+1)I_k-kI_{k-1}$ (see (6)) and Lemma 2.3 together with the observation

$$\begin{aligned}
&\frac{d}{dt}\big((k+1)E_k-kE_{k-1}\big)(\omega,\omega_{\varphi_t})\\
&\quad=-\frac{1}{V}\int_M\dot\varphi_t\omega_{\varphi_t}^n-\frac{d}{dt}\left(\frac{1}{V}\int_M f_{\omega_{\varphi_t}}(\operatorname{Ric}\omega_{\varphi_t})^k\wedge\omega_{\varphi_t}^{n-k}\right).
\end{aligned}\tag{A.4}$$

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