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The Quotient Field of an Intersection of Integral Domains

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INTRODUCTION

If K is a field and $\mathscr{S} = \{D_\lambda\}$ is a family of integral domains with quotient field K, then it is known that the family $\mathscr{S}$ may possess the following "bad" properties with respect to intersection:

(1) There may exist a finite subset $\{D_i\}_{i=1}^n$ of $\mathscr{S}$ such that $\bigcap_{i=1}^n D_i$ does not have quotient field K;

(2) for some D_λ in $\mathscr{S}$ and some subfield E of K, the quotient field of $D_\lambda \cap E$ may not be E.

In this paper we examine more closely conditions under which (1) or (2) occurs. In Section 1, we work in the following setting: we fix a subfield F of K and an integral domain J with quotient field F, and consider the family $\mathscr{S}$ of K-overrings[1] of J with quotient field K. We then ask for conditions under which $\mathscr{S}$ is closed under intersection, or finite intersection, or under which $D \cap E$ has quotient field E for each D in $\mathscr{S}$ and each subfield E of K

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[1] If R is a subring of the commutative ring S, then an S-*overring* of R is a subring of S containing R.

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containing F. Taking J to be the prime subring of K yields a determination of fields K for which properties (1) and (2) do not hold for the family $\mathscr{S}$ of all integral domains with quotient field K. As might be expected, if $\mathscr{S}$ is the family of all integral domains with quotient field K, we show that properties (1) and (2) do not hold if and only if K is algebraic over its prime subfield. Thus, if K is a proper algebraic extension field of the field of rational numbers, then $\mathscr{S}$ is closed under finite intersection, but not closed under arbitrary intersection.

In Sections 2 and 3, we present some less definitive results in a more general setting. If $\mathscr{S}$ is the family of valuation rings with quotient field K, then it is well known that properties (1) and (2) do not occur [6, Sects. 11.11, 11.15] or [3, Sects. 22.8, 19.16]. Whether property (1) can occur for the family of one-dimensional quasi-local domains with quotient field K seems to be an open question [4, Sect. 1.6]; we consider this question in Section 2. Given an integral domain D_1 with quotient field K, we consider such questions as whether there exists an integral domain D_2 with quotient field K such that $D_1 \cap D_2$ has a smaller quotient field; or if D_1^* denotes an integral or almost integral extension of D_1 in K, whether $D_1^* \cap D_2$ having quotient field K implies that $D_1 \cap D_2$ has quotient field K. In another direction, we consider in Section 3 the question for a given subfield E of K of the existence of a D_λ as in (2) such that $D_\lambda \cap E$ has a smaller quotient field than E.

1. Global Considerations

We fix the following notation for this section: F is a subfield of the field K, J is an integral domain with identity and with quotient field F, and $\mathscr{S} = \{D_\lambda\}_{\lambda \in \Lambda}$ is the family of K-overrings of J with quotient field K.

Proposition 1.1. *Let D_λ be an element of $\mathscr{S}$. There exists an element D of $\mathscr{S}$ such that D is contained in D_λ and D has a free J-module basis containing* 1; *such a D has the property that $D \cap F = J$.*

Proof. We consider the family $\mathscr{M}$ of subsets B of D_λ such that $1 \in B$ and B is a free J-module basis for the ring $J[B]$. The singleton set $\{1\}$ is in $\mathscr{M}$, and $\mathscr{M}$ is a partially ordered inductive set under inclusion. Let B_0 be a maximal element of $\mathscr{M}$ and $D_0 = J[B_0]$; to prove that D_0 satisfies the desired conditions of the Proposition, we need only prove that the quotient field K_0 of D_0 is equal to K. If not, then D_λ is not contained in K_0 and we take t in $D_\lambda - K_0$. If t is transcendental over K_0, then $B_0' = \{b_0 t^i \mid b_0 \in B_0 \text{ and } i \geqslant 0\}$ is a free J-module base for $J[B_0']$ properly containing B_0, and this contradicts the maximality of B_0. If t is algebraic over K_0 of degree n, then let $f(X)$ be a polynomial of degree n in $D_0[X]$ such that $f(t)=0$. If d is the leading coefficient of $f(X)$ and if $\theta = dt$, then $B_0^* = \{b_0\theta^i \mid b_0 \in B_0,\ 0 \leqslant i \leqslant n-1\}$ is a free J-module basis for $J[B_0^*] = J[B_0][\theta] \subseteq D_\lambda$, and again B_0 is properly contained in B_0^*. We conclude that $K_0 = K$, as asserted.

It is clear that $D \cap F = J$ for each D in $\mathscr{S}$ that has a free J-module basis containing 1.[2]

THEOREM 1.2. *Let* $T = \bigcap \{D_\lambda \mid \lambda \in \Lambda\}$.

(a) *If* $J \neq F$, *then* $T = J$.

(b) *If* $J = F$ *and if* K/F *is not algebraic, then* $T = J$.

(c) *If* $J = F$ *and if* K/F *is algebraic, then* $T = K$.

Proof. If $\{V_\alpha\}$ is the family of valuation rings on K that contain J, then $\bigcap_\alpha V_\alpha$ is the integral closure of J in K [3, p. 230]. Therefore T is integral over J, and Proposition 1.1 implies that $T \cap F = J$. To prove (a), we therefore need only prove that if $\theta \in K - F$ and if θ is integral over J, then $\theta \notin D_\lambda$ for some λ. Let $n = [F(\theta) : F]$. The minimal polynomial for θ over F is in $J'[X]$, where J' is the integral closure of J; hence $\{\theta^i\}_0^{n-1}$ is a free J'-module basis for $J'[\theta]$, and if d is a nonunit of J (and hence of J'), then $\{d^i\theta^i\}_0^{n-1}$ is a free J'-module basis for $J'[d\theta]$. Thus $\theta \notin J'[d\theta]$ and Proposition 1.1 implies that $J'[d\theta] = D_\lambda \cap F(\theta)$ for some D_λ in $\mathscr{S}$ so that $\theta \notin D_\lambda$. This completes the proof of (a).

If $J = F$ and if K/F is not algebraic, then as in (a), we can establish (b) by showing that if θ is an element of $K - F$ algebraic over F, then θ fails to belong to some D_λ. Let t be an element of K that is transcendental over F. Then, as in the preceding paragraph, the assertion that $\theta \notin D_\lambda$ for some λ follows from the fact that $F + tF(\theta)[t]$ is a domain with quotient field $F(\theta, t)$ not containing θ.

As (c) is well known, the proof of Theorem 1.2 is complete.

COROLLARY 1.3. *The family* $\mathscr{S}$ *is closed under arbitrary intersection if and only if* $F = K$, *or* $J = F$ *and* K/F *is algebraic. Thus, the family of all integral domains with identity and with quotient field* K *is closed under arbitrary intersection if and only if* K *is the field of rational numbers, or* K *is an absolutely algebraic field of nonzero characteristic.*

As expected, the set $\mathscr{S}$ is rarely closed under arbitrary intersection; we investigate more closely conditions under which $\mathscr{S}$ is closed under finite intersection. Following the terminology of Enochs in [2], we say that a ring

[2] The equality $D \cap F = J$ is equivalent to the condition that each principal ideal of J is the contraction of its extension in D. This holds under weaker conditions than that D has a free J-module basis containing 1; for example, it is true if J is a direct summand of D as a J-module, or if D is a faithfully flat J-module.

T is a *tight* extension of its subring R if each nonzero ideal of T contains a nonzero element of R; this is equivalent to the condition that each homomorphism of T restricting to an isomorphism on R is an isomorphism on T, and if R is an integral domain it is equivalent to the condition that each nonzero prime ideal of T contains a nonzero element of R. This follows from the fact that for R an integral domain, the nonzero elements of R form a multiplicative system of T, and ideals of T maximal with respect to not meeting this multiplicative system are prime. The concept of tightness is related to our investigations through the following proposition.

PROPOSITION 1.4. *Assume that R_1 and R_2 are subrings of the field K and that R_i has quotient field K_i.*

(1) *If R_1 is a tight extension of $R_1 \cap R_2$ and if R_2 is contained in K_1, then $R_1 \cap R_2$ has quotient field K_2.*

(2) *If each R_i is a tight extensions of $R_1 \cap R_2$, then $R_1 \cap R_2$ has quotient field $K_1 \cap K_2$.*

Proof. Let N be the set of nonzero elements of $R_1 \cap R_2$. The quotient field of $R_1 \cap R_2$ is $(R_1 \cap R_2)_N = (R_1)_N \cap (R_2)_N$. If R_1 is a tight extension of $R_1 \cap R_2$, then $(R_1)_N = K_1$. The statements now follow.

COROLLARY 1.5. *Assume that D_1 and D_2 are integral domains with quotient field K. The following conditions are equivalent.*

(1) *$D_1 \cap D_2$ has quotient field K.*

(2) *Each D_i is a tight extension of $D_1 \cap D_2$.*

(3) *D_1 is a tight extension of $D_1 \cap D_2$.*

The next result provides a sufficient condition in order that a ring extension should be tight.

PROPOSITION 1.6. *Assume that the ring T is algebraic over its subring R. If A is a regular ideal of T, then $A \cap R$ is nonzero. Hence if T is an integral domain, then T is a tight extension of R.*

Proof. Let s be a regular element of A and let $f(X) = a_n X^n + \cdots + a_0$ be a nonzero polynomial in $R[X]$ of minimal degree such that $f(s) = 0$. If $a_0 = 0$, then the regularity of s implies that $a_n s^{n-1} + \cdots + a_1 = 0$, contrary to the choice of $f(X)$. Hence a_0 is a nonzero element of $A \cap R$.

COROLLARY 1.7. *If K/F is algebraic and if R_1 and R_2 are K-overrings of J, then the quotient field of $R_1 \cap R_2$ is $K_1 \cap K_2$, where K_i is the quotient field of R_i.*

Proof. By Proposition 1.6, each nonzero ideal A_i of R_i contains a nonzero element s_i of J. Hence $A_1 \cap A_2$ contains $s_1 s_2 J$. Therefore, each R_i is a tight extension of $R_1 \cap R_2$, and the result follows from Proposition 1.4.

COROLLARY 1.8. *If D_1 and D_2 are integral domains with quotient field K and if K is algebraic over the quotient field of $D_1 \cap D_2$, then $D_1 \cap D_2$ has quotient field K.*

COROLLARY 1.9. *If R_1 and R_2 are nonzero subrings of the field of algebraic numbers, then the quotient field of $R_1 \cap R_2$ is the intersection of the quotient fields of R_1 and R_2.*

It follows from Corollary 1.8 that $\mathscr{S}$ is closed under finite intersection if K/F is algebraic; we proceed to establish the converse of this result.

PROPOSITION 1.10. *Assume that k is a subfield of the field L such that L/k is transcendental. There exist domains J_1, J_2 with quotient field L such that $J_1 \cap J_2 = k'$, the algebraic closure of k in L.*

Proof. Let $B = \{b_\alpha\}$ be a transcendence basis for L/k, and let v be the valuation on $k(B)$ over k induced by the mapping $b_\alpha \to -1$ of B into Z [3, p. 212]; thus v is the rank one discrete valuation that associates with each nonzero polynomial in $k[B]$ the negative of its degree, as a polynomial in the set B of indeterminates over the field k. If V is the valuation ring of v, then $k[B]$ and V are integrally closed domains with quotient field $k(B)$ such that $k = k[B] \cap V$. If $'$ denotes integral closure in L, then we note that $k' = k[B]' \cap V'$. To see that $k[B]' \cap V'$ is contained in k' (the other inclusion is clear), take an element $\theta \in k[B]' \cap V'$ and let $f(X)$ be the minimal polynomial for θ over $k(B)$. Since $k[B]$ and V are integrally closed, $f(X) \in k[B][X] \cap V[X] = (k[B] \cap V)[X] = k[X]$, and hence $\theta \in k'$. Because $L/k(B)$ is algebraic, the domains $k[B]'$ and V' have quotient field L.

THEOREM 1.11. *The set $\mathscr{S}$ is closed under finite intersection if and only if K/F is algebraic.*

Proof. Apply Proposition 1.10 and Corollary 1.8.

COROLLARY 1.12. *The set of all integral domains with quotient field K is closed under finite intersection if and only if K is algebraic over its prime subfield.*

In connection with Theorem 1.2, a question arises as to whether $T = \bigcap \{D_\lambda \mid D_\lambda \in \mathscr{S}\}$ can be realized as a finite intersection of elements of $\mathscr{S}$.

It follows from Theorems 1.2 and 1.11 that necessary conditions in order that T be a finite intersection of elements of $\mathscr{S}$ are either (1) K/F is transcendental, or (2) K/F is algebraic and $J = F$, or (3) $K = F$. It is clear that each of (2) and (3) implies that T is a finite intersection of elements of $\mathscr{S}$, for in these cases T belongs to $\mathscr{S}$. We show presently that condition (1) is also sufficient to imply T is a finite intersection of elements of $\mathscr{S}$.

PROPOSITION 1.13. *If K/F is transcendental, then J can be expressed as the intersection of two elements of $\mathscr{S}$.*

Proof. Let B be a transcendence basis for K/F. The proof of Proposition 1.10 implies that there exist elements D_1, D_2 of $\mathscr{S}$ such that D_1 contains $F[B]$ and $D_1 \cap D_2 = F'$, the algebraic closure of F in K. Let J_3 be the domain $J + A$, where A is the ideal of $F'[B]$ generated by B. Note that $F'(B)$ is the quotient field of J_3. We let D_3 be an element of $\mathscr{S}$ such that $D_3 \cap F'(B) = J_3$. Then $D_1 \cap D_2 \cap D_3 = F' \cap D_3 = (F' \cap F'(B)) \cap D_3 = F' \cap (J + BF'[B]) = J$; moreover, $D_1 \cap D_3$ is in $\mathscr{S}$, for $BF[B]$ is contained in $D_1 \cap D_3$ so that K is algebraic over the quotient field of $D_1 \cap D_3$. Thus $D_1 \cap D_3$ and D_2 are elements of $\mathscr{S}$ with intersection J.

The question of determining conditions under which $D_\lambda \cap E$ has quotient field E for D_λ in $\mathscr{S}$ and an intermediate field E is related to that of determining if $\mathscr{S}$ is closed under finite intersection.

For example, the following result follows from Propositions 1.4 and 1.6.

PROPOSITION 1.14. *Assume that D is an integral domain with quotient field K and E is a subfield of K. If D is algebraic over $D \cap E$, then E is the quotient field of $D \cap E$.*

It follows that $D_\lambda \cap E$ has quotient field E for each D_λ in $\mathscr{S}$ and each intermediate field E if K/F is algebraic. Our next result establishes the converse.

PROPOSITION 1.15. *If K/F is transcendental, then there exists an intermediate field E transcendental over F and an element D of $\mathscr{S}$ such that $D \cap E = J$.*

Proof. Assume that t is an element of K transcendental over F. We first observe that $F[t] \cap F(t + t^{-1}) = F$. Since $t + t^{-1} = (t^2 + 1)/t$, the field $F(t)$ is Galois over $F(t + t^{-1})$ of degree 2 and $X^2 - (t + t^{-1})X + 1$ is the minimal polynomial for t over $F(t + t^{-1})$. Hence t and t^{-1} are conjugate over $F(t + t^{-1})$ and there is an $F(t + t^{-1})$-automorphism σ of $F(t)$ such that $\sigma(t) = t^{-1}$. It follows that $F[t] \cap F(t + t^{-1}) = \sigma(F[t]) \cap \sigma(F(t + t^{-1})) = F[t^{-1}] \cap F(t + t^{-1})$, and hence $F[t] \cap F(t + t^{-1}) = F[t] \cap F(t + t^{-1}) \cap F[t^{-1}] \cap F(t + t^{-1}) = F$. We let $E = F(t + t^{-1})$, $J^* = J + tF[t]$, and let D be an element of $\mathscr{S}$ such that $D \cap F(t) = J^*$. Then $D \cap E = D \cap F(t) \cap E = J^* \cap F[t] \cap E = J^* \cap F = J$, and the proof of Proposition 1.15 is complete.

THEOREM 1.16. *In order that $D_\lambda \cap E$ should have quotient field E for each D_λ in $\mathscr{S}$ and each intermediate field E, it is necessary and sufficient that K/F is algebraic. Hence for the family of all integral domains with quotient field K, property* (2) *of the introduction holds for some subfield E of K if and only if K is not algebraic over its prime subfield.*

Proposition 1.1 suggests the following question. Does there exist $D_\lambda \in \mathscr{S}$ such that D_λ is a tight extension of J and $D_\lambda \cap F = J$? Propositions 1.1 and 1.6 show that this question has an affirmative answer in the case where K is algebraic over F. A combination of the algebraic case and Zorn's Lemma shows that to answer the question posed, it would suffice to answer it in the case where $K = F(X)$ is a simple transcendental extension of F; moreover, without loss of generality, $\mathscr{S}$ can be replaced by the smaller set $\mathscr{S}_0$ consisting of the set of overrings of $J[X]$. Under these hypotheses, we have been able to show that the question has an affirmative answer if either (1) J is integrally closed, (2) the conductor of J in its integral closure is nonzero, or (3) the integral closure of J is a Prüfer domain, but we have not obtained an answer in the case of an arbitrary domain J.

2. THE QUOTIENT FIELD OF $D_1 \cap D_2$

If D_1 and D_2 are integral domains with quotient field K, there are few results in the literature that provide general conditions under which $D_1 \cap D_2$ has quotient field K. In particular, as indicated in the introduction, the following question appears to be open.

QUESTION 2.1. *If D_1 and D_2 are one-dimensional quasi-local domains with quotient field K, must $D_1 \cap D_2$ have quotient field K?*

In relation to Question 2.1, we note that if M_i is the maximal ideal of D_i, then by Corollary 1.5, $D_1 \cap D_2$ has quotient field K if and only if $M_1 \cap D_2 \neq (0)$. Hence if $D_1 \cap D_2$ does not have quotient field K, then $D_1 \cap D_2$ is a field. Also, if E is a subfield of K, then it is easily seen that $D_i \cap E$ is either a one-dimensional quasi-local domain or a field; moreover, if $M_i \cap E \neq (0)$, then by Proposition 1.4, $D_i \cap E$ has quotient field E. Thus if there is an example answering Question 2.1 in the negative, then such an example exists with $K = F(x, y)$, F a field, $x \in M_1$, $y \in M_2$, and $D_1 \cap D_2 = F$.

In connection with property (1) of the Introduction, an interesting family of integral domains to consider is the set of integral domains that Kaplansky in [5, p. 12] terms G-domains. In Gilmer's terminology [3, p. 58], these are the domains with nonzero pseudoradical. An integral domain D with quotient field K is a G-domain if K is a finitely generated ring extension of D, or equivalently, if the nonzero prime ideals of D have a nonzero intersection. For an arbitrary field K, if $\{D_\lambda\} = \mathscr{S}$ is the family of G-domains with quotient field K, it would seem unlikely that $\mathscr{S}$ need be closed under finite intersection, but we know of no example showing this. As noted in [4, Sect. 1.7] the set of quasi-local domains with quotient field K need not be closed under finite intersection.

Let D and D' be integral domains with quotient field K and suppose D^* is a K-overring of D. Under what conditions does $D^* \cap D'$ having quotient field K imply $D \cap D'$ has quotient field K?

PROPOSITION 2.2. *If there is a nonzero conductor C from D to D^*, then $D^* \cap D'$ having quotient field K implies $D \cap D'$ has quotient field K.*

Proof. By Corollary 1.5, $D \cap D'$ has quotient field K if and only if for each nonzero ideal A of D, $A \cap D'$ is nonzero. We have $CA \subset A$, and CA is nonzero ideal of both D and D^*. Since $D^* \cap D'$ has quotient field K, $CA \cap D'$ is nonzero, so $A \cap D'$ is nonzero.

We recall that an element x of the quotient field K of D is said to be *almost integral* over D if $D[x]$ is contained in a finite D-module, or, equivalently, if the conductor of D in $D[x]$ is nonzero.

PROPOSITION 2.3. *Let D and D' be integral domains with quotient field K, and suppose that D is a G-domain. If $D \subseteq D^* \subseteq K$, with every element of D^* almost integral over D, then $D^* \cap D'$ having quotient field K implies that $D \cap D'$ has quotient field K.*[3]

Proof. Since D is a G-domain, each nonzero prime ideal of D contains a prime ideal of height one [3, p. 388]. If P is a prime ideal of height one of D, then there exists a rank-one valuation ring V of K such that $D \subseteq V \subseteq K$ and such that V has center P on D. Since every element of D^* is almost integral over D, we have $D^* \subseteq V$. Hence P is the contraction of a prime ideal of D^*. Therefore, if D_α is an integral domain with $D \subseteq D_\alpha \subseteq D^*$, and if N_α is the pseudoradical of D_α—that is, N_α is the intersection of the nonzero primes of D_α, then $N_\alpha \cap D = N$ is the pseudoradical of D. Suppose $D \cap D'$ does not have quotient field K. By Corollary 1.5, there exists a prime ideal P of D such that $P \cap D' = (0)$. Consider the set $\{D_\alpha\}$ of integral domains such that $D \subseteq D_\alpha \subseteq D^*$ and $D_\alpha \cap D'$ does not have quotient field K. Since $D_\alpha \cap D'$ does not have quotient field K if and only if $N_\alpha \cap D' = (0)$, where N_α is the

[3] This result is stated in [4, Sect. 1.8], but the proof given there is not complete.

pseudoradical of D_α, the set $\{D_\alpha\}$, partially ordered with respect to inclusion, forms an inductive set. By Zorn's lemma, there is a maximal element D_1 for the set $\{D_\alpha\}$. We have D_1 properly contained in D^* since $D^* \cap D'$ has quotient field K. But, if $x \in D^* \backslash D_1$, then there is a nonzero conductor between D_1 and $D_1[x]$. The maximality of D_1 implies $D_1[x] \cap D'$ has quotient field K. Hence, by Proposition 2.2, $D_1 \cap D'$ has quotient field K. This contradiction completes the proof of Proposition 2.3.

The assumption that D is a G-domain in 2.3 can be replaced by the assumption that D' is a G-domain as we note in 2.4, but it is not true in general that $D \cap D'$ has quotient field K provided $D^* \cap D'$ has quotient field K, where D^* is a K-overring of D and each element of D^* is almost integral over D. We illustrate this in Example 2.6. We do not know if there exists such an example with D^* integral over D.

PROPOSITION 2.4. *Let D and D' be integral domains with quotient field K. If $D \subseteq D^* \subseteq K$ with every element of D^* almost integral over D, and if D' is a G-domain, then $D^* \cap D'$ having quotient field K implies that $D \cap D'$ has quotient field K.*

Proof. Let N' be the pseudoradical of D'. If D_α is an integral domain with $D \subseteq D_\alpha \subseteq D^*$, then $D_\alpha \cap D'$ has quotient field K if and only if $D_\alpha \cap N' \neq (0)$. The proof can now be completed as in the proof of 2.3, so we omit the details.

Proposition 2.4 can be used to shed some additional light on Question 2.1. As previously observed, if Question 2.1 has a negative answer, then there exists a field F, one-dimensional quasi-local domains (D_1, M_1) and (D_2, M_2), and elements $x \in M_1$, $y \in M_2$ such that D_1 and D_2 have quotient field $F(x, y)$ and $D_1 \cap D_2 = F$. We note that these conditions imply that the set $\{x, y\}$ is algebraically independent over F, for if not, then the Krull–Akizuki theorem (see [6, Sect. 33.2]) shows that D_1 and D_2 are Noetherian. Hence D_1' and D_2', the integral closures of D_1 and D_2, respectively, are finite intersections of rank-one valuation rings on $F(x, y)$ [6, Sect. 33.10], so that $D_1' \cap D_2'$ has quotient field $F(x, y)$ [6, Sect. 11.11]. Applying Proposition 2.4 twice, we conclude that $D_1 \cap D_2$ has quotient field $F(x, y)$, contrary to assumption. Therefore $\{x, y\}$ is algebraically independent over F, as asserted.

We note the following fact concerning the intersection of a finite number of one-dimensional quasi-local domains.

PROPOSITION 2.5. *If $D_1, \ldots, D_n$ are one-dimensional quasi-local subrings of a field then $D = \bigcap_{i=1}^n D_i$ is of dimension $\leqslant 1$, and if M_i is the maximal ideal of D_i, then $\{M_i \cap D\}$ contains the set of nonzero prime ideals of D.*

Proof. If K is the quotient field of D, then either $K \subseteq D_i$, or $D_i \cap K$ is a one-dimensional quasi-local domain with quotient field K. Hence we may assume that each D_i has quotient field K. For any multiplicative system S in D we have $D_S = \bigcap_{i=1}^n (D_i)_S$, and $(D_i)_S$ is either D_i or K for each i. From this it follows that $\{M_i \cap D\}_{i=1}^n$ contains the set of nonzero prime ideals of D. Suppose that D contains a prime ideal P of height > 1. By passing from D to D_P, we may assume that D is quasi-local with maximal ideal P. Since D has only a finite number of prime ideals, we can choose $x \in P$ such that x is in no other prime ideal of D, and $y \neq 0$ in the pseudoradical of D. Since P has height > 1, $D[1/x] \neq K$, and y is in the pseudoradical of $D[1/x]$. Hence for any positive integer s, $x^s + y$ is a unit of $D[1/x]$. Let $V = \bigcap \{D_i \mid M_i \cap D = P\}$. Then $D = V \cap D[1/x]$, and x is in the pseudoradical of V. But if we choose a positive integers s so that x^s is contained in yV, then $(x^{s+1} + y)\, V = yV$. Hence $y/(x^{s+1} + y)$ is a unit of V, and $y/(x^{s+1} + y) \in V \cap D[1/x] = D$. Moreover, $y/(x^{s+1} + y) \notin P$, since $PV \neq V$. Therefore $y/(x^{s+1} + y)$ is a unit of D and $yD = (x^{s+1} + y)D$. This implies that x is in the pseudoradical of D, a contradiction to our choice of x. We conclude that D has dimension $\leqslant 1$.

EXAMPLE 2.6. This is an example to show the necessity of the G-domain hypothesis of 2.3 and 2.4. Let F be a field and let $K = F(\{X_i\}_{i=1}^\infty)$, where $\{X_i\}_{i=1}^\infty$ is a set of elements algebraically independent over F. We define a valuation ring D on K/F with value group $G = \bigoplus_{i=1}^\infty \mathbb{Z}_i$. where $\mathbb{Z}_i$ is a copy of the additive group of integers. We order G with reverse lexicographic ordering, viz., if $a = (a_i)$ and $b = (b_i)$ are in G, then $a < b$ if and only if $a_i < b_i$ for the largest i such that $a_i \neq b_i$. Let e_i denote the element of G with 1 in the $\mathbb{Z}_i$-coordinate and zeros elsewhere. Giving X_i the value $-e_i$ completely determines a valuation ring D on D/F such that $D \cap F[\{X_i\}_{i=1}^\infty] = F$. Moreover, every element of the field K is almost integral over D. With $D^* = K$ and $D' = F[\{X_i\}]$, we have that $D^* \cap D'$ has quotient field K, but $D \cap D'$ does not have quotient field K.

3. THE NONGLOBAL CASE

Several questions that naturally arise in connection with our results in Section 1 are the following. We maintain the notation of Section 1 for J, F, K, and $\mathscr{S}$.

(i) If D is a fixed element of $\mathscr{S}$, under what conditions does there exist a D' in $\mathscr{S}$ such that $D \cap D'$ does not have quotient fixed K?

(ii) For D a fixed element of $\mathscr{S}$, under what conditions does there exist a field E between F and K such that $D \cap E$ does not have quotient field E?

(iii) Given a field E properly between F and K, under what conditions does there exist a D in $\mathscr{S}$ such that $D \cap E$ does not have quotient field E?

We conclude with some remarks concerning these questions.

Remark 3.1. If $J = F$ and K/F is a finitely generated field extension of transcendence degree one, then for any D in $\mathscr{S}$, D not equal to K, there exists a D' in $\mathscr{S}$ such that $D \cap D'$ does not have quotient field K. For let V be a nontrivial valuation ring of K/F such that $D \subseteq V$, and let D' be the intersection of the valuation rings of K/F other than V. By Riemann's theorem (see [1, p. 22]), we see that D' has quotient field K; and since $D \cap D'$ is contained in all the valuation rings of K/F, $D \cap D'$ is algebraic over F.

Remark 3.2. If $F(X)$ is a simple transcendental extension of F, K is the algebraic closure of $F(X)$, and D is a rank 1 valuation ring of K/F, then for any D' in $\mathscr{S}$, $D \cap D'$ has quotient field K. For, suppose $D \cap D'$ does not have quotient field K. By Corollary 1.8, D is not algebraic over the quotient field of $D \cap D'$. Hence for any field E with $F(X) \subseteq E \subseteq K$, it follows that $D \cap D' \cap E$ does not have quotient field E. Since D is a rank one valuation ring of K/F, $D \cap E$ is a nontrivial valuation ring of E/F. Let us consider E such that E/F is finitely generated. Applying Riemann's theorem again, we see that $D' \cap E$ is contained in all the valuation rings of E/F except possibly $D \cap E$. Since K is algebraically closed, D is not the only extension of $D \cap E$ to K. Hence there exists a finite algebraic extension L of E such that $D \cap L$ is not the only extension of $D \cap E$ to L. Since $D' \cap L$ is contained in all the valuation rings of L/F except possibly $D \cap L$, we see that $D' \cap L$ is contained in an extension of $D \cap E$ to L. Therefore $D' \cap E$ is contained in all the valuation rings of E/F. Hence $D' \cap E$ is algebraic over F. Since this holds for any field E with $F(X) \subseteq E \subseteq K$ and E/F finitely generated, we conclude that if D' is an integral domain with $F \subseteq D' \subseteq K$ and if D' has quotient field K, then $D \cap D'$ has quotient field K.

Remark 3.3. Suppose that K/F is a finitely generated extension of transcendence degree one, and D is an integral domain with $F \subseteq D \subseteq K$ and D having quotient field K. If D is contained in all but a finite number of the valuation rings of K/F, then there exists a field E with $F \subset E \subset K$ and K/E algebraic such that $D \cap E$ does not have quotient field E. For, suppose $V_1, \ldots, V_n$ are the valuation rings of K/F that do not contain D, and let W be a rank one valuation ring of K/F that does contain D. By the independence theorem on valuation rings [5, p. 38], [3, p. 280], or [1, p. 11], there exists a nonzero element X of K such that X is in the maximal ideals of W and each of the V_i. Therefore, X is transcendental over F, and $W \cap F(X) = V_i \cap F(X)$ for each i. It follows that $D \cap F(X)$ is contained in all of the valuation rings of $F(X)/F$ so that $D \cap F(X) = F$.

Remark 3.4. If $F \subset E \subset K$ are fields with K/E not algebraic, then there exists an integral domain D with quotient field K such that $D \cap E$ does not have quotient field E. To see this, we may enlarge F if necessary so that E/F is algebraic, but $F \neq E$. By Proposition 1.13, there exists domains D_1 and D_2 with quotient field K such that $D_1 \cap D_2 = F$. By Corollary 1.8, $D_1 \cap E$ or $D_2 \cap E$ does not have quotient field E.

On the other hand, if $F \subset E \subset K$, with K/E algebraic, then in considering (iii), there is no loss generality in assuming that K/E is separable, in the following sense: if K_s is the separable part of K/E, then there exists a domain D with quotient field K such that $D \cap E$ does not have quotient field E if and only if there exists a domain D_s with quotient field K_s such that $D_s \cap E$ does not have quotient field E. The proof of this statement can be obtained from Proposition 1.1 and the fact that $D \cap K_s$ has quotient field K_s for each domain D with quotient field K, since K/K_s is purely inseparable.

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