The Lower Central Series of Groups of a Special Class*

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1. Introduction

The structure of the lower central series of free groups of finite rank is well known ([1], [7]). To describe the factor groups of the lower central series of an arbitrary group is, however, very difficult. Recently this has been done for one-relator groups ([5], [10]). Such factor groups are important in Burnside's problem ([3], [4]). It seems desirable to obtain results for other groups.

In this paper we shall study the factor groups arising from groups, $G$, of a special class through the use of basic commutators. We shall assume that $G$ is a free product of finitely many groups, $G(i)$, and that every $G(i)$ is the direct product of finitely many groups of prime order. The factor groups are then either of order one, or they are direct products of finitely many groups of prime order, just as the groups $G(i)$. We shall discuss the computation of the number of groups of a given prime order in such a direct product. For this purpose we shall introduce special sets of basic commutators all of which have the same interesting structure.

2. Group Theoretical Foundations

We start by giving some notations, definitions, known results, and immediate consequences of these results.

Let $G$ be a group. Let $a, b \in G$. Then the commutator

$$ (a, b) = a^{-1}b^{-1}ab. $$

(2.1)

The lower central series

$$ G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots $$

(2.2)

is the sequence of subgroups defined as follows:

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$G_1 = G$. $G_n$ is generated by all commutators $(a, b_{n-1})$ where $a \in G$ and $b_{n-1} \in G_{n-1}$.

We say that the element $c \neq 1$ has weight $n = W(c)$ if $c \in G_n$ but $c \notin G_{n+1}$. It is evident that $a \in G_n$, implies that if $W(a)$ is defined, then $W(a) \geq n$.

The following properties of the lower central series are well-known ([1], [7]):

If $(a, b) \neq 1$, then
\[ W((a, b)) \geq W(a) + W(b). \] (2.3a)

If $W(a_i) = w_1$, $W(b_j) = w_2$, then
\[ \left( \prod_{i=1}^{l} a_i^{a_i} \prod_{j=1}^{l} b_j^{b_j} \right) \equiv \prod_{i=1}^{l} \prod_{j=1}^{l} [(a_i, b_j)]^{a_i b_j} \text{ mod } G_{w_1 + w_2 + 1}. \] (2.3b)

If $a \equiv c \text{ mod } G_{w(a)+1}$, $b \equiv d \text{ mod } G_{w(b)+1}$, then
\[ (a, b) \equiv (c, d) \text{ mod } G_{w(a), w(b)+1}. \] (2.3c)

The Jacobi identity
\[ ((a, b), c)((b, c), a)((c, a), b) = 1 \text{ mod } G_{w+n}, \] (2.3d)

where $W = W(a) + W(b) + W(c)$.

In this paper we define basic commutators according to the natural linear ordering given in [9]. We shall need them to study the lower central series of a group, $G$, with which we shall associate a free group, $F$. To distinguish between $F$ and $G$, we shall call the weight in $F$ of an element $a \in F$, its dimension and denote it by $D(a)$; we shall reserve the phrase weight of $b$ and the notation $W(b)$ for the weight in $G$ of the element $b \in G$. ($G = F$ in a special case, where dimension and weight have the same meaning.)

**Definition 2.1.** The basic commutators of dimension one are the free generators of the free group $F$ in the order
\[ c_1 < c_2 < \cdots < c_r. \] (2.4)

Having defined basic commutators of dimension less than $n$ and ordered them, we use these to get the ones of dimension $n$. The basic commutators of dimension $n$ are $c_m = (c_i, c_j)$ where $c_i$ and $c_j$ are basic commutators such that

(i) $D(c_i) + D(c_j) = n$,

(ii) $c_i > c_j$,

(iii) if $c_i = (c_a, c_b)$, then $c_j \geq c_i$.

(For $c_m$ we shall write $c_i = (c_m)^{a}$, $c_j = (c_m)^{b}$. Let $c_{m_1} = (c_{i_1}, c_{i_2})$ and $c_{m_2} = (c_{i_3}, c_{i_4})$ such that $D(c_{m_1}) = D(c_{m_2})$. Then $c_{m_1} > c_{m_2}$ if $c_{i_1} > c_{i_2}$ or
A basic commutator of dimension \( n \) is greater than any of smaller dimension. (In this definition we are using the word dimension according to its general meaning, since a basic commutator of dimension \( n \) is in \( \mathcal{F}_n \), but not in \( \mathcal{F}_{n+1} \).) Having ordered all basic commutators, we assume that their subscripts are chosen so that \( c_i \) is the \( i \)th basic commutator.

To proceed we now introduce an auxiliary definition.

**Definition 2.2.** Let \( G \) have the presentation

\[
G = \langle c_1, c_2, \ldots, c_r; s_1, s_2, \ldots, s_t \rangle. \tag{2.5}
\]

(Then \( G \) is the factor group \( \mathcal{F}/N \), where \( N \) is the normal closure of the subgroup of \( \mathcal{F} \) generated by the words \( s_1, s_2, \ldots, s_t \). In particular when \( t = 1, s_1 = 1 \) then \( G = \mathcal{F} \).) Let the basic commutator \( c_m \) be the element of \( \mathcal{F} \) of Definition 2.1, as well as its image in \( G \) under the homomorphism \( \mathcal{F} \to G = \mathcal{F}/N \); we shall, however, always mean by the dimension of \( c_m \) the number of Definition 2.1. The element \( a \in G_w \) is said to be basic-commutator representable (b.c.-representable) if

\[
a = c_{i_1}^{e_1} c_{i_2}^{e_2} \cdots c_{i_h}^{e_h} \mod G_{w+1}, \tag{2.6}
\]

where the \( e_i \) are elements of \( G \) as well as basic commutators of dimension \( w \), \( c_{i_1} < c_{i_2} < \cdots < c_{i_h} \) if \( h > 1 \), and \( e_1, e_2, \ldots, e_h \) are nonzero exponents. The product on the right-hand side of (2.6) will be called a basic-commutator representation (or b.c.-representation).

Before going further we should note an important inequality: If \( a \in \mathcal{F} \) and \( a \) is its image under the homomorphism \( \mathcal{F} \to G = \mathcal{F}/N \), then

\[
D(a) \leq W(\bar{a}) \tag{2.7}
\]

when \( W(\bar{a}) \) is defined.

The name basic commutator is appropriate in the sense of the following well-known theorem [1]:

**Theorem 2.1.** Every group \( \mathcal{F}_{n+1} \) is a normal subgroup of \( \mathcal{F}_k \) where \( 1 \leq h \leq n \), and every factor group \( \mathcal{F}_n = \mathcal{F}_n/\mathcal{F}_{n+1} \) is a free abelian group. The basic commutators of dimension \( n \) (\( n \geq 1 \)) are mapped into a basis of \( \mathcal{F}_n \) (under the homomorphism \( \mathcal{F}_n \to \mathcal{F}_n = \mathcal{F}_n/\mathcal{F}_{n+1} \) such that every element \( a \in \mathcal{F} \), which \( \neq 1 \), has a unique dimension and a unique b.c.-representation. If \( a, b \) are distinct basic commutators, then \( D((a, b)) = D(a) + D(b) \). Moreover \( \mathcal{F}_n \) is the normal closure in \( \mathcal{F} \) of that subgroup which is generated by the basic commutators of dimension \( n \).

By the definition of the lower central series we obtain at once the following corollary for the group \( G \) presented in (2.5).
Corollary 2.1. Every group $G_{n+1}$ is a normal subgroup of $G_k$ where $1 \leq k \leq n$, and every factor group $G_n = G_n/G_{n+1}$ is an abelian group. The basic commutators of dimension $n$ are mapped into generators of $G_n$ (under the homomorphism $G_1 \to G_n = G_n/K_{n+1}$) such that every element $a$ of weight $n > 0$ is b.c.-representable. Moreover, $G_n$ is the normal closure in $G$ of that subgroup which is generated by the basic commutators of dimension $n$.

To compute a b.c.-representation of a group element we have the well-known "collection-procedure" [1] available.

For the above properties of basic commutators, our natural linear ordering is not required [1]. It is, however, order preserving under commutation [9]. Before stating this result we give some preliminary definitions.

Definition 2.3. Let $a \in \mathcal{F}$ and have dimension $n > 0$. The maximal component of $a$, $M(a)$, is then the largest commutator in the b.c.-representation (2.6), i.e. $M(a) = c_{i_a}$.

Definition 2.4. Let $a, b \in \mathcal{F}$. The inequalities $a > b$ and $a \geq b$ shall mean that $M(a) > M(b)$ and $M(a) \geq M(b)$, respectively. We shall also write $a \approx b$ and $a \neq b$ to stand for $M(a) = M(b)$ and $M(a) \neq M(b)$, respectively.

The following result [9] is of importance in this paper.

Theorem 2.2. Let the elements $a, b, c \in \mathcal{F}$ be basic commutators such that $a > b$, $a \neq c$ and $b \neq c$. Then $(a, c) > (b, c)$.

It is evident from Theorem 2.1 that Theorem 2.2 has the alternate, more general formulation:

Let $a, b, c \in \mathcal{F}$, such that $a > b$, $c \neq 1$, $a \neq c$, $b \neq c$. Then $(a, c) > (b, c)$.

To apply Theorem 2.2 we shall need more machinery. We shall introduce for every basic commutator $c$ its "regular sequence", $[c]$, i.e.,

$$[c] = [d_1, d_2, \ldots, d_h] \quad (2.8)$$

Definition 2.5. The sequence on the right-hand side of (2.8) consists of $c$ only when $D(c) = 1$. Having defined the regular sequences of all basic commutators of dimension $< n$, we define $[c]$ for $D(c) = n$. The sequence $[c] = [e_1, e_2, \ldots, e_p, c^F]$, where $[c^F] = [e_1, e_2, \ldots, e_p]$.

At this point we are ready for the important

Lemma 2.1. Let $C$ and $c$ be basic commutators such that (i) $D(C) > 1$, (ii) $C > c$, (iii) $[C] = [d_1, d_2, \ldots, d_h]$. Then $[M(C, c)] = [d_1, e_1, e_2, \ldots, e_h]$, such that $e_1 \leq e_2 \leq \cdots \leq e_h$ is a rearrangement of $d_2, \ldots, d_h, c$.

For a proof of this Lemma see [10].
3. Analysis of the Lower Central Series of $G$ in the General Case

We shall obtain results by working with special sets of basic commutators. We introduce first the $F$-simple basic commutators.

**Definition 3.1.** A basic commutator, $c$, is $F$-simple if either $D(c) = 1$ or $D(c) > 1$ but $D(c^F) = 1$.

The following is an immediate consequence of Definitions 2.1 and 3.1 and of Theorem 2.1.

**Lemma 3.1.** The $F$-simple basic commutators of dimension $\leq n$ map into generators of $\tilde{F}^n = F_{n+1} \cap F_n$ under the homomorphism $F \to F_n$ where $n \geq 1$. Moreover, those of dimension $> 1$ but $\leq n$ map into generators of the subgroup $\tilde{F}^n = F_{n+1} \cap F_n$, where $n > 1$.

In this paper we do not investigate $F$, but rather a group $G$ as described in the introduction. We shall see that there are subsets of the $F$-simple commutators of dimension $> 1$ which map into sets of generators of $\tilde{G}^n = G_{n+1}/G_n$. We shall be especially interested in a distinguished subset which helps our analysis of the groups $\tilde{G}_k = G_k/G_{k+1}$.

To introduce our generators of $\tilde{G}$ we need first of all additional notation to describe $G$. We supposed in the introduction that

$$G = G(1) \ast G(2) \ast \cdots \ast G(s).$$

(3.1)

We also supposed that the generators $c_1, c_2, \ldots, c_r$ of $G$ have prime order $p(1), p(2), \ldots, p(r)$, respectively. Let

$$p_1 < p_2 < \cdots < p_q$$

(3.2)

be the set of distinct primes which occur among $p(1), p(2), \ldots, p(r)$. We know that there exists integers

$$0 = n_0 < n_1 < n_2 < \cdots < n_s = r$$

(3.3)

so that $c_{n_i-1+1}, c_{n_i-1+2}, \ldots, c_{n_i}$ generate $G(i)$ where $i = 1, 2, \ldots, s$. Without lack of generality, let us order these generators of $G(i)$ so that $c_j$ has order $p(j) = p_k$ if $n_{i,k-1} < j \leq n_{i,k}$ where

$$n_{i-1} = n_{i0} \leq n_{i1} \leq \cdots \leq n_{iq} = n_i.$$  

(3.4)

Since the groups $G(i)$ were assumed to be abelian we know that if the $F$-simple basic commutator, $c$, is not mapped into 1 under the homomorphism $F \to G$, then it must satisfy criterion 1 according to Definitions 2.1 and 3.1.
CRITERION 1. If \( D(c) > 1 \), then \( c = (\ldots(c_{j_1}, c_{j_2}), \ldots, c_{j_r}) \) is such that there exists a positive integer \( n_c \) of the set (3.3) which is \(< r \) so that \( j_2 \leq n_c < j_1 \); i.e. \( c_{j_1} \) and \( c_{j_2} \) are not generators of the same abelian group \( G(i) \).

Before proceeding to the next criterion\(^1\) required in our construction of the generators of \( G^n \), we introduce two preliminary definitions.

**Definition 3.2.** A generator, \( c_i \) (\( i = 1, 2, \ldots, r \)), is a 1-commutator. Having defined all \( j \)-commutators (\( 1 \leq j < k \)), we use these to get the \( k \)-commutators, \( d \), where \( k \) is any integer \( >1 \). Any \( k \)-commutator, \( d = (e, f) \), where \( e \neq f \), \( e \) and \( f \) are \( s \)- and \( t \)-commutators, respectively, such that \( s + t = k \).

**Definition 3.3.** A 1-commutator, \( c \), has the generator-sequence \( \langle c \rangle \), consisting of itself. Having defined the generator-sequences for all \( k \)-commutators when \( 1 \leq k < \omega \), we do so for the \( \omega \)-commutators. Let \( d = (e, f) \) be a \( \omega \)-commutator such that \( e \) and \( f \) have generator sequences \( \langle e \rangle = \langle e_1, e_2, \ldots, e_s \rangle \) and \( \langle f \rangle = \langle f_1, f_2, \ldots, f_t \rangle \), respectively. Then \( d \) has the generator sequence \( \langle d \rangle = \langle e_1, e_2, \ldots, e_s, f_1, f_2, \ldots, f_t \rangle \). (If the basic commutator \( c \) is \( \mathcal{F} \)-simple, then \( [c] \) and \( \langle c \rangle \) are evidently identical.)

We are now ready to state criterion 2.

**Criterion 2.** \( c \) is such that every generator, \( c_\sigma \) (\( 1 \leq \sigma \leq \omega \)), occurring in its generator sequence, \( \langle c_1, c_2, \ldots, c_\omega \rangle \), has the same prime order \( p \).

Criterion 2 arises in connection with Theorem 3.1 below for which we need to define sets \( \Sigma_{a,b} \). We shall say that the \( k \)-commutator, \( d, e \) to the set \( \Sigma_{a,b} \) if the distinct generators \( a \) and \( b \) occur in \( \langle d \rangle \).

**Theorem 3.1.** Suppose that \( d \) is a \( k \) commutator which \( e \Sigma_{a,b} \) such that \( a \) and \( b \) have order \( p \) and \( q \), respectively, where \( p \neq q \). Then \( d \in G_{\omega+\sigma} \) for \( \sigma = 0, 1, 2, \ldots \).

As a consequence of Definitions 2.1 and 3.1, Corollary 2.1 and Theorem 3.1 we conclude at once:

**Lemma 3.2.** The \( \mathcal{F} \)-simple basic commutators of dimension \( >1 \) but \( \leq n \), which also satisfy both criteria 1 and 2, map into generators of the subgroup \( \tilde{G}^n = G_2/G_{n+1} \) of \( \tilde{G}^n = G/G_{n+1} \), under the sequence of homomorphisms \( \mathcal{F} \rightarrow G = \mathcal{F}/N ightarrow G^n \).

\(^1\) The following discussions up to Corollary 3.2 can be replaced by a treatment based on the results of [6]. We continue, however, with commutators in the spirit of this paper.
The proof of Theorem 3.1 given below, depends upon the well-known commutator identities [1, 7]:

\[
(ab, c) = (a, c)((a, c), b)(b, c),
\]

\[
(a, bc) - (a, c)(a, b)((a, b), c).
\]

Proof of Theorem 3.1. Consider \( \langle d \rangle = \langle d_1, d_2, ..., d_k \rangle \). Let us suppose without lack of generality that \( a = d_i \) and \( b = d_j \) so that \( 1 \leq i < j \leq k \). Then

\[
d = w(d_1, d_2, ..., d_k)
\]

where \( w \) is obtained by commutation from the elements of \( \langle d \rangle \). By hypothesis \( d_i^p = d_j^q = 1 \); hence,

\[
A = w(d_1, d_2, ..., d_{i-1}, d_i^p, d_{i+1}, ..., d_k) = 1,
\]

and

\[
B = w(d_1, d_2, ..., d_{j-1}, d_j^q, d_{j+1}, ..., d_k) = 1.
\]

But by repeated use of the identities (3.5) and (3.6) we compute easily that

\[
A = d_i^p d_j^q, \quad B = d_i^p d_j^q
\]

where the \( d_i^p \), \( d_j^q \) are words in \( k \)-commutators, \( f_s \), which are such that (i) \( f_s \in \Sigma_{a,b} \) and (ii) \( k_s > k \). But \( p \) and \( q \) are relatively prime by hypothesis, and thus there exist integers \( \alpha \) and \( \beta \) so that \( \alpha p + \beta q = 1 \). We find then that once that

\[
1 = A^\alpha B^\beta = d_i^p d_j^q.
\]

We have shown that every \( k \)-commutator, \( d \), which \( d \in \Sigma_{a,b} \) is a word in \( k \)-commutators with properties (i) and (ii). Making use of Definition 3.1 and relation (2.3a), we arrive at the conclusion of our theorem.

We are now ready to discuss further consequences of Theorem 3.1. First we state without proof the obvious

**Lemma 3.3.** If every \( F \)-simple basic commutator of dimension \( >1 \) but \( \leq n \) does not satisfy both criteria 1 and 2 simultaneously, then

\[
\tilde{G}^n = G_2/G_{n+1} \Rightarrow G^n = G/G_{n+1},
\]

\[
G_2 = G_2/G_3, \quad G_3 = G_3/G_4, ..., \quad G_n = G_n/G_{n+1} \text{ are all groups of order } 1.
\]

To discuss the more interesting case where \( \tilde{G}^n \) does not have order 1 we require additional notation first.

Let \( \mathcal{F}(j) \) be that subgroup of \( G \) which is generated by precisely those elements among \( c_1, c_2, ..., c_r \) which have order \( p_j \), where \( 1 \leq j \leq q \). Let \( \mathcal{F}(j) \) be that subgroup of \( G \) which is generated by precisely those elements among \( c_1, c_2, ..., c_r \) which have order \( p_j \), where \( 1 \leq j \leq q \). Let

\[
(\mathcal{F}(j))_n = \mathcal{F}(j) [\mathcal{F}(j)]_{n+1}, \text{ where } n \geq 1.
\]

We are ready now to proceed to
THEOREM 3.2. $\tilde{G}^n$ is the direct product of the q groups $[G(j)]^n$, i.e.,

$$\tilde{G}^n = [G(1)]^n \times [G(2)]^n \times \cdots \times [G(q)]^n.$$  \hspace{1cm} (3.11)

The elements of $[G(j)]_n/[G(j)]_{n+1} = [G(j)]_n$ have order $p_j$ or 1.

Proof. It is easy to see that (3.11) is the consequence of two facts: (i) $\tilde{G}^n$ has the generators of Lemma 3.2. (ii) Let $d$ be a $k$-commutator such that $k > 1$ and $\langle d \rangle$ contains elements $e$ and $f$ of order $p$ and $q$, respectively, with $p \neq q$. Then $d \in G_{n+1}$ by Theorem 3.1.

That the elements of $[G(j)]_n$ have order $p_j$ or 1 follows at once from Corollary 2.1 and relation (2.3b).

For the subgroup $G_n$ of $G^n$ we have the immediate

COROLLARY 3.2.

$$G_n = G_n/G_{n+1} = [G(1)]_n \times [G(2)]_n \times \cdots \times [G(q)]_n.$$  \hspace{1cm} (3.12)

We see now that the structure of any factor group $\tilde{G}^n$ is completely determined by the structure of the corresponding factor groups $[G(j)]^n$ for $j = 1, 2, \ldots, q$. Accordingly, in the next section of this paper, we shall investigate only the special case where the generators $c_1, c_2, \ldots, c_r$ of $G = \mathcal{G}$ have the same prime order $p$.

4. THE LOWER CENTRAL SERIES OF THE GROUP \( \mathcal{G} \) GENERATED BY ELEMENTS OF THE SAME PRIME ORDER \( p \)

The $\mathcal{F}$-simple basic commutators are mapped into generators of $\mathcal{F}^n$ according to Lemma 3.1. In an analogous manner, we shall define the set of $\mathcal{G}$-simple basic commutators in Definition 4.1 and establish their role as generators of $\mathcal{F}^n$ in Lemma 4.1.

The basic commutators of dimension $n$ are either $\mathcal{F}$-simple basic commutators or commutators of $\mathcal{F}$-simple basic commutators; they are mapped into a basis of $\mathcal{F}^n$ according to Theorem 2.1. In analogy we shall discuss the construction of bases of $\mathcal{G}^n$ and show that the basis-elements are images of words in $\mathcal{G}$-simple basic commutators of the special form (4.15a).

DEFINITION 4.1. $c$ is a $\mathcal{G}$-simple basic commutator if it satisfies four conditions:

(i) $c$ is $\mathcal{F}$ simple, i.e., either $c$ is a generator, $c_i$, or $c = (...)c_{j_1}, c_{j_2}, \ldots, c_{j_\omega})$ such that $c_r > c_{j_1} > c_{j_2} > c_1$ and $c_{j_2} < c_{j_2} < \cdots < c_{j_\omega} < c_r$. 


(ii) **Criterion 1** (see also Section 3): If \( D(c) > 1 \), then \( (c_1, c_2) \neq 1 \) in \( \mathcal{G} \).

(iii) **Criterion 3**: If \( c_i \) occurs \( \sigma \) times in \( \langle c \rangle \), then \( 1 \leq \sigma < p \).

(iv) **Criterion 4**: If \( D(c) > 1 \) and \( (c_1, c_2, \ldots, c_r) = 1 \) in \( \mathcal{G} \) for \( 1 < \tau \leq \omega \), then \( c_1 \geq c_\tau \).

**Lemma 4.1.** The \( \mathcal{F} \)-simple basic commutators of dimension \( \leq n \) map into generators of \( \mathcal{G}^n = \mathcal{G}/\mathcal{G}_{n+1} \) under the homomorphism \( \mathcal{F} \to \mathcal{G}^n \), where \( n \geq 1 \). Moreover those of dimension \( > 1 \), but \( \leq n \) map into generators of the subgroup \( \mathcal{G}^n = \mathcal{F}_2/\mathcal{G}_{n+1} \subset \mathcal{G}_n \), where \( n > 1 \).

We shall prove Lemma 4.1 through a sequence of preliminary Lemmas. To prove these lemmas we must first obtain consequences of the identity (3.6) and introduce additional notations.

Let \( P_0 = P_0(a, b) = a, P_1 = P_1(a, b) = (P_0(a, b), b)_1 \ldots, P_\mu = P_\mu(a, b) = \langle (P_\mu-1(a, b), ) \ldots \rangle \). By repeated use of (3.6) we find at once that

\[
(a, b) = P_1^{2}P_2
\]

and

\[
(a, b^\nu) = P_1^{2}Q_\nu P_\nu \quad \text{for} \quad \nu > 2,
\]

where \( Q_\nu \) is a word in \( P_1, P_2, \ldots, P_{\nu-1} \) which is easily computed by induction on \( \nu \). Evidently \( P_1 \) occurs exactly \( \nu \) times on the right-hand side of (4.1) and \( Q_\nu \) does not contain any \( P_1^{-1} \). In particular, for \( a \) and \( b \) generators of \( \mathcal{F} \) and \( a > b \), we find by repeated use of the identity \( AP_1 = P_1A(A, P_1) \) that

\[
(a, b^\nu) = [P_1(a, b)]^\nu H_\nu(a, b)
\]

where \( H_\nu(a, b) \) is a word in basic commutators of dimension \( \geq 2 \) all of which \( c \Sigma_{a,b} \); here \( \Sigma_{a,b} \) has the meaning of Theorem 3.1.

We are now ready to state our first preliminary Lemma.

**Lemma 4.2.** Suppose \( c \) is an \( \mathcal{F} \)-simple basic commutator of dimension \( \omega > 1 \) which satisfies criterion 1 but not criterion 3. Then \( c = \omega \), where \( \omega \) is a word in \( \mathcal{F} \)-simple basic commutators of dimension \( < \omega \) all of which satisfy both criteria 1 and 3.

For the proof of Lemma 4.2 we first require the identities (4.3) and (4.5) below as well as an auxiliary lemma.

If \( A = \prod_{i=1}^{l} a_i^\nu_i \), then

\[
(A, b) = A^{-1} \prod_{i=1}^{l} [a_i(a, b)]^\nu_i.
\]

The identity

\[
((a, b), c)((b, c), a)(((c, a), b)
= (b, a)(c, a) a^{-1}(c, b) a(a, b) b^{-1}(a, c) b a^{-1}(b, c) a(a, c) b^{-1}(c, a)b
\]

(4.4)
is well-known ([1], [7]). We shall write it in the form

\[ ((a, b), c) = (b, a)(c, a)(c, b)((c, b), a)((a, b), a)(a, c)(b, c). \]  
(4.5)

Making use of (4.3) and (4.5) we establish now the auxiliary

**Lemma 4.3.** Suppose that \( c \) and \( d \) are \( \mathcal{F} \)-simple basic commutators such that \( \langle c \rangle = \langle c_{j_1}, c_{j_2}, \ldots, c_{j_\omega} \rangle, \omega > 1, D(d) = 1 \) and \( d < c_{j_\omega} \). The identity

\[ (c, d) = \Pi_1 M((c, d)) \Pi_2 \]  
(4.6)

is then valid in \( \mathcal{F} \), where \( \Pi_1 \) and \( \Pi_2 \) are words in \( \mathcal{F} \)-simple basic commutators \( v_i \) such that:

(i) \( 1 < D(v_i) \leq \omega + 1 \); (ii) if \( \langle v_i \rangle = \langle w_1, w_2, \ldots, w_n \rangle \), then \( w_1, w_2, \ldots, w_n \) is a rearrangement of a subsequence of \( c_{i_1}, c_{i_2}, \ldots, c_{i_\omega}, d \); (iii) \( v_i \prec (c, d) \).

**Proof.** We shall proceed by induction on the place of \( c \) in the ordering of Definition 2.1. For \( \omega = 2 \) we note that \( c_{i_1} > d \) since necessarily \( c_{i_1} > c_{i_2} \); hence, \( M[(c, d)] = (c_{i_1}, d), c_{i_2} \) by Lemma 2.1. By (4.3) and (4.5) we find then that

\[ (c, d) = (c_{i_1}, c_{i_2})^{-1}(c_{i_1}, d)^{-1}(c_{i_2}, d)^{-1}(c_{i_1}, d)^{-1} \times (c_{i_1}, c_{i_2})(c_{i_1}, d)((c_{i_1}, d), c_{i_2})(c_{i_2}, d) \]  
(4.7)

which evidently yields our conclusion in the present case according to Definitions 2.1 and 3.1. Next suppose inductively that the lemma has been demonstrated for \( c < c_k \) with \( D(c_k) > 2 \). We proceed then to the smallest \( \mathcal{S} \)-simple basic commutator, \( c \), which is \( \geq c_k \). Let \( c^L = A, c^R = b \). Then

\[ (c, d) = c^{-1}(A, d)^{-1}A^{-1}(b, d)^{-1}(b, d)^{-1}Ac(A, d)((A, d), b)(b, d) \]  
(4.8)

by (4.5). Now \( A < c \) by Definition 2.1, and \( (A, d) < c \) by Theorem 2.2; moreover, \( z = (A, d) \) is then either an \( \mathcal{F} \)-simple basic commutator or it is a word in \( \mathcal{F} \)-simple basic commutators, \( z_i \), of the form (4.6) according to the induction hypothesis such that all the \( z_i < c \). Applying Lemma 2.1, the induction hypothesis and relations (4.3) and (4.8) we then easily obtain our conclusion for \( (c, d) \). Our induction proof has been completed.

We are ready for the

**Proof of Lemma 4.2.** We shall again proceed by induction on the place of \( c \) in the ordering of Definition 2.1. Evidently \( \omega \geq p + 1 \) by Definition 3.1. For \( \omega = p + 1 \) we must consider two cases: (i) \( c = P_p(a, b) \), (ii) \( c = P_{p-1}(a, b, a) \).

In both cases \( a > b \) and \( (a, b) \neq 1 \). For cases (i) we note that \( P_p(a, b) \) is a word in \( P_1(a, b), P_2(a, b), \ldots, P_{p-1}(a, b) \) according to (4.1) since \( b^p = 1 \); we have thus shown that the lemma holds for \( P_p(a, b) \). In case (ii) we find by
repeated application of (4.3) to $P_1((a, b), a) = P_1((b, a)^{-1}, a)$, $P_2((a, b), a) = P_2((b, a)^{-1}, a)$, ..., $P_v((a, b), a) = P_v((b, a)^{-1}, a)$ that

$$P_v((a, b), a) = S_v(P_1(b, a), P_2(b, a), ..., P_{v-1}(b, a)) \tag{4.9}$$

where $S_v$ is a word in $P_1, P_2, ..., P_{v-1}$. For $v = p - 1$ in particular we find then just as in case (i) that $P_{p-1}((a, b), a)$ is a word in the commutators $P_1(b, a)$, $P_2(b, a)$, ..., $P_{p-1}(b, a)$. Hence, $P_{p-1}((a, b), a)$ is also a word in $P_0((a, b), a)$, $P_1((a, b), a)$, ..., $P_{p-2}((a, b), a)$ according to (4.3), and we have demonstrated the lemma for $\omega = p + 1$. Next suppose inductively that the lemma has been demonstrated for $\omega < p + 1$. We proceed then to the smallest $\mathcal{F}$-simple basic commutator, $\gamma$, which is $\geq c_k$ and satisfies the hypotheses of the lemma. Here we must also distinguish between two cases: (i) $c^\mathcal{F}$ does not satisfy criterion 3, (ii) $c^\mathcal{F}$ satisfies criterion 3. In case (i) $c^\mathcal{F}$ is by the induction hypothesis a word in $\mathcal{F}$-simple basic commutators of dimension $< D(c) - 1 = \omega - 1$; hence, $\gamma$ is by identity (4.3) and Lemma 4.3 a word in $\mathcal{F}$-simple basic commutators of dimension $< \omega$. By Definition 2.1 and our induction hypothesis we then obtain our conclusion for case (i).

In case (ii) we start from the generator sequence $\langle \gamma \rangle = \langle c_{j_1}, c_{j_2}, ..., c_{j_\omega} \rangle$. Evidently $c_{j_\omega}$ occurs exactly $p$ times in $\langle \gamma \rangle$. If $c_{j_1} \neq c_{j_\omega}$ then it is easily seen that $\gamma$ has the form $P_2(e, c_{j_\omega})$ and is thus a word in $P_1(e, c_{j_\omega})$, $P_2(e, c_{j_\omega})$, ..., $P_{p-1}(e, c_{j_\omega})$ by relation (4.1). It remains to suppose that $c_{j_1} = c_{j_\omega}$ which we shall do in the rest of this proof. We observe that $c_{j_1} \neq c_{j_\omega}$ and $c_{j_2} \neq c_{j_\omega}$ since $D(\gamma) > p + 1$ by Definition 2.1, and it is then easily seen that $d = (\ldots((c_{j_1}, c_{j_2}), c_{j_3}), c_{j_4}, ..., c_{j_\omega})$ is a basic commutator which is $\mathcal{F}$-simple, is $< \gamma$ and satisfies criterion 1, but not criterion 3. Making use of the induction hypothesis, Definition 2.1, Lemma 2.1, Lemma 4.3 and the identity (4.3), we compute then $\langle d, c_{j_2} \rangle$ and find that $\gamma$ is a word in $\mathcal{F}$-simple basic commutators of dimension $< \omega$ which satisfy both criteria 1 and 3. Our induction proof of the lemma is now finished.

We are ready to proceed to our last two auxiliary lemmas. It is evident that the important Lemma 4.1 is a trivial consequence of Lemmas 3.1 and 4.2 and of

**Lemma 4.4.** Every $\mathcal{F}$-simple basic commutator, $\gamma$, of dimension $\omega > 1$ which satisfies both criteria 1 and 3, is $= \omega$, where $\omega$ is a word in $\mathcal{F}$-simple basic commutators of dimension $> 1$ but $\leq \omega$.

To prove Lemma 4.4 it is evidently sufficient to consider the special case of

**Lemma 4.5.** Let $\gamma$ be an $\mathcal{F}$-simple basic commutator which satisfies criteria 1 and 3, but not criterion 4. Let $\langle \gamma \rangle = \langle c_{i_1}, c_{i_2}, ..., c_{i_\omega} \rangle$. Let $\tau$ be the largest
integer such that \( c_t \) occurs in \( \langle c \rangle \) and \( (c_{i_1}, c_{i_1}) = 1 \), and let \( \delta = \omega - \tau \). For \( \delta = 0 \), let the \( S \)-simple basic commutator, \( d \), be such that

\[
\langle d \rangle = \langle c_{i^{\omega}}, d_2, d_3, \ldots, d_\omega \rangle,
\]

where \( d_2, d_3, \ldots, d_\omega \) is that rearrangement of \( c_{i_1}, c_{i_2}, \ldots, c_{i_{\omega-1}} \) for which \( d_2 \leq d_3 \leq \cdots \leq d_\omega \). Then \( c \) is a word, \( w \), in \( S \)-simple basic commutators, \( u_i \), of dimension \( \geq 1 \) but \( \leq \omega \) and such that if \( \langle u_i \rangle = \langle w_1, w_2, \ldots, w_n \rangle \), then \( w_1, w_2, \ldots, w_n \) is a rearrangement of a subsequence of \( c_{i_1}, c_{i_2}, \ldots, c_{i_\omega} \). Moreover, \( w \) is a word in \( d \) and in \( S \)-simple basic commutators of dimension \( < \omega \), when \( \delta = 0 \).

Proof. We shall again proceed by induction on the place of \( c \) in our natural linear ordering of Definition 2.1. Evidently \( D(c) = \omega \geq 3 \). For \( \omega = 3 \), we have

\[
c = ((c_{i_1}, c_{i_1}), c_{i_1}) = ((c_{i_2}, c_{i_2}), c_{i_2}) \quad \text{with} \quad (c_{i_1}, c_{i_1}) = 1 \quad \text{and} \quad c_{i_2} > c_{i_1};
\]

thus (4.5) yields

\[
\text{(4.10)}
\]

and we have verified the conclusion of our lemma in the present case. Next suppose that we have already proven our lemma for \( c < c_k \), where \( D(c_k) > 3 \). We proceed next to the smallest commutator, \( c \), which satisfies the hypotheses of our lemma and \( \geq c_k \). For \( \delta > 0 \) we note that \( e = \ldots((c_{i_1}, c_{i_1}), c_{i_1}), \ldots, c_{i_{\omega-1}} \) is by the induction hypothesis and Definition 2.1 a word in \( S \)-simple basic commutators, \( z_t \), such that every \( z_t \) has dimension \( >1 \) but \( \leq \omega - \delta \), and also has \( z_t^R \leq c_{i_{\omega-1}} \). Applying relation (4.3) repeatedly we compute easily that \( c \) itself is a word, \( w \), which has all the properties required by our conclusion. It remains to examine the case \( \delta = 0 \) for which we write \( (c^L)^R = A \), \( (c^L)^R = c_{i_{\omega-1}} = g \), \( c^R = c_{i_\omega} = h \). Then

\[
c = (A, g)^{-1}(A, h)^{-1}A^{-1}(h, g)A(A, g)(A, h)((A, h), g)(h, g)^{-1}
\]

by the identity (4.5). But a commutator among \( A \), \( (A, g) \), \( (A, h) \) and \( (h, g) \) is either \( =1 \) or is by Definition 2.1 an \( S \)-simple basic commutator of dimension \( >1 \) but \( < \omega \), so that (i) its generator sequence is a rearrangement of a subsequence of \( \langle c \rangle \), (ii) it satisfies criteria 1 and 3; by the induction hypothesis it is then a word in \( S \)-simple basic commutators of dimension \( >1 \) but \( < \omega \) all of which have the properties of our conclusion. It remains to rewrite \( Z = ((A, h), g) \) as a word in \( S \)-simple basic commutators. We have just seen that \( (A, h) \) is a word in \( S \)-simple basic commutators, \( s_{s_o} \), with all of the special properties of our conclusion; and thus \( Z \) is by (4.3) a word in all of the commutators \( s_{s_o} \) and \( (s_{s_o}, g) \). But by Lemmas 2.1 and 4.3 and the induction hypothesis we find easily that \( (s_{s_o}, g) \) itself is a word in \( S \)-simple basic commutators of dimension \( >1 \) but \( < \omega \), all of which have the required
properties of our conclusion, except for \( s_\alpha = s = ((c_{\omega_1}, e_2), e_3), ..., e_{\omega-1}) \)
where \( e_\alpha \leq e_\alpha \leq \cdots \leq e_{\omega-1} \) is a rearrangement of \( c_{\omega_1}, c_{\omega_2}, ..., c_{\omega-1} \). When
\( c_{\omega_1} \leq g = c_{\omega-1} \), then \( (s, g) \) evidently \( =d \). We must still express \( (s, g) \) by
\( \mathcal{G} \)-simple basic commutators in the case \( c_{\omega_1} > g \) for which we make use of
identity (4.5) again. We find that
\[
(s, g) = s^{-1}(s^L, g)^{-1}(s^L, g)^{-1}(s^L, g)^{-1}(s^L, g)((s^L, g), s^R)(s^R, g).
\]
(4.12)
Now \( s^R = c_j \) by Definition 2.1. It is then easy to verify from our knowledge of
\( \langle s \rangle \) that \( s, s^L, (s^L, g) \) and \( (s^R, g) \) are either \( =1 \) or are \( \mathcal{G} \)-simple basic commutators
of dimension \( >1 \) but \( <\omega \) with the required properties. Also it is evident
that \( ((s^L, g), s^R) = d \). We have thus proven that \( c \) can be rewritten under all
circumstances as a word in \( \mathcal{G} \)-simple basic commutators in the manner of our
conclusion. Our induction proof is now complete.

Having finally established the role of \( \mathcal{G} \)-simple basic commutators as
generators of groups \( \mathcal{G}^n \), we are ready to continue the development of the
analogy between \( \mathcal{F} \) and \( \mathcal{G} \). It was stated in Theorem 2.1 that every element \( a \)
of \( \mathcal{F} \) of dimension \( n > 0 \) can be represented uniquely by basic commutators
of dimension \( n \); we also know by Definition 2.1 that these basic commutators
are words in \( \mathcal{F} \)-simple basic commutators. We shall show that any element
\( b \) of \( \mathcal{G} \) of weight \( n > 0 \) can be represented uniquely by appropriate basis
elements of \( \mathcal{G}^n \) which are words in \( \mathcal{G} \)-simple basic commutators.

Consider an element \( e \in \mathcal{G} \) which \( \neq 1 \); let us express it as a word in the
generators \( c_j \) \((1 \leq j \leq r)\). All of the \( c_j \) have order \( p \) and we find by
"collecting" first \( c_1 \) on the left, then \( c_2 \) on the left,..., finally \( c_r \) on the left that
\[
e = (c_1^{a_1}c_2^{a_2} \cdots c_r^{a_r}) Q
\]
(4.13)
where \( 0 \leq a_i \leq p - 1 \) \((i = 1, 2, ..., r)\) and \( Q \in \mathcal{G}_2 \). But by Lemma 4.1
\[
Q = w(c_{i_1}, c_{i_2}, ..., c_{i_n}) \mod \mathcal{G}_{n+1}
\]
(4.14)
where \( n > 1 \) and \( w \) is a word in the \( \mathcal{G} \)-simple basic commutators
\( c_{i_1} < c_{i_2} < \cdots < c_{i_n} \) of dimension \( >1 \) but \( \leq n \). Let us continue with the
"collection-procedure" and rewrite the word \( w \). To state the result of this
rewriting we require now a preliminary definition.

**Definition 4.2.** The generators \( c_j \) \((1 \leq j \leq r)\) are the fundamental
commutators of dimension 1. Having defined all fundamental commutators
of dimension \( \geq 1 \) but \( <\omega \), we define those of dimension \( \omega \). A basic
commutator, \( c \), of dimension \( \omega \) is a fundamental commutator if either \( c \) is
\( \mathcal{G} \)-simple or else \( c \) is not \( \mathcal{F} \)-simple, but \( c^L \) and \( c^R \) are both fundamental. (We
shall be interested in the fundamental commutators as elements of \( \mathcal{G} \). Since
they are basic commutators we speak of their dimension, rather than their weight in accordance with our general usage. However, if \( D(r) = \omega \), then \( c \in \mathcal{G}_\omega \) and \( W(c) \geq \omega \) by inequality (2.7).

Let \( c_1 < c_2 < \cdots < c_\mu \) be the fundamental commutators of dimension \( \leq \omega \). "Collecting" first \( c_{r+1} \) on the left of \( \omega \), then \( c_{r-1}, \ldots, \) finally \( c_\mu \), we find by inequality (2.7) that

\[
eq \prod_{\tau=1}^{\mu} c_{i_\tau}^{n_\tau} \mod \mathcal{G}_{n+1}
\]  

where \( c_{i_\tau} = c_\tau \) and \( \eta_\tau = \alpha_\tau \) for \( \tau = 1, 2, \ldots, r \). Theorem 3.2 asserts that the elements of \( \mathcal{G}_\omega (\omega \geq 1) \) have order 1 or \( p \). Thus if \( x \in \mathcal{G}_\omega \), then \( x^{p(x)} \in \mathcal{G}_{\omega + a} \) for \( a = 0, 1, 2, \ldots \); in particular if \( D(c) = \omega \), where \( c \) is a fundamental commutator, then \( c^{p(x)} \in \mathcal{G}_{\omega + a} \). We conclude that we may restrict the \( \eta_\tau \) by the inequalities

\[
0 \leq \eta_\tau < p \quad \text{for} \quad D(c_{\tau}) = 1,
\]

\[
0 \leq \eta_\tau < p^{n+1-\omega} \quad \text{for} \quad n \geq D(c_{\tau}) = \omega > 1.
\]

Hence, if there are \( \Phi(\omega) \) fundamental commutators of dimension \( \omega \), then

\[
\Phi(n) \leq p^r \prod_{j=2}^{n} p^{(n+1-j)(\phi(j))}
\]

for \( n > 1 \), where \( \Phi(n) \) is the order of \( \mathcal{G}_n \).

Next let us turn our attention to the subgroup \( \mathcal{G}_n \) of \( \mathcal{G}_n \). It is clearly finite and abelian; by Theorem 3.2 its elements have order \( p \) or 1. It is well-known [1] that such a group has finite bases (\( = \) independent generating sets) and a unique rank, \( \lambda = \lambda_n \), where by rank we mean the number of elements in a basis. By the preceding discussion we know that any element in a basis must be the image in \( \mathcal{G}_n \) under the homomorphism \( \mathcal{G} \rightarrow \mathcal{G}/\mathcal{G}_{n+1} = \mathbb{F}^n \), of a word in \( \mathcal{G} \)-simple basic commutators, in analogy to the basic commutators in \( \mathcal{G} \); this word must have the special form

\[
\prod_{\tau=r+1}^{\mu} c_{i_\tau}^{n_\tau}
\]  

where the \( \eta_\tau \) are restricted by (4.16). Furthermore, if we call the words \( w_1, w_2, \ldots, w_\lambda \) which map into such a basis, \( \mathcal{G} \)-basic commutators of weight \( n \), then it follows from the independence of the basis elements that any element
a of weight \( n \) has a unique "b.c. representation" in analogy with Theorem 2.1, i.e.,

\[
a = \prod_{\theta=1}^{\lambda} \pi_{\theta}^{n_{\theta}} \mod \mathcal{G}_{n+1}
\]  

(4.18)

where the exponents \( n_{\theta} \) are unique and are \( >0 \) but \( <p \).

As a first step in determining the rank of \( \mathcal{G}_n \), when \( n > 1 \), let us derive an inequality for \( \lambda_n \). By the "collection-procedure" we evidently find that all of the exponents \( \eta_1, \eta_2, \ldots, \eta_r \) in the expression (4.15) vanish, when \( e \in \mathcal{G}_2 \).

Since \( \mathcal{G}_n \subseteq \mathcal{G}_j \) for \( n > 1 \) and since \( \mathcal{G}_n \) is abelian, we conclude that \( \mathcal{G}_n \) is a subgroup of an abelian group, \( \mathcal{A} \), which is generated by \( \mu - r \) elements. Thus any set of independent elements of \( \mathcal{A} \) consists of at most \( \mu - r \) elements [1]. Hence,

\[
\lambda_n \leq \mu - r = \sum_{j=2}^{n} \phi(j)
\]

(4.19)

Having proven the inequality (4.19), we wish to find a specific basis of \( \mathcal{G}_n \) and thus replace (4.19) by an equality. For this purpose we propose to construct first a representation of \( \mathcal{G}_n \) and then to obtain from it a basis of the subgroup \( \mathcal{G}_n \). This last step may be accomplished according to Corollary 2.1 by applying the representation of \( \mathcal{G}_n \) to the group generated by the images in \( \mathcal{G}_n \) of the basic commutators of dimension \( n \).

Let \( c_{\mu+1}, c_{\mu+2}, \ldots, c_{\mu+r} \) be the basic commutators of dimension \( n+1 \). Lemma 4.1 tells us in particular that any such basic commutator \( c_{\gamma} \) has the form

\[
c_{\gamma} = w_{\gamma} z_{\gamma},
\]

(4.20)

where \( w_{\gamma} \) is a word in \( \mathcal{G} \)-simple basic commutators of dimension \( >1 \) but \( \leq n + 1 \) and \( z_{\gamma} \in \mathcal{G}_{n+2} \). Let \( \mathcal{G} \) have the representation (2.5). By Corollary 2.1 we know then that \( \mathcal{G}_n \) has the representation

\[
\mathcal{G}_n = \langle c_1, c_2, \ldots, c_r; s_1, s_2, \ldots, s_t, w_{\mu+1} z_{\mu+1}, w_{\mu+2} z_{\mu+2}, \ldots, w_{\mu+r} z_{\mu+r} \rangle
\]

(4.21)

It is reasonable to suppose that our program of finding a basis can be carried out in general with the help of (4.21) after computing all the \( w_{\gamma} \). Indeed this has been done in the special case \( p = 2 \), [8], by making use of properties of our linear ordering of basic commutators and above all of the simple relation

\[
1 = (a, b^p) = (a, b)^p((a, b), b)
\]

(4.22)

which the identity (3.8) becomes for \( b^p = 1 \). It was found that a basis of \( \mathcal{G}_n(n \geq 2) \) consists of the images of all elements \( e^{2^{n-D(c)}} \), where \( c \) is a fundamental commutator of dimension \( \geq 2 \) but \( \leq n \). Hence, \( \lambda_n = \sum_{j=2}^{n} \phi(j) \) in this case, in place of inequality (4.19).
The author has been unable to generalize the above special result to arbitrary primes, $p$. He conjectures, however, that one can always construct a basis of $\mathcal{F}_n$ which includes the images in $\mathcal{F}_n$ of all of the fundamental commutators of dimension $n$. This implies among other things the independence of the images in $\mathcal{F}_n$ of the $F$-simple basic commutators of dimension $n$ in analogy to $\mathcal{F}$; the images in $\mathcal{F}_n$ of the $F$-simple basic commutators of dimension $n$ are certainly independent, since these images constitute part of a basis of $\mathcal{F}_n$.

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