The Heyde theorem for locally compact Abelian groups

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Abstract

We prove a group analogue of the well-known Heyde theorem where a Gaussian measure is characterized by the symmetry of the conditional distribution of one linear form given another. Let \( X \) be a locally compact second countable Abelian group containing no subgroup topologically isomorphic to the circle group \( T \), \( G \) be the subgroup of \( X \) generated by all elements of order 2, and \( \text{Aut}(X) \) be the set of all topological automorphisms of \( X \). Let \( \alpha_j, \beta_j \in \text{Aut}(X), \ j = 1, 2, \ldots, n, \ n \geq 2, \) such that \( \beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X) \) for all \( i \neq j \). Let \( \xi_j \) be independent random variables with values in \( X \) and distributions \( \mu_j \) with non-vanishing characteristic functions. If the conditional distribution of \( L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n \) given \( L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \) is symmetric, then each \( \mu_j = \gamma_j * \rho_j \), where \( \gamma_j \) are Gaussian measures, and \( \rho_j \) are distributions supported in \( G \).

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1. Introduction

By the well-known Skitovich–Darmois theorem a Gaussian measure on the real line can be characterized by the independence of two linear forms of independent random variables. A similar result was obtained by C.C. Heyde, where a Gaussian measure is characterized by the symmetry of the conditional distribution of one linear form given another. He proved the following theorem ([9], see also [10, §13.4]).
Theorem A. Let $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables, let $\alpha_j, \beta_j$ be nonzero constants such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ is symmetric, then all $\xi_j$ are Gaussian.

In the last years much attention has been devoted to generalizing of the classical characterization theorems into various algebraic structures such as locally compact Abelian groups, Lie groups, quantum groups, symmetric spaces (see e.g. [7,11], and [5] where one can find additional references). The present article continues these researches.

Let $X$ be a locally compact second countable Abelian group and let $Y = X^*$ be its character group. Denote by $(x, y)$ the value of a character $y \in Y$ at an element $x \in X$. Let $\text{Aut}(X)$ be the set of all topological automorphisms of $X$. Denote by $\mathbb{T}$ the circle group (the one-dimensional torus) $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$. Let $M^1(X)$ be the convolution semigroup of probability distributions on $X$. For $\mu \in M^1(X)$ denote by

$$\hat{\mu}(y) = \int (x, y) \, d\mu(x), \quad y \in Y$$

its characteristic function (the Fourier transform). If $\mu = \mu_1 * \mu_2$, $\mu_j \in M^1(X)$, then $\mu_j$ are called factors of $\mu$. A probability measure $\mu \in M^1(X)$ is called Gaussian (in the sense of Parthasarathy) [12, Ch. IV, §6] if its characteristic function can be represented in the form

$$\hat{\mu}(y) = (x, y) \exp\{-\varphi(y)\}, \quad y \in Y,$$

where $x \in X$ and $\varphi$ is a continuous nonnegative function satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \quad (1)$$

Taking into account that in the article we will deal only with Gaussian measures in the sense of Parthasarathy we will name them Gaussian. Denote by $\Gamma(X)$ the set of Gaussian measures on the group $X$.

Let $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables with values in $X$ and distributions $\mu_j$ with non-vanishing characteristic functions. Let $\alpha_j, \beta_j \in \text{Aut}(X)$ such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X)$ for all $i \neq j$. Consider linear forms $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ and $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$. It was proved in [4] that the symmetry of the conditional distribution of $L_2$ given $L_1$ implies that all $\mu_j \in \Gamma(X)$ if and only if $X$ contains no elements of order 2. If a group $X$ contains elements of order 2, then the following natural problem arises:

Problem 1. Let $X$ be a locally compact second countable Abelian group, and assume that $X$ contains elements of order 2. Let $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables with values in $X$ and distributions $\mu_j$ with non-vanishing characteristic functions. Let $\alpha_j, \beta_j \in \text{Aut}(X)$ such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X)$ for all $i \neq j$. What distributions $\mu_j$ are characterized by the symmetry of the conditional distribution of $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$?

This problem was solved in [4] for the case when $X$ is the two-dimensional torus $\mathbb{T}^2$. Namely, the following theorem holds: Let $\xi_1, \xi_2$ be independent random variables with values in $\mathbb{T}^2$
and distributions \( \mu_j \) with non-vanishing characteristic functions. Consider linear forms \( L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 \) and \( L_2 = \beta_1 \xi_1 + \beta_2 \xi_2 \), where \( \alpha_j, \beta_j \in \text{Aut}(\mathbb{T}^2) \) such that \( \beta_1 \alpha_1^{-1} \pm \beta_2 \alpha_2^{-1} \in \text{Aut}(\mathbb{T}^2) \).

If the conditional distribution of \( L_2 \) given \( L_1 \) is symmetric, then \( \mu_j = \gamma_j \ast \rho_j \), where \( \gamma_j \in \Gamma(\mathbb{T}^2) \), \( \rho_j \in M^1(G) \), \( j = 1, 2 \), and \( G \) is the subgroup of \( \mathbb{T}^2 \) generated by all elements of order 2.

The aim of this article is to solve Problem 1 for groups \( X \) containing no subgroup topologically isomorphic to \( \mathbb{T} \). It turned out that the answer is the same as in case when \( X \) is the two-dimensional torus \( \mathbb{T}^2 \). We will prove the following theorem.

**Theorem 1.** Let \( X \) be a locally compact second countable Abelian group containing no subgroup topologically isomorphic to \( \mathbb{T} \). Let \( G \) be the subgroup of \( X \) generated by all elements of order 2. Let \( \alpha_j, \beta_j \in \text{Aut}(X) \), \( j = 1, 2, \ldots, n, n \geq 2 \), such that \( \beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X) \) for all \( i \neq j \). Let \( \xi_j \) be independent random variables with values in \( X \) and distributions \( \mu_j \) with non-vanishing characteristic functions. If the conditional distribution of \( L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n \) given \( L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \) is symmetric, then each \( \mu_j = \gamma_j \ast \rho_j \), where \( \gamma_j \in \Gamma(X) \), \( \rho_j \in M^1(G) \).

The proof of Theorem 1 can be reduced to solving of a functional equation in the class of continuous positive definite functions on a locally compact Abelian group.

To prove Theorem 1 we shall use some results of the structure theory for locally compact Abelian groups and the duality theory (see [8]). If \( L \) is a subgroup of \( Y \), then denote by \( A(X, L) = \{ x \in X : (x, y) = 1 \text{ for all } y \in L \} \) its annihilator. Denote by \( c_X \) the connected component of zero of \( X \) and by \( b_X \) the subgroup of \( X \) consisting of all compact elements of \( X \). For each natural number \( n \) let \( f_n : X \mapsto X \) be a homomorphism \( f_n(x) = nx \), and \( X^{(n)} = \text{Im} f_n \), \( X^{(n)} = \text{Ker} f_n \). For \( \mu \in M^1(X) \) we define the distribution \( \hat{\mu} \in M^1(X) \) by the formula \( \hat{\mu}(E) = \mu(-E) \) for all Borel sets \( E \subset X \). Observe that \( \hat{\mu}(y) = \overline{\mu(y)} \). We denote by \( \sigma(\mu) \) the support of \( \mu \in M^1(X) \). It should be noted that if \( \hat{\mu}(y) = 1 \) for all \( L \), then \( \sigma(\mu) \subset A(X, L) \). If \( \alpha \in \text{Aut}(X) \), then the adjoint automorphism \( \tilde{\alpha} \in \text{Aut}(Y) \) is defined by the formula \( (x, \tilde{\alpha}y) = (\alpha x, y) \) for all \( x \in X, y \in Y \). A subgroup \( G \) of \( X \) is called characteristic if \( \alpha(G) = G \) for any \( \alpha \in \text{Aut}(X) \).

Let \( \psi : Y \mapsto \mathbb{C} \) be an arbitrary function, and let \( h \in Y \). We denote by \( \Delta_h \) the finite difference operator

\[
\Delta_h \psi(y) = \psi(y + h) - \psi(y), \quad y \in Y.
\]

A continuous function \( \psi \) on \( Y \) is called a polynomial if for some natural \( m \) the equality

\[
\Delta_{-h}^m \psi(y) = 0
\]

is fulfilled for all \( y, h \in Y \). The minimal \( m \) for which this equality holds is called the degree of \( \psi \). Observe that a nonzero function satisfying (1) is a polynomial of degree 2.

2. Proof of lemmas

To prove Theorem 1 we need some lemmas.

**Lemma 1.** (See [4].) Let \( Y \) be a locally compact Abelian group, \( \psi_j, j = 1, 2, \ldots, n, n \geq 2 \), be functions on \( Y \) satisfying the equation

\[
\Delta_{-h}^m \psi(y) = 0
\]
\[ \sum_{j=1}^{n} [\psi_j(u + \delta_j v) - \psi_j(u - \delta_j v)] = 0, \quad u, v \in Y, \]  
\[ \text{(2)} \]

where \( \hat{\delta}_j \in \text{Aut}(Y) \) such that \( \hat{\delta}_i \pm \hat{\delta}_j \in \text{Aut}(Y) \) for \( i \neq j \). Then each function \( \psi_j \) satisfies the equation

\[ \Delta_k \Delta_h^{2n-2} \psi_j(y) = 0, \]

\[ \text{(3)} \]

where \( k, h \) and \( y \) are arbitrary elements of \( Y \).

**Lemma 2.** Let \( n \) be a natural number and \( X \) be a locally compact Abelian group satisfying the condition \( X^{(n)} = c_X \). Then \( X = c_X \times \tilde{X} \), where \( \tilde{X} \) is a subgroup of \( X \) such that \( \tilde{X}_{(n)} = \tilde{X} \).

**Proof.** Assume first that \( X \) is a compact group. Then \( Y = X^* \) is a discrete group. Set \( L = X/c_X \). It follows from the condition of the lemma that \( L_{(n)} = L \). This implies that \( (L^n)_{(n)} = L^n \). Taking into account that \( L^n \cong A(Y, c_X) = b_Y \), we see that \( b_Y \) is a bounded subgroup of \( Y \). Since \( b_Y \) is a pure subgroup of \( Y \), by the Kulikov theorem (see [6, §27]) \( b_Y \) is a direct factor of \( Y \), i.e.,

\[ Y = D \times b_Y, \]

\[ \text{(4)} \]

where \( D \cong Y/b_Y \cong (c_X)^n \). Taking into account that \( c_X = A(X, b_Y) \), it follows from (4) that \( X = c_X \times \tilde{X} \), where \( \tilde{X} \cong (b_Y)^n \). Thus, the statement of the lemma is proved for a compact group \( X \).

Let \( X \) be an arbitrary locally compact Abelian group. By the structure theorem \( X \cong \mathbb{R}^m \times G \), where \( m \geq 0 \) and \( G \) contains a compact open subgroup. Without restricting the generality, we may assume that \( X = \mathbb{R}^m \times G \). This implies that \( c_X = \mathbb{R}^m \times c_G \). Let \( K \) be a compact open subgroup of \( G \). It is obvious that \( c_K = c_G \) and \( K_{(n)} = c_K \). As has been shown above \( K = c_K \times \tilde{K} \), where \( \tilde{K}_{(n)} = \tilde{K} \). Let \( \pi \) be a continuous homomorphism \( \pi : K \mapsto c_K \), such that the restriction of \( \pi \) to \( c_K \) is the identity automorphism. Since the group \( c_K \) is divisible, the homomorphism \( \pi \) can be extended to a homomorphism \( \tilde{\pi} : G \mapsto c_K = c_G \) (see [8, §A.7]). Taking into account that the homomorphism \( \tilde{\pi} \) is continuous on an open subgroup \( K \), we conclude that \( \tilde{\pi} \) is continuous on \( G \), and then \( G = c_G \times \tilde{G} \) [8, §6.22]. Hence, \( X = \mathbb{R}^m \times c_G \times \tilde{G} = c_X \times \tilde{X} \). Lemma 2 is proved. \( \square \)

**Lemma 3.** (See [4].) Let \( X \) be a locally compact second countable Abelian group. Let \( \xi_1, \ldots, \xi_n, n \geq 2 \), be independent random variables with values in \( X \) and distributions \( \mu_j \). Assume that \( \delta_j \in \text{Aut}(X) \). The conditional distribution of \( L_2 = \delta_1 \xi_1 + \cdots + \delta_n \xi_n \) given \( L_1 = \xi_1 + \cdots + \xi_n \) is symmetric if and only if the characteristic functions of the distributions \( \mu_j \) satisfy the equation

\[ \prod_{j=1}^{n} \hat{\mu}_j(u + \delta_j v) = \prod_{j=1}^{n} \hat{\mu}_j(u - \delta_j v), \quad u, v \in Y. \]

\[ \text{(5)} \]

**Lemma 4.** Let \( X \) be a locally compact second countable Abelian group. Let \( \alpha_j, \beta_j \in \text{Aut}(X), j = 1, 2, \ldots, n, n \geq 2 \), such that \( \beta_i \alpha_j^{-1} = \beta_j \alpha_i^{-1} \in \text{Aut}(X) \) for all \( i \neq j \). Let \( \xi_j \) be independent random variables with values in \( X \) and distributions \( \mu_j \) with non-vanishing characteristic functions. Assume that the conditional distribution of \( L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n \) given
Proof. Passing to the random variables \( \tilde{\xi}_j = \alpha_j \xi_j \) we can assume without loss of generality that \( L = \xi_1 + \cdots + \xi_n \) and \( L' = \delta_1 \xi_1 + \cdots + \delta_n \xi_n \), where \( \delta_j \in \text{Aut}(X) \) such that \( \delta_i \pm \delta_j \in \text{Aut}(X) \) for \( i \neq j \). By Lemma 3, the characteristic functions \( \hat{\mu}_j(y) \) satisfy Eq. (5). It is clear that the characteristic functions of the distributions \( \hat{\mu}_j \) satisfy Eq. (5) too. Hence, the characteristic functions of the distributions \( v_j = \mu_j \ast \hat{\mu}_j \) also satisfy Eq. (5). Observe that \( \hat{v}_j(y) = |\hat{\mu}_j(y)|^2 > 0 \), and set \( \psi_j(y) = -\ln \hat{v}_j(y) \). It follows from (5) that the functions \( \psi_j \) satisfy Eq. (2) and hence by Lemma 1, the functions \( \psi_j \) satisfy Eq. (3). This implies that the function \( \psi_j \) is a polynomial on the subgroup \( Y^{(2)} \). Particularly, the function \( \psi_j \) is a polynomial on the subgroup \( b_{Y^{(2)}} \). It is well known (see [2]) that any polynomial on the subgroup consisting of all compact elements of a group is a constant. So, we have \( \psi_j(y) = \psi_j(0) = 0 \) for \( y \in b_{Y^{(2)}} \). Hence, \( \hat{v}_j(y) = 1 \) for all \( y \in b_{Y^{(2)}} \) and for this reason \( \sigma(v_j) \subseteq K = A(X,b_{Y^{(2)}}) \). Since \( c_X = A(X,b_Y) \), we have \( K = \{ x \in X: (x,2y) = 1 \text{ for all } y \in b_Y \} = \{ x \in X: (2x,y) = 1 \text{ for all } y \in b_Y \} = \{ x \in X: 2x \in c_X \} \). Since \( \hat{v}_j(y) = 1 \) for \( y \in b_{Y^{(2)}} \), the restrictions of the characteristic functions \( \hat{\mu}_j(y) \) to the subgroup \( b_{Y^{(2)}} \) are some characters of the subgroup \( b_{Y^{(2)}} \). We can extend these characters to some characters of the group \( Y \). By the duality theorem, there exist elements \( x_j \in X \) such that \( \hat{\mu}_j(y) = (x_j,y) \) for \( y \in b_{Y^{(2)}} \). Set \( x'_j = -\delta_j^{-1} \sum_{j=2}^n \delta_j x_j \) and \( x_j = x_j, j = 2, \ldots, n \). Since \( \sum_{j=1}^n \delta_j x_j = 0 \), the functions \( f_j(y) = (x'_j,y) \) satisfy Eq. (15) on the group \( Y \). Put \( \hat{\lambda}_j(y) = \hat{\mu}_j(y)(x'_j,y), j = 1,2,\ldots,n \). It is clear that the characteristic functions \( \hat{\lambda}_j(y) \) possess the following properties:

(i) \( \hat{\lambda}_j(y) \) satisfy Eq. (5) on the group \( Y \);
(ii) \( \hat{\lambda}_j(y) = 1 \) for \( y \in b_{Y^{(2)}} \) and hence, \( \sigma(\lambda_j) \subseteq K, j = 2, \ldots, n \);
(iii) \( \hat{\lambda}_1(y) = (x''_1,y) \), where \( x''_1 = x_1 + \delta_1^{-1} \sum_{j=2}^n \delta_j x_j, y \in b_{Y^{(2)}} \).

Inasmuch as \( b_{Y^{(2)}} \) is a characteristic subgroup of the group \( Y \), it follows from (i) and (ii) that the restriction of \( \hat{\lambda}_1(y) \) to the subgroup \( b_{Y^{(2)}} \) is of the form

\[
\hat{\lambda}_1(u + \delta_1 v) = \hat{\lambda}_1(u - \delta_1 v), \quad u, v \in b_{Y^{(2)}}.
\]

Putting here \( u = \delta_1 v \) we conclude that \( \hat{\lambda}_1(y) = 1 \) for \( y \in b_{Y^{(2)}} \). This implies that \( \sigma(\lambda_1) \subseteq F = A(X,b_{Y^{(4)}}) = \{ x \in X: 4x \in c_X \} \). Since \( K \subseteq F \) and taking into account that \( F \) is a characteristic subgroup of the group \( X \), we can assume from the beginning that the group \( X \) satisfies the condition \( X^{(4)} = c_X \). By Lemma 2, \( X = c_X \times \bar{X} \), where \( \bar{X}^{(4)} = X \). It follows from this that \( Y = D \times b_Y \), where \( D \cong c_X^* \). Represent the element \( x''_1 \) in the form \( x''_1 = c + \bar{x} \), where \( c \in c_X, \bar{x} \in \bar{X} \). This implies that \( (x''_1,y) = (\bar{x},y) \) for \( y \in b_Y \). Set \( g_1(y) = (\bar{x},y), g_j(y) = 1, y \in Y, j = 2, \ldots, n \), and \( \hat{\mu}'_j(y) = \hat{\lambda}_j(y)g_j(y), j = 1,2,\ldots,n \). It follows from (iii) that \( \hat{\mu}'_j(y) = 1 \) for \( y \in b_{Y^{(2)}} \). Moreover \( \hat{\mu}'_j(y) = 1 \) for \( y \in b_{Y^{(2)}}, j = 2, \ldots, n \), and hence, \( \sigma(\mu'_j) \subseteq A(X,b_{Y^{(2)}}) = K, j = 1,2,\ldots,n \). Since \( \hat{\mu}'_j(y) = 1 \) for \( y \in b_{Y^{(2)}}, j = 1,2,\ldots,n \), the characteristic functions \( \hat{\mu}'_j(y) \) are invariant with respect to the subgroup \( b_{Y^{(2)}} \) and hence, they satisfy Eq. (5) on the subgroup \( b_Y \).

Taking into account that the characteristic functions \( \hat{\mu}'_j(y) \) and \( \hat{\lambda}_j(y) \) satisfy Eq. (5) on the sub-
group $b_Y$ we conclude that the functions $g_j(y)$ also satisfy Eq. (5) on the subgroup $b_Y$. It follows from this that

$$(\tilde{x}, u + \tilde{\delta}_1 v) = (\tilde{x}, u - \tilde{\delta}_1 v), \quad u, v \in b_Y,$$

and hence, $2\tilde{x} = 0$. This implies that the characteristic functions $\hat{\mu}_j'(y)$ satisfy Eq. (5) on the group $Y$. By Lemma 3, if independent random variables $\xi_j'$ take values in $X$ and have distributions $\mu_j'$, then the conditional distribution of $L_2' = \beta_1 \xi_1' + \cdots + \beta_n \xi_n'$ given $L_1' = \alpha_1 \xi_1' + \cdots + \alpha_n \xi_n'$ is symmetric. Lemma 4 is proved.

**Remark 1.** Taking into account that $K = \{x \in X: 2x \in c_X\}$ is a characteristic subgroup of $X$, we draw the following conclusion from Lemma 4. Let $\xi_j$, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables with values in a locally compact second countable Abelian group $X$ and distributions $\mu_j$ with non-vanishing characteristic functions. Let $\alpha_j, \beta_j \in \text{Aut}(X)$ such that $\beta_i \alpha_i^{-1} \pm \beta_j \alpha_j^{-1} \in \text{Aut}(X)$ for all $i \neq j$. Assume that the conditional distribution of $L_2 = \beta_1 \xi_1 + \cdots + \beta_n \xi_n$ given $L_1 = \alpha_1 \xi_1 + \cdots + \alpha_n \xi_n$ is symmetric. Studying the possible distributions $\mu_j$ we can suppose, without restricting of generality, that $X^{(2)} = c_X$.

**Lemma 5.** (See [2].) If a locally compact second countable Abelian group $X$ contains no subgroup topologically isomorphic to $\mathbb{T}$ and the characteristic function of a distribution $\mu \in M_1(X)$ is of the form

$$\hat{\mu}(y) = \exp\{-\psi(y)\},$$

where $\psi$ is a polynomial, then $\mu \in \Gamma(X)$.

**Lemma 6.** Let $Y$ be a locally compact Abelian group, let $\psi$ be a continuous function on $Y$ satisfying the equation

$$\Delta_2 \Delta_h^2 \psi(y) = 0, \quad h, k, y \in Y,$$

and $\psi(-y) = \psi(y)$, $\psi(0) = 0$. Then the function $\psi$ can be represented in the form

$$\psi(y) = \varphi(y) + r_\alpha, \quad y \in y_\alpha + Y^{(2)},$$

where $\varphi$ is a continuous function on $Y$ satisfying Eq. (1), and $Y = \bigcup_{\alpha} (y_\alpha + Y^{(2)})$, $y_0 = 0$, is a decomposition of the group $Y$ with respect to the subgroup $Y^{(2)}$.

**Proof.** It should be noted that any function $\varphi$ on the subgroup $Y^{(2)}$ satisfying Eq. (1) can be extended to the function $\tilde{\varphi}$ on the group $Y$ by the formula

$$\tilde{\varphi}(y) = \frac{1}{4} \varphi(2y), \quad y \in Y.$$

The function $\tilde{\varphi}$ also satisfies Eq. (1). Analogously, any additive function $l$, i.e. any function satisfying the Cauchy equation
\[
I(u + v) = I(u) + I(v) \quad (9)
\]
on the subgroup \(Y^{(2)}\) can be extended to an additive function \(\tilde{I}\) on the group \(Y\) by the formula
\[
\tilde{I}(y) = \frac{1}{2} I(2y), \quad y \in Y. \quad (10)
\]
Observe that any polynomial \(f\) on \(Y\) of degree \(\leq 2\) can be represented as a sum
\[
f(y) = \varphi(y) + I(y) + c,
\]
where \(\varphi\) is a continuous function satisfying Eq. (1), \(I\) is a continuous function satisfying Eq. (9), and \(c\) is a constant.

Note that there exists one-to-one correspondence between functions \(\varphi\) satisfying (1) and biadditive functions \(\Phi(u, v)\). Namely, \(\Phi(u, v) = \frac{1}{2} [\varphi(u + v) - \varphi(u) - \varphi(v)]\), \(\varphi(y) = \Phi(y, y)\).

Consider an arbitrary coset \(y_{\alpha} + Y^{(2)}\) of the group \(Y\) with respect to the subgroup \(Y^{(2)}\). The function \(\psi(y_{\alpha} + y)\), as a function of \(y\) satisfies Eq. (6). For this reason its restriction to the subgroup \(Y^{(2)}\) is a polynomial of degree \(\leq 2\). Hence, we have the following representation
\[
\psi(y_{\alpha} + y) = \varphi_{\alpha}(y) + l_{\alpha}(y) + c_{\alpha}, \quad y \in \overline{Y^{(2)}}, \quad (11)
\]
where the function \(\varphi_{\alpha}\) satisfies Eq. (1), the function \(l_{\alpha}\) satisfies Eq. (9), and \(c_{\alpha}\) is a constant.

Extend functions \(\varphi_{\alpha}\) and \(l_{\alpha}\) from the subgroup \(Y^{(2)}\) to the functions \(\tilde{\varphi}_{\alpha}\) and \(\tilde{l}_{\alpha}\) on the group \(Y\) by formulas (8) and (10) respectively. Passing from the function \(\tilde{\varphi}_{\alpha}\) to the corresponding biadditive function \(\Phi_{\alpha}\) we find from (11)
\[
\psi(-y_{\alpha} - y) = \psi(y_{\alpha} + (-2y_{\alpha} - y))
\]
\[
= \tilde{\varphi}_{\alpha}(-2y_{\alpha} - y) + \tilde{l}_{\alpha}(-2y_{\alpha} - y) + c_{\alpha}
\]
\[
= 4\Phi_{\alpha}(y_{\alpha}, y_{\alpha}) + 4\Phi_{\alpha}(y_{\alpha}, y) + \Phi_{\alpha}(y, y)
\]
\[
- 2\tilde{l}_{\alpha}(y_{\alpha}) - \tilde{l}_{\alpha}(y) + c_{\alpha}, \quad y \in \overline{Y^{(2)}}. \quad (12)
\]
Since
\[
\psi(y_{\alpha} + y) = \Phi_{\alpha}(y_{\alpha}, y) + \tilde{l}_{\alpha}(y) + c_{\alpha}, \quad y \in \overline{Y^{(2)}}, \quad (13)
\]
and \(\psi(-y) = \psi(y)\), we conclude from equalities (12) and (13) that
\[
2\Phi_{\alpha}(y_{\alpha}, y) = \tilde{l}_{\alpha}(y), \quad y \in \overline{Y^{(2)}}. \quad (14)
\]
It follows from (13) and (14) that
\[
\psi(y_{\alpha} + y) = \Phi_{\alpha}(y_{\alpha} + y, y_{\alpha} + y) - 2\Phi_{\alpha}(y_{\alpha}, y) - \Phi_{\alpha}(y_{\alpha}, y_{\alpha}) + \tilde{l}_{\alpha}(y) + c_{\alpha}
\]
\[
= \tilde{\varphi}_{\alpha}(y_{\alpha} + y) + r_{\alpha}, \quad y \in \overline{Y^{(2)}}, \quad (15)
\]
where \(r_{\alpha}\) is a constant.
Putting \( k = h \) in (6) we obtain the equation
\[
\psi(y + 4h) - 2\psi(y + 3h) + 2\psi(y + h) - \psi(y) = 0, \quad y, h \in Y.
\] (16)

Substituting \( y = 0, h = y_\alpha + 2u, u \in Y \) in (16) we arrive at
\[
\psi(4y_\alpha + 8u) - 2\psi(3y_\alpha + 6u) + 2\psi(y_\alpha + 2u) = 0, \quad u \in Y.
\]

Taking into account (15) we infer that
\[
\tilde{\varphi}_0(4y_\alpha + 8u) - 2\tilde{\varphi}_\alpha(3y_\alpha + 6u) + 2\tilde{\varphi}_\alpha(y_\alpha + 2u) = 0, \quad u \in Y.
\]

Passing here from the functions \( \tilde{\varphi}_0 \) and \( \tilde{\varphi}_\alpha \) to the corresponding biadditive functions \( \Phi_0 \) and \( \Phi_\alpha \) we obtain
\[
4(\Phi_0(u, u) - \Phi_\alpha(u, u)) + 4(\Phi_0(y_\alpha, u) - \Phi_\alpha(y_\alpha, u)) + (\Phi_0(y_\alpha, y_\alpha) - \Phi_\alpha(y_\alpha, y_\alpha)) = 0, \quad u \in Y.
\]

This implies that \( \tilde{\varphi}_\alpha(y) \equiv \tilde{\varphi}_0(y) \), and representation (7) follows from (15). Lemma 6 is proved. \( \Box \)

Denote by \( \mathbb{R}^{N_0} \) the space of all sequences of real numbers equipped with the product topology. We note that the topological group \( \mathbb{R}^{N_0} \) is not locally compact. Denote by \( \mathbb{R}^{N_0*} \) the space of all finitary sequences of real numbers with the topology of strictly inductive limit of spaces \( \mathbb{R}^n \). The topological group \( \mathbb{R}^{N_0*} \) is not locally compact either. Let \( t = (t_1, \ldots, t_n, \ldots) \in \mathbb{R}^{N_0} \) and \( s = (s_1, \ldots, s_n, 0, \ldots) \in \mathbb{R}^{N_0*} \). Set \( \langle t, s \rangle = \sum_{j=1}^{\infty} t_j s_j \), and put \( \langle t, s \rangle = \exp(i \langle t, s \rangle) \).

Let \( \mu \) be a distribution on the group \( \mathbb{R}^{N_0} \). We define the characteristic function of \( \mu \) by the formula \( \tilde{\mu}(s) = \int_{\mathbb{R}^{N_0}} (t, s) d\mu(t) \), \( s \in \mathbb{R}^{N_0*} \). Let \( A = (\alpha_{ij})_{i,j=1}^{\infty} \) be a symmetric positive semidefinite matrix, i.e. the square form \( \langle As, s \rangle = \sum_{i,j=1}^{\infty} \alpha_{ij} s_i s_j \) is nonnegative for all \( s \in \mathbb{R}^{N_0*} \). We can define now a Gaussian measure on the group \( \mathbb{R}^{N_0} \) (see, e.g. in [3, §5]). A probability measure \( \mu \) on the group \( \mathbb{R}^{N_0} \) is called Gaussian if its characteristic function is represented in the form
\[
\tilde{\mu}(s) = (t, s) \exp\{-\langle As, s \rangle\}, \quad s \in \mathbb{R}^{N_0*},
\]
where \( t \in \mathbb{R}^{N_0} \), and \( A = (\alpha_{ij})_{i,j=1}^{\infty} \) is a symmetric positive semidefinite matrix. Denote by \( \Gamma(\mathbb{R}^{N_0}) \) the set of Gaussian measures on the group \( \mathbb{R}^{N_0} \).

**Lemma 7.** (See [1].) Let \( X \) be a locally compact second countable Abelian group containing no subgroup topologically isomorphic to \( \mathbb{T} \), let \( E \) be either \( \mathbb{R}^m \) or \( \mathbb{R}^{N_0} \), let \( \gamma \) be a symmetric Gaussian measure on \( X \). Then there exists a continuous monomorphism \( \pi : E \rightarrow X \), such that \( \gamma = \pi(N) \), \( N \in \Gamma(E) \).

**Lemma 8.** (See [4].) Assume that a group \( X \) is of the form \( X = E \times G \), where either \( E = \mathbb{R}^m \) or \( E = \mathbb{R}^{N_0} \) and \( G \) is a locally compact second countable Abelian group such that all nonzero elements of \( G \) have order 2. If \( \tau \in M^1(X) \), \( \tau = \lambda \ast \omega \), where \( \lambda \in \Gamma(E) \), \( \omega \in M^1(G) \), and the
characteristic function of $\omega$ does not vanish, then each factor $\tau_1$ of $\tau$ can be represented in the form $\tau_1 = \lambda_1 * \omega_1$, where $\lambda_1 \in \Gamma(E)$ and $\omega_1 \in M^1(G)$.

Remark 2. Denote by $\mathbb{Z}(2)$ the two-element cyclic group. It is relevant to remark that if $G$ is a locally compact Abelian group and all nonzero elements of $G$ have order 2, then by the structure theorem for such groups [8, §25.29] we have

$$G \cong (\mathbb{Z}(2))^\mathfrak{M} \times (\mathbb{Z}(2))^\mathfrak{N},$$

where $\mathfrak{M}$ and $\mathfrak{N}$ are cardinal numbers, the group $(\mathbb{Z}(2))^\mathfrak{M}$ is a direct product of the group $\mathbb{Z}(2)$ considering in the product topology, and the group $(\mathbb{Z}(2))^\mathfrak{N}$ is a weak direct product of the group $\mathbb{Z}(2)$ considering in the discrete topology.

3. Proof of Theorem 1

Proof of Theorem 1. Passing to the random variables $\xi'_j = \alpha_j \xi_j$ we can assume without loss of generality that $L_1 = \xi_1 + \cdots + \xi_n$ and $L_2 = \delta_1 \xi_1 + \cdots + \delta_n \xi_n$, where $\delta_j \in \text{Aut}(X)$ such that $\delta_i \neq \delta_j \in \text{Aut}(X)$ for $i \neq j$. Put $v_j = \mu_j * \tilde{\mu}_j$ and $\psi_j(y) = -\ln \tilde{v}_j(y)$. By Lemma 3, the symmetry of the conditional distribution of $L_2$ given $L_1$ implies that the characteristic functions $\tilde{\mu}_j(y)$ satisfy Eq. (5). Hence, the characteristic functions $\tilde{v}_j(y)$ also satisfy Eq. (5). It follows from this that the functions $\psi_j$ satisfy Eq. (2). By Lemma 1, equality (3) holds true. It follows from this that the restriction of the function $\psi_j$ to the subgroup $Y^{(2)}$ is a polynomial.

We will prove at first that any distribution $v_j$ is represented as a convolution of a Gaussian measure and a distribution supported in $G$. By Lemma 4 and Remark 1, we can suppose that the group $X$ possesses the property: $X^{(2)} = c_X$, i.e. the mapping $\pi : X \mapsto c_X$, $\pi(x) = 2x$ is an epimorphism, $\text{Ker} \pi = G$ and $c_X \cong X/G$. It follows from this that $(c_X)^* \cong A(Y, G) = Y^{(2)}$. We can assume that $c_X \neq \{0\}$, for otherwise the statement of Theorem 1 follows directly from Lemma 4. Taking into account that $c_X$ contains no subgroup topologically isomorphic to $\mathbb{T}$ and $(c_X)^* \cong Y^{(2)}$ we can apply Lemma 5 to the restriction of the function $\exp(-\psi_j(y))$ to $Y^{(2)}$. We obtain that this restriction is the characteristic function of a Gaussian measure on $X/G \cong c_X$. This implies that the restriction of the function $\psi_j$ to $Y^{(2)}$ satisfies Eq. (1). If $\psi_j(y) = 0$ for $y \in Y^{(2)}$, then $\sigma(v_j) \subset A(X, Y^{(2)}) = G$, and the statement of Theorem 1 for the distribution $\mu_j$ follows from the fact that $\mu_j$ is a factor of $v_j$. Therefore we can assume that $\psi_j(y) \neq 0$ on $Y^{(2)}$. Thus, the restriction of the function $\psi_j$ to $Y^{(2)}$ is a polynomial of degree 2. Hence, for any fixed $k \in Y$ the restriction of the function $\Delta_{2k} \psi_j(y)$ to $Y^{(2)}$ is a polynomial of degree 1. On the other hand (3) implies that the function $\Delta_{2k} \psi_j(y)$ is a polynomial on $Y$. It follows from this that $\Delta_{2k} \psi_j(y)$ is a polynomial of degree 1 on $Y$. So, the function $\psi_j$ satisfies Eq. (6). Applying Lemma 6 we obtain representation (7) for the function $\psi_j$, where the function $\varphi$ and $r_\alpha$ depend on $\psi_j$. Let $\gamma$ be a Gaussian measure on the group $X$ with the characteristic function

$$\hat{\gamma}(y) = \exp\{-\varphi(y)\}, \quad y \in Y. \quad (17)$$

Consider on the group $Y$ the function

$$g(y) = \exp\{-r_\alpha\}, \quad y \in y_\alpha + Y^{(2)}. \quad (18)$$
It follows from \( g(y) = \hat{v}_j(y)/\hat{y}(y) \) that the function \( g \) is continuous. Observe also that \( g \) is invariant with respect to the subgroup \( \overline{Y}(2) \). We will verify that \( g \) is a positive definite function. For this purpose we take arbitrary elements \( t_1, \ldots, t_n \in Y \), such that \( t_j \in y_{\alpha_j} + \overline{Y}(2) \) and verify that for arbitrary \( z_1, \ldots, z_n \in \mathbb{C} \) the inequality

\[
\sum_{i,j=1}^{n} g(t_i - t_j) z_i \bar{z}_j \geq 0
\]  

holds true. Consider a subgroup \( H \) in \( Y \) generated by cosets \( y_{\alpha_j} + \overline{Y}(2) \), \( j = 1, \ldots, n \). Note that \( Y/\overline{Y}(2) \cong (A(X, \overline{Y}(2)))^* = G^* \). We conclude from this that \( H \) consists of a finite number of cosets. Set \( K = H^*, L = A(X, H) \). Then \( K \cong X/L, L \subset G \). Consider the restriction \( g|_H \) of the function \( g \) to \( H \). This restriction is invariant with respect to the subgroup \( \overline{Y}(2) \) and hence, it can be considered as a function on the factor-group \( H/\overline{Y}(2) \). Note that the factor-group \( H/\overline{Y}(2) \) is finite and all nonzero elements of it have order 2. This implies that any real valued function on \( H/\overline{Y}(2) \) is the characteristic function of a signed measure. From what has been said it follows that \( g|_H \) is the characteristic function of a signed measure \( \beta \) on a finite subgroup \( F = A(K, \overline{Y}(2)) \). It follows from (7), (17) and (18) that the restriction of the function \( \hat{v}_j(y) \) to \( H \) is the characteristic function of a convolution of a Gaussian measure \( \alpha \) on \( K \) and a signed measure \( \beta \) on \( F \). We will verify that \( \beta \) is a probability distribution and hence, (19) will be proved.

We remind that we assume that \( X(2) = c_X \). This property implies that \( c_X \cong X/G \) and hence, \( c_X \cong (X/L)/(G/L) \). Taking into account that \( c_X \) contains no subgroup topologically isomorphic to \( \mathbb{T} \) it follows from this that the group \( K \) also contains no subgroup topologically isomorphic to \( \mathbb{T} \). By Lemma 7, there exists a continuous monomorphism \( p : E \rightarrow K \), where either \( E = \mathbb{R}^m \) or \( E = \mathbb{R}^{N_0} \) such that \( \alpha = p(N), N \in \Gamma(E) \). Thus, the Gaussian measure \( \alpha \) is concentrated on a Borel subgroup \( B = p(E) \) of \( K \). Note that all nonzero elements of subgroup \( F \) have order 2 and hence, \( B \cap F = \{0\} \). Since \( \alpha \ast \beta \) is a probability distribution, we conclude that \( \beta \) is a probability distribution too. Thus, \( g \) is a positive definite function. By the Bochner theorem there exists a distribution \( \rho \in M^1(X) \) such that \( \hat{\rho}(y) = g(y) \). Since \( g(y) = 1 \) for all \( y \in \overline{Y}(2) \), we see that \( \sigma(\rho) \subset A(X, \overline{Y}(2)) = G \), i.e. \( \rho \in M^1(G) \). We inferred that \( v_j = \gamma \ast \rho \), where \( \gamma \in \Gamma(X), \rho \in M^1(G) \).

Prove now that the distribution \( \mu_j \) has the required representation. By Lemma 7, there exists a continuous monomorphism \( q : E \rightarrow X \), where either \( E = \mathbb{R}^m \) or \( E = \mathbb{R}^{N_0} \), such that \( \gamma = q(N), N \in \Gamma(E) \). Since \( q(E) \cap G = \{0\} \), the monomorphism \( q \) can be extended to a continuous monomorphism \( \tilde{q} : E \times G \rightarrow X \) by the formula \( \tilde{q}(t, \xi) = q(t) + \xi, t \in E, \xi \in G \). Then \( v_j = \tilde{q}(N \ast \rho) \) and hence, the distribution \( v_j \) is concentrated on the Borel subgroup \( q(E) + G \) of \( X \).

The distribution \( \mu_j \) is a factor of \( v_j \). Substituting, if it is necessary, the distribution \( \mu_j \) by its shift, we can assume that \( \mu_j \) is also concentrated on the Borel subgroup \( q(E) + G \). For this reason \( \mu_j = \tilde{q}(M_j) \), where \( M_j \in M^1(E \times G) \) and \( M_j \) is a factor of \( N \ast \rho \). By Lemma 8, \( M_j = N_j \ast \rho_j \), where \( N_j \in \Gamma(E), \rho_j \in M^1(G) \). Hence, \( \mu_j = \tilde{q}(M_j) = \tilde{q}(N_j \ast \rho_j) = \tilde{q}(N_j) \ast \tilde{q}(\rho_j) = \gamma_j \ast \rho_j \), where \( \gamma_j = \tilde{q}(N_j) \). Since \( \gamma_j \in \Gamma(X) \), Theorem 1 is proved. \( \square \)

We note that the following statement results from Lemma 6 and the proof of Theorem 1.
Proposition 1. Let $X$ be a locally compact second countable Abelian group containing no subgroup topologically isomorphic to $\mathbb{T}$, $G$ be the subgroup of $X$ generated by all elements of order 2. Let $\mu \in M^1(X)$, $\nu = \mu * \bar{\mu}$, and the characteristic function of the distribution $\nu$ be of the form

$$\hat{\nu}(y) = \exp\{-\psi(y)\},$$

where the function $\psi$ satisfies the equation

$$\Delta_k \Delta_h^m \psi(y) = 0, \quad k, h, y \in Y.$$ 

Then $\mu = \gamma * \rho$, where $\gamma \in \Gamma(X)$, and $\rho \in M^1(G)$.

Proposition 1 can be regarded as one more group analogue of the well-known Marcinkiewicz theorem (compare with Lemma 5).

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References