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Grassmann Algebras as Hilbert Space[1]

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INTRODUCTION

If $\mathscr{H}_M$ is an M-dimensional Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ over the complex numbers $\mathbf{C}$, then $\wedge\, \mathscr{H}_M$, the 2^M-dimensional Grassmann algebra generated by $\mathscr{H}_M$, can also be made into a Hilbert space. In fact, we define the inner product between decomposable vectors $x_1 \wedge x_2 \wedge \cdots \wedge x_k$, and $y_1 \wedge y_2 \wedge \cdots \wedge y_k \in \wedge^k \mathscr{H}_M$, $x_i, y_i \in \mathscr{H}_M$, by

$$\langle x_1 \wedge x_2 \wedge \cdots \wedge x_k , y_1 \wedge y_2 \wedge \cdots \wedge y_k \rangle_k = \det(\langle x_i , y_j \rangle), \qquad (0.1)$$

the determinant of the $k \times k$ matrix whose ijth entry is the scalar $\langle x_i , y_j \rangle$. Since $\wedge\, \mathscr{H}_M$ is the direct sum of the spaces

$$\mathbf{C}, \wedge^1 \mathscr{H}_M = \mathscr{H}_M ,..., \wedge^k \mathscr{H}_M ,..., \wedge^M \mathscr{H}_M ,$$

we extend the inner product (0.1) to all of $\wedge\, \mathscr{H}_M$ by setting these direct summand spaces $\wedge^k \mathscr{H}_M$, $k = 0, 1,..., M$, orthogonal to each other.

Since, for $1 < n < M$, $\wedge^n \mathscr{H}_M$ is a Hilbert space, by virtue of (0.1), we may consider the Grassmann algebra $\bar{\wedge}\, (\wedge^n \mathscr{H}_M)$ generated by it, and give it an inner product $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_k$ in like fashion to (0.1). That is, for vectors $\mathbf{x}^1, \mathbf{x}^2,..., \mathbf{x}^k, \mathbf{y}^1, \mathbf{y}^2,..., \mathbf{y}^k \in \wedge^n \mathscr{H}_M$, we define the inner product $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_k$ by

$$\langle\!\langle \mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{y}^1 \bar{\wedge}\, \mathbf{y}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{y}^k \rangle\!\rangle_k = \det(\langle \mathbf{x}^i, \mathbf{y}^j \rangle_n). \qquad (0.1)'$$

Note that the symbol $\bar{\wedge}$ is used for the product in the Grassmann algebra $\bar{\wedge}\, (\wedge^n \mathscr{H}_M)$ generated by Hilbert space $\wedge^n \mathscr{H}_M$ in contrast to the symbol $\wedge$ which denotes the product in the Grassmann algebra $\wedge\, \mathscr{H}_M$ generated by Hilbert space $\mathscr{H}_M$. To underscore the distinction between $\wedge$ defined for $\wedge\, \mathscr{H}_M$ and $\bar{\wedge}$ defined for $\bar{\wedge}\, (\wedge^n \mathscr{H}_M)$, consider the following example: Let e_1, e_2, e_3, e_4 be vectors of $\mathscr{H}_M$, $M \geqslant 4$. Then $\mathbf{x} = e_1 \wedge e_2$ and $\mathbf{y} = e_3 \wedge e_4$

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are vectors in $\Lambda^2 \mathscr{H}_M$. Now $\mathbf{x} \wedge \mathbf{y}$ is a decomposable vector of degree 4 in $\Lambda^4 \mathscr{H}_M$, while $\mathbf{x} \,\bar{\wedge}\, \mathbf{y}$ is a decomposable vector of degree 2 in $\bar{\Lambda}^2 (\Lambda^2 \mathscr{H}_M)$. In fact, if $\{e_1, e_2, e_3\}$ is orthonormal in $\mathscr{H}_M$, and $e_4 = e_2$, we have that $\mathbf{x} \wedge \mathbf{y} = (e_1 \wedge e_2) \wedge (e_3 \wedge e_2) = 0$ in $\Lambda^4 \mathscr{H}_M$, whereas (from (0.1)') $\mathbf{x} \,\bar{\wedge}\, \mathbf{y} = (e_1 \wedge e_2) \,\bar{\wedge}\, (e_3 \wedge e_2)$ is a *unit* vector of $\bar{\Lambda}^2 (\Lambda^2 \mathscr{H}_M)$.

It is the purpose of this paper to relate properties of the inner products $\langle \cdot , \cdot \rangle_k$, $\langle \cdot , \cdot \rangle_{k+1}$ defined on $\Lambda^k \mathscr{H}_M$ and $\Lambda^{k+1} \mathscr{H}_{M+1}$, respectively (Theorem 1.2). A similar comparison is made for $\bar{\Lambda} (\Lambda^n \mathscr{H}_M)$, the Grassmann algebra generated by $\binom{M}{n}$-dimensional Hilbert space $\Lambda^n \mathscr{H}_M$, (Theorem 2.1). The inner products $\langle \cdot , \cdot \rangle$ and $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ on the Grassmann algebras generated by $\mathscr{H}_M$, and $\Lambda^n \mathscr{H}_M$, respectively, are tied together in Corollary 2.4. Finally, applications of these results to determinantal inequalities on positive definite matrices, are obtained.

We shall use the symbol $Q_{M,k}$ to denote the $\binom{M}{k}$-element set of order-preserving functions σ sending the set $\{1, 2,..., k\}$ into the set $\{1, 2,..., k,..., M\}$. That is, $\sigma \in Q_{M,k}$ if and only if $1 \leqslant \sigma(1) < \sigma(2) < \cdots < \sigma(k) \leqslant M$. Thus, if $\{x_1, x_2, ..., x_M\}$ is a basis for $\mathscr{H}_M$, the $\binom{M}{k}$ decomposable vectors $x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)}$, $\sigma \in Q_{M,k}$, form a basis for the subspace, denoted by $\Lambda^k \mathscr{H}_M$. For convenience, we use the abbreviation

$$\mathbf{x}_\sigma = x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \cdots \wedge x_{\sigma(k)} .$$

If $\mathbf{T}$ is a linear operator on $\mathscr{H}_M$, then $C_k(\mathbf{T})$, the kth compound of $\mathbf{T}$, is a linear operator on $\Lambda^k \mathscr{H}_M$ defined by

$$C_k(\mathbf{T})\, x_1 \wedge x_2 \wedge \cdots \wedge x_k = \mathbf{T}x_1 \wedge \mathbf{T}x_2 \wedge \cdots \wedge \mathbf{T}x_k$$

for all $x_1, x_2, ..., x_k \in \mathscr{H}_M$. We are ready to present our results.

1. The Algebra $\Lambda \mathscr{H}_M$ as Hilbert Space

Proposition 1.1. *Let $P_{\mathscr{M}}$ denote the orthogonal projection onto $\mathscr{M}$, where $\mathscr{M}$ is the subspace of $\mathscr{H}_M$, with orthonormal basis $\{u_1, u_2, ..., u_r\}$. Then $C_n(P_{\mathscr{M}^\perp})$ is the orthogonal projection onto the $\binom{r}{n}$-dimensional subspace of $\Lambda^n \mathscr{H}_M$ spanned by the orthonormal set*

$$\{u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge \cdots \wedge u_{\sigma(n)} : \sigma \in Q_{r,n}\}.$$

Proof. The result follows directly from the definition of C_n. A more general statement concerning partial isometries on $\mathscr{H}_M$ is to be found in ([2], Lemma 3.1).

THEOREM 1.2. *Consider the kn-element ordered set of linearly independent vectors*

$$\{x_1^1, x_2^1,..., x_n^1, x_1^2, x_2^2,..., x_n^2,..., x_1^k, x_2^k,..., x_n^k\}$$

in $\mathscr{H}_M$, *and denote by* $\mathscr{M}$, *the subspace of* $\mathscr{H}_M$ *spanned by these vectors. If* $\mathbf{x}^1, \mathbf{x}^2,..., \mathbf{x}^k$ *are the decomposable vectors in* $\wedge^n \mathscr{H}_M$ *defined by*

$$\mathbf{x}^i = x_1^i \wedge x_2^i \wedge \cdots \wedge x_n^i, \quad i = 1, 2,..., k,$$

then for all $\mathbf{y}, \mathbf{z} \in \wedge^n \mathscr{H}_M$, *we have*

$$\begin{aligned}&\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z}\rangle_{(k+1)n} \\ &\quad = \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn} \langle C_n(P_{\mathscr{M}^\perp})\, \mathbf{y}, \mathbf{z}\rangle_n, \qquad (1.1)\end{aligned}$$

where $\mathscr{M}^\perp$ *is the orthogonal complement of the subspace* $\mathscr{M}$ *in* $\mathscr{H}_M$ *spanned by* $\{x_j^i : i = 1, 2,..., k, j = 1, 2,..., n\}$.

Proof. We first observe that the equality remains valid if we multiply the vectors x_j^i, $i = 1, 2,..., k, j = 1, 2,..., n$, by various scalars. Hence, we may, if necessary, multiply the x_j^i's by the desired scalars which will force the Gram-Schmidt orthogonalization procedure to yield *unit* vectors

$$e_1^1, e_2^1,..., e_n^1,..., e_1^k, e_2^k,..., e_n^k,$$

which form an orthonormal basis for $\mathscr{M}$. Next, extend this basis to the orthonormal basis

$$\{e_1^1, e_2^1,..., e_n^1,..., e_1^k, e_2^k,..., e_n^k, u_1, u_2,..., u_p\}, \quad kn + p = M, \quad (1.2)$$

for the whole space $\mathscr{H}_M$. Thus, the subspace $\mathscr{M}$ has for a basis, either of the sets $\{x_1^1,..., x_n^1,..., x_1^k,..., x_n^k\}$ or $\{e_1^1,..., e_n^1,..., e_1^k,..., e_n^k\}$, while $\mathscr{M}^\perp$, its orthogonal complement, has for a basis, the orthonormal set

$$\{u_1, u_2,..., u_p\}.$$

Since each e_j^i is equal to x_j^i plus a certain linear combination of the preceding vectors x_s^r, it is immediate, then, that

$$\begin{aligned}x_1^1 &= e_1^1 \\ x_1^1 \wedge x_2^1 &= e_1^1 \wedge e_2^1 \\ &\vdots \\ \mathbf{x}^1 = x_1^1 \wedge x_2^1 \wedge \cdots \wedge x_n^1 &= e_1^1 \wedge e_2^1 \wedge \cdots \wedge e_n^1 = \mathbf{e}^1 \\ &\vdots \\ \mathbf{x}^1 \wedge \mathbf{x}^2 &= \mathbf{e}^1 \wedge \mathbf{e}^2 \\ &\vdots \\ \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \cdots \wedge \mathbf{x}^k &= \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^k.\end{aligned}$$

In a natural fashion, we have used the symbol $\mathbf{e}^i$, $i = 1, 2,..., k$, to represent the decomposable (unit) vector $e_1{}^i \wedge e_2{}^i \wedge \cdots \wedge e_n{}^i$. Thus,

$$\mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y} = (\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^k) \wedge \sum_{i=1}^{N} \langle \mathbf{y}, \mathbf{f}_i \rangle_n \mathbf{f}_i , \qquad N = \binom{M}{n}, \tag{1.3}$$

where $\{\mathbf{f}_1 , \mathbf{f}_2 ,..., \mathbf{f}_N\}$ is the orthonormal basis of decomposable vectors (of degree n) for $\wedge^n \mathscr{H}_M$, formed from the orthonormal basis (1.2) for $\mathscr{H}_M$. That is, a typical vector $\mathbf{f}_i$ is of the form

$$\mathbf{f}_i = e_{j_1}^{i_1} \wedge \cdots \wedge e_{j_r}^{i_r} \wedge u_{t_1} \wedge \cdots \wedge u_{t_s}, \qquad r + s = n.$$

But the non-zero terms of (1.3) occur only when $\mathbf{f}_i$ is a decomposable vector of the form

$$\mathbf{f}_i = u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge \cdots \wedge u_{\sigma(n)} ,$$

where

$$1 \leqslant \sigma(1) < \sigma(2) < \cdots < \sigma(n) \leqslant M - kn = p.$$

In other words, we may rewrite (1.3) as follows:

$$\mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y} = \sum_{\sigma \in Q_{p,n}} \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma , \tag{1.4a}$$

where $p = M - kn$, and $\mathbf{u}_\sigma = u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge \cdots \wedge u_{\sigma(n)}$ for each σ in the $\binom{p}{n}$-element set $Q_{p,n}$. Similarly, for $\mathbf{z} \in \wedge^n \mathscr{H}_M$,

$$\mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z} = \sum_{\tau \in Q_{p,n}} \langle \mathbf{z}, \mathbf{u}_\tau \rangle_n \, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\tau . \tag{1.4b}$$

The equalities (1.4a) and (1.4b) lead us to

$$\begin{aligned}\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z} \rangle_{(k+1)n} &= \sum_{\sigma,\tau \in Q_{p,n}} \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \overline{\langle \mathbf{z}, \mathbf{u}_\tau \rangle_n} \\ &\quad \cdot \langle \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma , \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\tau \rangle_{(k+1)n} .\end{aligned} \tag{1.5}$$

Now since the $(k + 1)n$ vectors

$$\{e_1{}^1, e_2{}^1,..., e_n{}^k, u_{\sigma(1)} ,..., u_{\sigma(n)}\}$$

form an orthonormal set in $\mathscr{H}_M$, for each $\sigma \in Q_{p,n}$, their exterior products

$$\{\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma = e_1{}^1 \wedge e_2{}^1 \wedge \cdots \wedge e_n{}^k \wedge u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)} : \sigma \in Q_{p,n}\}$$

form an orthonormal set in $\wedge^{(k+1)n} \mathscr{H}_M$. This fact allows us to set $\sigma = \tau$ in (1.5), since

$$\langle \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\tau \rangle$$

equals zero when $\sigma \neq \tau$. In continuing (1.5), then, we have

$$\begin{aligned}
&\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z} \rangle_{(k+1)n} \\
&= \sum_{\sigma \in Q_{p,n}} \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \overline{\langle \mathbf{z}, \mathbf{u}_\sigma \rangle_n} \\
&\qquad \cdot \underbrace{\langle \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \mathbf{u}_\sigma \rangle_{(k+1)n}}_{=1} \\
&= \sum_{\sigma \in Q_{p,n}} \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \overline{\langle \mathbf{z}, \mathbf{u}_\sigma \rangle_n} \underbrace{\langle \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \rangle_{kn}}_{=1} \\
&= \sum_{\sigma \in Q_{p,n}} \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \overline{\langle \mathbf{z}, \mathbf{u}_\sigma \rangle_n} \underbrace{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn}}_{=1} \\
&= \sum_{\sigma, \tau \in Q_{p,n}} \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn} \\
&\qquad \cdot \langle \langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n \mathbf{u}_\sigma, \langle \mathbf{z}, \mathbf{u}_\tau \rangle_n \mathbf{u}_\tau \rangle_n \qquad (1.6) \\
&\qquad\qquad \text{since} \quad \langle \mathbf{u}_\sigma, \mathbf{u}_\tau \rangle_n = 0 \quad \text{when} \quad \sigma \neq \tau \\
&\qquad\qquad\qquad\qquad\qquad = 1 \quad \text{when} \quad \sigma = \tau. \\
&= \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn} \langle P(S)\, \mathbf{y}, \mathbf{z} \rangle,
\end{aligned}$$

where $P(S)$ is the orthogonal projection onto the subspace S of $\wedge^n \mathscr{H}_M$, with orthonormal basis $\{\mathbf{u}_\sigma : \sigma \in Q_{p,n}\}$. Proposition 1.1 now tells us that $P(S) = C_n(P_{\mathscr{M}^\perp})$, where $\mathscr{M}^\perp$, spanned by the orthonormal set $\{u_1, u_2, ..., u_p\}$, is the orthogonal complement of the space $\mathscr{M}$ in $\mathscr{H}_M$. Recall that $\mathscr{M}$ has for a basis, the set $\{x_1^1, x_2^1, ..., x_n^1, ..., x_1^k, ..., x_n^k\}$ (see (1.2)). Substituting $C_n(P_{\mathscr{M}^\perp})$ for $P(S)$ in (1.6) yields the final equation

$$\begin{aligned}
&\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z} \rangle_{(k+1)n} \\
&\quad = \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn} \langle C_n(P_{\mathscr{M}^\perp})\, \mathbf{y}, \mathbf{z} \rangle_n .
\end{aligned}$$

In multiplying the vectors x_j^i, $i = 1, 2, ..., k, j = 1, 2, ..., n$, by arbitrary scalars, we leave this equation intact; at the same time, we remove the restriction that the Gram-Schmidt orthogonalizing vectors from the set $\{x_j^i\}$ are each of length one. That ends the proof of the theorem.

2. The Algebra $\bar{\wedge}\,(\wedge^n)\,\mathscr{H}_M$ as Hilbert Space

We turn our attention to $\bar{\wedge}\,(\wedge^n \mathscr{H}_M)$, the Grassmann algebra generated by the $\binom{M}{n}$-dimensional Hilbert space $\wedge^n \mathscr{H}_M$, whose inner product $\langle \cdot\,, \cdot \rangle_n$ is defined in (0.1). For the exterior product of (not necessarily decomposable) vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k \in \wedge^n \mathscr{H}_M$, we write

$$\mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \tag{2.1}$$

employing the wedge symbol $\bar{\wedge}$ to distinguish this product in the algebra $\bar{\wedge}\,(\wedge^n \mathscr{H}_M)$ from the exterior product $\wedge$ in the algebra $\wedge\, \mathscr{H}_M$. The linear span of all decomposable vectors of the form (2.1) is denoted by $\bar{\wedge}^k\,(\wedge^n \mathscr{H}_M)$. Analogous to (0.1), $\bar{\wedge}^k\,(\wedge\, \mathscr{H}_M)$ is given a Hilbert space structure, by defining the inner product of the decomposable vector $\mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k$ against $\mathbf{y}^1 \bar{\wedge}\, \mathbf{y}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{y}^k$ as follows:

$$\langle\!\langle \mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{y}^1 \bar{\wedge}\, \mathbf{y}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{y}^k \rangle\!\rangle_k = \det(\langle \mathbf{x}^i, \mathbf{y}^j \rangle_n), \tag{2.2}$$

the determinant of the $k \times k$ matrix whose ijth entry is the scalar $\langle \mathbf{x}^i, \mathbf{y}^j \rangle_n$, where each $\mathbf{x}^l, \mathbf{y}^l \in \wedge^n \mathscr{H}_M$, $l = 1, 2, \ldots, k$. The direct sum of the Hilbert spaces

$$\mathbf{C},\ \bar{\wedge}^1\,(\wedge^n \mathscr{H}_M) = \wedge^n \mathscr{H}_M,\ \bar{\wedge}^2\,(\wedge^n \mathscr{H}_M), \ldots, \bar{\wedge}^k\,(\wedge^n \mathscr{H}_M), \ldots, \bar{\wedge}^{\binom{M}{n}}\,(\wedge^n \mathscr{H}_M),$$

denoted $\bar{\wedge}(\wedge^n \mathscr{H}_M)$, is the Grassmann (Hilbert) algebra generated by the space $\wedge^n \mathscr{H}_M$.

Our next result flows from Theorem 1.2.

Theorem 2.1. *Let $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ be a linearly independent set of (not necessarily decomposable) vectors in $\wedge^n \mathscr{H}_M$. Then for any vectors $\mathbf{y}$ and $\mathbf{z} \in \wedge^n \mathscr{H}_M$, we have*

$$\begin{aligned} &\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \bar{\wedge}\, \mathbf{y}, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \bar{\wedge}\, \mathbf{z} \rangle\!\rangle_{k+1} \\ &\quad = \langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k \langle\!\langle P(S^\perp)\, \mathbf{y}, \mathbf{z} \rangle\!\rangle_1, \end{aligned} \tag{2.3}$$

where S is the k-dimensional subspace of $\wedge^n \mathscr{H}_M$ spanned by the set $\{x^1, x^2, \ldots, x^k\}$, and $P(S^\perp)$ is the orthogonal projection onto $S^\perp$, the orthogonal complement of S.

Proof. Note that the statement of Theorem 1.2 has it that

$$\begin{aligned} &\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{z} \rangle_{(k+1)n} \\ &\quad = \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn} \langle C_n(P_{\mathscr{M}^\perp})\, \mathbf{y}, \mathbf{z} \rangle_n \end{aligned} \tag{2.4}$$

for *decomposable* vectors $\mathbf{x}^i = x_1{}^i \wedge \cdots \wedge x_n{}^k$ in $\wedge^n \mathscr{H}_M$, and for *arbitrary* vectors $\mathbf{y}$, $\mathbf{z}$ in $\Lambda^n \mathscr{H}_M$, where $P_{\mathscr{M}^\perp}$ is the orthogonal projection onto the subspace $\mathscr{M}^\perp$ of $\mathscr{H}_M$ and $\mathscr{M}$ is spanned by the kn vectors

$$x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2, \ldots, x_n^k.$$

Now set $n = 1$ and replace $\mathscr{H}_M$ by the Hilbert space $\Lambda^p \mathscr{H}_M$. Theorem 1.2 and (2.4) then read: For any linearly independent set of vectors, $\mathbf{x}^1, \ldots, \mathbf{x}^k$, in $\Lambda^p \mathscr{H}_M$, and for any $\mathbf{y}, \mathbf{z} \in \Lambda^p \mathscr{H}_M$, we have

$$\begin{aligned}&\langle\!\langle \mathbf{x}^1 \,\bar{\wedge}\, \cdots \,\bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{y}, \mathbf{x}^1 \,\bar{\wedge}\, \cdots \,\bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{z} \rangle\!\rangle_{k+1} \\ &\quad = \langle\!\langle \mathbf{x}^1 \,\bar{\wedge}\, \cdots \,\bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \,\bar{\wedge}\, \cdots \,\bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k \, \langle\!\langle P_{\mathscr{M}^\perp}\mathbf{y}, \mathbf{z} \rangle\!\rangle_1 \qquad (2.5)\end{aligned}$$

where $\mathscr{M}^\perp$ is the orthogonal complement in $\Lambda^p \mathscr{H}_M$ to the subspace $\mathscr{M}$ spanned by the vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k$. (Note that $C_n(P_{\mathscr{M}^\perp})$ in (2.4) becomes $P_{\mathscr{M}^\perp}$ when $n = 1$). We modify (2.5) by replacing the symbol p with n, and by setting $\mathscr{M} = S$ to obtain the desired form. So ends the proof.

In Theorems 1.2 and 2.1, it was not assumed that the integer n, which appeared in each, was the same. The following proposition says, in effect, that if we make this assumption, then the hypotheses of Theorem 1.2 force the hypotheses of Theorem 2.1 to obtain.

PROPOSITION 2.2. *Consider the kn-element set of linearly independent vectors*

$$\{x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2, \ldots, x_1^k, \ldots, x_n^k\} \subset \mathscr{H}_M.$$

Let $\mathbf{x}^i = x_1^i \wedge \cdots \wedge x_n^i$, $i = 1, 2, \ldots, k$. *Then the set*

$$\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\} \subset \Lambda^n \mathscr{H}_M$$

is necessarily linearly independent.

Proof. Extend the set $\{x_j^i : i = 1, 2, \ldots, k, \; j = 1, 2, \ldots, n\}$ to the M-element basis

$$\{x_1^1, \ldots, x_n^1, \ldots, x_1^k, \ldots, x_n^k, u_1, u_2, \ldots, u_p\}$$

for $\mathscr{H}_M$. Then the wedge products of these vectors, taken n at a time, comprise a basis for $\Lambda^n \mathscr{H}_M$. Among these basis vectors are the k vectors $\mathbf{x}^i = x_1^i \wedge x_2^i \wedge \cdots \wedge x_n^i$, $i = 1, 2, \ldots, k$, which are therefore linearly independent.

We come to the theorem which compares the orthogonal projections $C_n(P_{\mathscr{M}^\perp})$, and $\mathscr{P}(S^\perp)$ which arise in Theorems 1.2, and Theorem 2.1, respectively.

THEOREM 2.3. *Let $\mathscr{M}$ be the kn-dimensional subspace of $\mathscr{H}_M$, with ordered* *basis* $\{x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2, \ldots, x_1^k, \ldots, x_n^k\}$. *Let* S *be the k-dimensional subspace of* $\wedge^n \mathscr{H}_M$ *which is spanned by the linearly independent vectors* $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$ *(see Proposition 2.2), where for each* $i = 1, 2, \ldots, k$, $\mathbf{x}^i = x_1^i \wedge x_2^i \wedge \cdots \wedge x_n^i$. *Then* $C_n(P_{\mathscr{M}^\perp}) < \mathscr{P}(S^\perp)$ *in the sense that the range of the projection* $C_n(P_{\mathscr{M}^\perp})$ is *a proper subspace of* $S^\perp$, *the range of the projection* $\mathscr{P}(S^\perp)$.

Proof. Let $\mathscr{M}^\perp$, the orthogonal complement of $\mathscr{M}$, have the set $\{u_1, u_2, \ldots, u_p\}$, $p + kn = M$, as an orthonormal basis. According to Proposition 1.1, the range $C_n(P_{\mathscr{M}^\perp})$ is spanned by the orthonormal set $\{\mathbf{u}_\sigma = u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)} : \sigma \in Q_{p,n}\}$ of $\wedge^n \mathscr{H}_M$.

On the other hand, every such vector $\mathbf{u}_\sigma$ is orthogonal to every decomposable vector $\mathbf{x}^i = x_1^i \wedge x_2^i \wedge \cdots \wedge x_n^i$. That $\langle \mathbf{x}^i, \mathbf{u}_\sigma \rangle_n = \langle\!\langle \mathbf{x}^i, \mathbf{u}_\sigma \rangle\!\rangle_1 = 0$ can be seen directly from the definition of the inner product, since each $u_{\sigma(r)}$ is orthogonal to each x_j^i by definition. That is, the range of the orthongonal projection $C_n(P_{\mathscr{M}^\perp})$ with basis $\{\mathbf{u}_\sigma : \sigma \in Q_{p,n}\}$ is a subspace of $S^\perp$, the orthogonal complement to the subspace S with basis $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$. In other words, $C_n(P_{\mathscr{M}^\perp}) < \mathscr{P}(S^\perp)$.

A simple dimension argument would establish that the inclusion of subspaces is proper, but it will suffice to exhibit a vector $\mathbf{y}$ in the range of $\mathscr{P}(S^\perp)$ which is not in the range of $C_n(P_{\mathscr{M}^\perp})$(not in the span of the set)

$$\{\mathbf{u}_\sigma : \sigma \in Q_{p,n}\}.$$

Consider, for example, the decomposable vector with "mixed" factors,

$$\mathbf{y} = x_1^1 \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_{k-1}.$$

It is a straight forward application of the definition (0.1) to show that $\langle \mathbf{x}^i, \mathbf{y} \rangle_n = 0$, so that $\mathbf{y} \in S^\perp$. Yet, $\mathbf{y}$ cannot be expressed as a linear combination of the vectors

$$\{\mathbf{u}_\sigma : \sigma \in Q_{p,n}\}.$$

In fact, since x_1^1 is orthogonal to each and every $u_{\sigma(i)}$, $i = 1, 2, \ldots, p$, we have

$$\langle \mathbf{y}, \mathbf{u}_\sigma \rangle_n = 0 \text{ for all } \sigma \in Q_{p,n}.$$

The proof of our theorem is complete.

Our next corollary pulls together the ideas of this section and sets the stage for applications to determinantal inequalities on positive definite matrices.

Corollary 2.4. *Let* $\{x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2, \ldots, x_1^k, \ldots, x_n^k\}$ *be a basis for the kn-dimensional subspace* $\mathscr{M}$ *of* $\mathscr{H}_M$. *If for each* $i = 1, 2, \ldots, k$, *we set* $\mathbf{x}^i = x_1^i \wedge x_2^i \wedge \cdots \wedge x_n^i$, *then we denote by* S, *the k-dimensional subspace of*

$\wedge^n \mathscr{H}_M$ spanned by the (necessarily) linearly independent set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$. Then for any $\mathbf{y} \in \wedge^n \mathscr{H}_M$, but not in S,

$$\frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}\rangle_{(k+1)n}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{y}, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{y}\rangle\!\rangle_{k+1}} \leqslant \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k\rangle\!\rangle_k} \leqslant 1, \tag{2.7}$$

where equality between the first pair of quotients obtains if, and only if, $\mathbf{y}$ belongs to the range of $C_n(P_{\mathscr{M}^\perp}) + S$, i.e., *if, and only if,*

$$\mathbf{y} = \sum_{i=1}^{k} a_i \mathbf{x}^i + \sum_{\sigma \in Q_{p,n}} b_\sigma \mathbf{u}_\sigma, \qquad a_i, b_\sigma \textit{ complex},$$

is a linear combination of the "pure" decomposable vectors $\mathbf{x}^i$, $i = 1, 2, \ldots, k$, and

$$\mathbf{u}_\sigma = u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge \cdots \wedge u_{\sigma(n)}, \ \sigma \in Q_{p,n},$$

where $\{u_1, u_2, \ldots, u_p\}$ is an o.n. basis for $\mathscr{M}^\perp$.

Proof. In combining Theorems 1.2 and 2.1, we have

$$\frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{y}\rangle_{(k+1)n}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{y}, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{y}\rangle\!\rangle_{k+1}} = \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k\rangle\!\rangle_k} \cdot \frac{\langle C_n(P_{\mathscr{M}^\perp})\, \mathbf{y}, \mathbf{y}\rangle_n}{\langle\!\langle \mathscr{P}(S^\perp)\, \mathbf{y}, \mathbf{y}\rangle\!\rangle_1}$$

where the range of $\mathscr{P}(S^\perp)$ is the orthogonal complement in $\wedge^n \mathscr{H}_M$, of the subspace S with basis $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k\}$. Now $\langle\!\langle \mathscr{P}(S^\perp)\, \mathbf{y}, \mathbf{y}\rangle\!\rangle_1 = \langle \mathscr{P}(S^\perp)\, \mathbf{y}, \mathbf{y}\rangle_n$ (see (2.2)). We see, now, that the quotient

$$\frac{\langle C_n(\mathscr{M}^\perp)\, \mathbf{y}, \mathbf{y}\rangle_n}{\langle \mathscr{P}(S^\perp)\, \mathbf{y}, \mathbf{y}\rangle_n} \leqslant 1 \tag{2.8}$$

since, as we have seen (Theorem 2.3), $S^\perp$, the range of $\mathscr{P}(S^\perp)$, properly contains the range of $C_n(P_{\mathscr{M}^\perp})$, which is spanned by the vectors $\mathbf{u}_\sigma = u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)}$, where $u_{\sigma(r)} \in \mathscr{M}^\perp$. Now to say that equality obtains in (2.8) for some $\mathbf{y} \in \wedge^n \mathscr{H}_M$, is to say that $\mathbf{y}$ lies in the kernel of the orthogonal projection

$$\mathscr{P}(S^\perp) - C_n(P_{\mathscr{M}^\perp}).$$

That is, $\langle \mathscr{P}(S^\perp) - C_n(P_{\mathscr{M}^\perp})\, \mathbf{y}, \mathbf{y}\rangle_n = 0$

$\rightleftarrows$ $\mathbf{y} \in \text{range}[1 - (\mathscr{P}(S^\perp) - C_n(P_{\mathscr{M}^\perp}))]$

$\rightleftarrows$ $\mathbf{y} \in \text{range}[\mathscr{P}(S^\perp)^\perp + C_n(P_{\mathscr{M}^\perp})]$

$\rightleftarrows$ $\mathbf{y} \in \text{range}[\mathscr{P}(S) + C_n(P_{\mathscr{M}^\perp})]$

$\rightleftarrows$ $\mathbf{y}$ is a linear combination of the "pure" vectors $\mathbf{x}^i = x_1{}^i \wedge \cdots \wedge x_n{}^i$, $i = 1, 2,..., k$, and $\mathbf{u}_\sigma = u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)}$, where each $u_{\sigma(j)} \in \mathscr{M}^\perp$.

This ends our proof.

The following result describes conditions under which

$$\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k = \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn}$$

for decomposable vectors $\mathbf{x}^1,..., \mathbf{x}^k \in \wedge^n \mathscr{H}_M$.

COROLLARY 2.5. *For decomposable vectors*

$$\mathbf{x}^i = x_1{}^i \wedge x_2{}^i \wedge \cdots \wedge x_n{}^i \in \wedge^n \mathscr{H}_M,$$

$i = 1, 2,..., k$, *we have*

$$\langle\!\langle \mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge}\, \mathbf{x}^2 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k$$
$$= \langle \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge^k \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn} \tag{2.9}$$

if and only if each $x_p{}^i$ *is orthogonal to each* $x_q{}^j$ *for* $i \neq j$. *In the case of equality for* (2.9) *we have both sides equal to*

$$\langle \mathbf{x}^1, \mathbf{x}^1 \rangle_n \langle \mathbf{x}^2, \mathbf{x}^2 \rangle_n \cdots \langle \mathbf{x}^k \mathbf{x}^k \rangle_n .$$

Proof. From Corollary 2.4, we may write

$$\frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^{k-1} \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^{k-1} \wedge \mathbf{x}^k \rangle_{kn}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^{k-1} \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^{k-1} \bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k}$$
$$\leqslant \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^{k-1}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^{k-1} \rangle_{(k-1)n}}{\langle\!\langle \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^{k-1}, \mathbf{x}^1 \bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^{k-1} \rangle\!\rangle_{k-1}} \leqslant 1$$

where equality obtains (in the first case above) if and only if the decomposable vector $\mathbf{x}^k = x_1{}^k \wedge x_2{}^k \wedge \cdots \wedge x_n{}^k$ is of the form (see (2.7))

$$\mathbf{x}^k = \sum_{i=1}^{k-1} a_i \mathbf{x}^i + \sum_{\sigma \in Q_{p,n}} b_\sigma \mathbf{u}_\sigma, \qquad a_i, b_\sigma \text{ complex}, \tag{2.10}$$

where the subspace $\mathscr{M}$ is spanned in this case by the $(k-1)n$ vectors $\{x_1^1,..., x_n^1,..., x_1^{k-1},..., x_n^{k-1}\}$ and $\{u_1, u_2,..., u_p\}$ is an orthonormal basis for $\mathscr{M}^\perp$. Now an interesting result of Klaus Vala ([7], Lemma 3, page 15) says that a sum of decomposable vectors (e.g., the right-hand side of (2.10) may be written with fewer terms if, and only if, the set of all the factors $\{x_1^1,..., x_n^{k-1}, u_1,..., u_p\}$ is linearly dependent in $\mathscr{H}_M$. In our case, our set of factors is linearly independent (in fact, a *basis*) in $\mathscr{H}_M$, so the number of terms in (2.10) cannot be reduced. Yet the left-hand side of (2.10) tells us that the sum is a single decomposable vector $\mathbf{x}^k$ which forces the sum

$$\sum_{i=1}^{k-1} a_i\mathbf{x}^i + \sum_\sigma b_\sigma \mathbf{u}_\sigma$$

to be only a one-term sum. Clearly, then, each complex a_i, $i = 1, 2,..., k-1$ in (2.10) must equal zero, so that

$$\mathbf{x}^k = x_1^k \wedge \cdots \wedge x_n^k = b_\sigma(u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(n)})$$

for some $\{u_{\sigma(1)},..., u_{\sigma(n)}\}$ in $\mathscr{M}^\perp$. The equality of the two decomposable vectors above tells us that $\{x_1^k,..., x_n^k\}$ and $\{u_{\sigma(1)},..., u_{\sigma(n)}\}$ span the same subspace of $\mathscr{H}_M$. Since by definition each $u_{\sigma(i)}$ is orthogonal to each vector in $\mathscr{M}$, where $\mathscr{M}$ is spanned by $\{x_1^1,..., x_n^{k-1}\}$, it follows that each x_i^k is orthogonal to each x_j^l in $\mathscr{M}$. We repeat this argument successively to conclude: Equality in (2.9) obtains if, and only if, each x_p^i is orthogonal to each x_q^j for $i \neq j$, $p, q = 1, 2,..., n$. This proves part of our corollary.

To see that

$$\begin{aligned}\langle\langle \mathbf{x}^1 \bar\wedge \cdots \bar\wedge \mathbf{x}^k, \mathbf{x}^1 \bar\wedge \cdots \bar\wedge \mathbf{x}^k\rangle\rangle_k &= \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn} \\ &= \langle \mathbf{x}^1, \mathbf{x}^1\rangle_n \langle \mathbf{x}^2, \mathbf{x}^2\rangle_n \cdots \langle \mathbf{x}^k, \mathbf{x}^k\rangle_n\end{aligned}$$

holds for the case of equality, it suffices to apply the definition (0.1) to show that $\mathbf{x}^i \perp \mathbf{x}^j$ for $i \neq j$ (since $x_p^i \perp x_q^j$), and then to use (0.1)' for $\langle\langle \mathbf{x}^1 \bar\wedge \cdots \bar\wedge \mathbf{x}^k, \mathbf{x}^1 \bar\wedge \cdots \bar\wedge \mathbf{x}^k\rangle\rangle_k$. This ends the proof.

3. Applications to Matrices

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$$

be a 2n × 2n positive definitive matrix, where each A_{ij}, $i, j = 1, 2$, is an

$n \times n$ matrix. It has been shown by W. N. Everitt [3] and in a more general context, by C. Davis ([1], Lemma 1), that

$$\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \leqslant \det\begin{pmatrix} \det(A_{11}) & \det(A_{12}) \\ \det(A_{21}) & \det(A_{22}) \end{pmatrix}. \tag{3.1}$$

For a generalization of this result where A_{12} and A_{21} need not be square, see F. T. Metcalf ([2], Theorem 10). R. C. Thompson proved [6], that (3.1) could be extended to

$$\det\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix} \leqslant \det\begin{pmatrix} \det(A_{11}) & \det(A_{12}) & \cdots & \det(A_{1m}) \\ \det(A_{21}) & \det(A_{22}) & \cdots & \det(A_{2m}) \\ \vdots & \vdots & & \vdots \\ \det(A_{m1}) & \det(A_{m2}) & \cdots & \det(A_{mm}) \end{pmatrix} \tag{3.2}$$

whenever the $mn \times mn$ matrix $A = (A_{ij})$ is positive definite. A generalization of Thompson's result was achieved by M. Marcus ([4], Theorem 3) which, in turn, is a special case of our Corollary 2.4.

We describe the result of Marcus.

Let $A_{(m)} = (A_{ij})$ be the $mn \times mn$ positive definite matrix, partitioned into m^2 matrices A_{ij}, $i, j = 1, 2, \ldots, m$, where each A_{ij} is an $n \times n$ matrix. Similarly, for $1 \leqslant k \leqslant m$, $A_{(k)} = (A_{ij})$ is the $kn \times kn$ submatrix of $A_{(m)}$, where $i, j = 1, 2, \ldots, k$. The symbol $\tilde{A}_{(k)} = (\det(A_{ij}))$ shall denote the "reduced" $k \times k$ matrix whose ijth entry is the scalar $\det(A_{ij})$. With this notation, Thompson's result (3.2) reads

$$\det(A_{(m)}) \leqslant \det(\tilde{A}_{(m)}).$$

The generalization by Marcus reads

$$\frac{\det(A_{(m)})}{\det(\tilde{A}_{(m)})} \leqslant \cdots \leqslant \frac{\det(A_{(k+1)})}{\det(\tilde{A}_{(k+1)})} \leqslant \frac{\det(A_{(k)})}{\det(\tilde{A}_{(k)})} \leqslant \cdots \leqslant \frac{\det(A_{(1)})}{\det(\tilde{A}_{(1)})} = 1, \tag{3.3}$$

for all $k = 1, 2, \ldots, m - 1$.

Incidentally, we can be sure that $\det(A_{(k)})$ is positive since $A_{(m)}$ (hence, $A_{(k)}$) is a positive-definite matrix. The author [2] has shown that, consequently, the $k \times k$ matrix

$$\begin{pmatrix} E_q(A_{11}) & \cdots & E_q(A_{1k}) \\ \vdots & & \vdots \\ E_q(A_{k1}) & \cdots & E_q(A_{kk}) \end{pmatrix}$$

is positive definite, where E_q is the qth elementary symmetric function. Hence, for $q = n(E_q = \det)$, we are assured that $\det(\tilde{A}_{(k)})$ is also positive.

We show how the Marcus-Thompson result follows from Corollary 2.4.

THEOREM 3.1. *(Marcus, Thompson) Let* $A = (A_{ij})$,

$$i, j = 1, 2, \dots, m,$$

be a positive definite $mn \times mn$ *matrix, where each* A_{ij} *is an* $n \times n$ *matrix. Then for all* $k = 1, 2, \dots, m-1$,

$$\frac{\det\begin{pmatrix} A_{11} & \cdots & A_{1k} & A_{1,k+1} \\ \vdots & & \vdots & \vdots \\ A_{k1} & & A_{kk} & A_{k,k+1} \\ A_{k+1,1} & \cdots & A_{k+1,k} & A_{k+1,k+1} \end{pmatrix}}{\det\begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} & \det A_{1,k+1} \\ \vdots & & \vdots & \vdots \\ \det A_{k1} & \cdots & \det A_{k,k} & \det A_{k,k+1} \\ \det A_{k+1,1} & \cdots & \det A_{k+1,k} & \det A_{k+1,k+1} \end{pmatrix}}$$

$$\leqslant \frac{\det\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}}{\det\begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{pmatrix}}.$$

Moreover, equality holds if, and only if, $A_{(k+1)} = A_{(k)} \oplus A_{k+1,k+1}$, *i.e., if, and only if, for each* $i = 1, 2, \dots, k$,

$$A_{k+1,i} = A_{i,k+1} = \textit{the zero } n \times n \textit{ matrix.}$$

Proof. We note that a square $M \times M$ matrix A is positive definitive if and only if the rsth entry of A is the scalar $\langle z_s, z_r \rangle$ for some basis $\{z_1, z_2, \dots, z_M\}$ of the underlying Hilbert space $\mathscr{H}_M$. In fact, for each $i = 1, 2, \dots, M$, $z_i = P^{\frac{1}{2}} e_i$, where P is the positive definite operator on $\mathscr{H}_M$, and A is the $M \times M$ matrix of P relative to the basis $\{e_1, e_2, \dots, e_M\}$. Since transposes of positive definite matrices are positive definite, we suppose the rsth entry of A to be the scalar $\langle z_r, z_s \rangle$.

Therefore, we may set $A = (A_{ij})$ to be the positive definite $mn \times mn$ matrix whose pqth entry of the n $\times$ n matrix A_{ij} is the scalar $\langle x_p{}^i, x_q{}^j \rangle$, where

$$\{x_1{}^1, \dots, x_n{}^1, x_1{}^2, \dots, x_n{}^2, \dots, x_1{}^m, \dots, x_n{}^m\}$$

is a basis for the underlying Hilbert space $\mathscr{H}_{mn}$. This tells us that for all $k = 1, 2, \dots, m$,

$$\det(A_{(k)}) = \det \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}$$

$$= \langle x_1{}^1 \wedge \cdots \wedge x_n{}^1 \wedge \cdots \wedge x_1{}^k \wedge \cdots \wedge x_n{}^k,$$

$$x_1{}^1 \wedge \cdots \wedge x_n{}^1 \wedge \cdots \wedge x_1{}^k \wedge \cdots \wedge x_n{}^k\rangle_{kn}$$

$$= \langle \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn}, \tag{3.4}$$

where $\mathbf{x}^i$ denotes the decomposable vector $x_1{}^i \wedge x_2{}^i \wedge \cdots \wedge x_n{}^i$, $i = 1, 2, \dots, k$.

Since the pqth entry of the $n \times n$ block matrix A_{ij} is just the scalar $\langle x_p{}^i, x_q{}^j\rangle$, we see immediately from (0.1) that

$$\det(A_{ij}) = \langle x_1{}^i \wedge x_2{}^i \wedge \cdots \wedge x_n{}^i, x_1{}^j \wedge x_2{}^j \wedge \cdots \wedge x_n{}^j\rangle_n$$

$$= \langle \mathbf{x}^i, \mathbf{x}^j\rangle_n .$$

Thus,

$$\det(\tilde{A}_{(k)}) = \det \begin{bmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{bmatrix}$$

$$= \det \begin{bmatrix} \langle \mathbf{x}^1, \mathbf{x}^1\rangle_n & \cdots & \langle \mathbf{x}^1, \mathbf{x}^k\rangle_n \\ \vdots & & \vdots \\ \langle \mathbf{x}^k, \mathbf{x}^1\rangle_n & \cdots & \langle \mathbf{x}^k, \mathbf{x}^k\rangle_n \end{bmatrix}$$

$$= \langle\!\langle \mathbf{x}^1 \mathbin{\bar\wedge} \mathbf{x}^2 \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} \mathbf{x}^k, \mathbf{x}^1 \mathbin{\bar\wedge} \mathbf{x}^2 \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} \mathbf{x}^k\rangle\!\rangle_k \tag{3.5}$$

[see (2.2)]. We now combine (3.4) and (3.5) to obtain

$$\frac{\det(A_{(k)})}{\det(\tilde{A}_{(k)})} = \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k\rangle_{kn}}{\langle\!\langle \mathbf{x}^1 \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} \mathbf{x}^k, \mathbf{x}^1 \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} \mathbf{x}^k\rangle\!\rangle_k}, \tag{3.6}$$

for all $k = 1, 2, \dots, m$.

Let us now rewrite the statement (2.7) of Corollary 2.4 for the special case where $\mathbf{y}$ is the decomposable vector $\mathbf{y} = \mathbf{x}^{k+1} = x_1^{k+1} \wedge x_2^{k+1} \wedge \cdots \wedge x_n^{k+1}$.

This leads us to

$$\frac{\det(A_{(k)})}{\det(\tilde{A}_{(k)})} = \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn}}{\langle\!\langle \mathbf{x}^1 \,\bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k, \mathbf{x}^1 \,\bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \rangle\!\rangle_k} \qquad \text{from (3.6)}$$

$$\geqslant \frac{\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{x}^{k+1}, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \wedge \mathbf{x}^{k+1} \rangle_{(k+1)n}}{\langle\!\langle \mathbf{x}^1 \,\bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{x}^{k+1}, \mathbf{x}^1 \,\bar{\wedge} \cdots \bar{\wedge}\, \mathbf{x}^k \,\bar{\wedge}\, \mathbf{x}^{k+1} \rangle\!\rangle_{k+1}}$$

from Corollary 2.4,

$$= \frac{\det(A_{(k+1)})}{\det(\tilde{A}_{(k+1)})} \qquad \text{from (3.4) and (3.5).}$$

$$1 = \frac{\det(A_{(1)})}{\det(\tilde{A}_{(1)})} \geqslant \cdots \geqslant \frac{\det(A_{(k)})}{\det(\tilde{A}_{(k)})} \geqslant \frac{\det(A_{(k+1)})}{\det(\tilde{A}_{(k+1)})}$$

$$\geqslant \cdots \geqslant \frac{\det(A_{(m)})}{\det(\tilde{A}_{(m)})}. \tag{3.7}$$

According to Corollary 2.4, equality can obtain in (3.7) if, and only if, $\mathbf{x}^{k+1} \in C_n(P_{\mathscr{M}^\perp})$, i.e., if, and only if, $\mathbf{x}^{k+1}$ is a linear combination of decomposable vectors

$$u_{\sigma(1)} \wedge u_{\sigma(2)} \wedge \cdots \wedge u_{\sigma(n)}\,,$$

where each $u_{\sigma(r)}$ is orthogonal to each of the kn vectors $x_1{}^1,\dots, x_n{}^1,\dots, x_1{}^k,\dots, x_n{}^k$. But under these circumstances we have, from (0.1), that $\langle \mathbf{x}^{k+1}, \mathbf{x}^i \rangle_n = 0$, for each $i = 1, 2,\dots, k$. Since $\mathbf{x}^{k+1} = x_1^{k+1} \wedge x_2^{k+1} \wedge \cdots \wedge x_n^{k+1}$, where each x_j^{k+1} is not in the span of the kn vectors $x_1{}^1,\dots, x_n{}^1,\dots, x_1{}^k,\dots, x_n{}^k$, this can only mean each x_g^{k+1} is itself orthogonal to each $x_j{}^i$, $i = 1, 2,\dots, k$, $j = 1, 2,\dots, n$, i.e. $\langle x_g^{k+1}, x_j{}^i \rangle = 0$.

In other words, equality for (3.7) obtains only when

$$A_{(k+1)} = \begin{pmatrix} A_{11} & \cdots & A_{1k} & 0 \\ \vdots & & \vdots & \vdots \\ A_{k1} & \cdots & A_{kk} & 0 \\ 0 & \cdots & 0 & A_{k+1,k+1} \end{pmatrix} \tag{3.8}$$

The converse statement, that $A_{(k+1)}$ given in the form (3.8) above, yields equality for (3.7), is trivial. This completes our proof.

As a final observation, we note that

$$\det(A_{(k)}) = \det \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} = \langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle_{kn}\,,$$

and

$$\det(\tilde{A}_{(k)}) = \det \begin{pmatrix} \det A_{11} & \cdots & \det A_{1k} \\ \vdots & & \vdots \\ \det A_{k1} & \cdots & \det A_{kk} \end{pmatrix} = \langle\!\langle \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k, \mathbf{x}^1 \wedge \cdots \wedge \mathbf{x}^k \rangle\!\rangle_k .$$

By reading Corollary 2.5 in this setting, we establish

COROLLARY 3.2. *For positive definite matrix* $A_{(k)} = (A_{ij})$, *each* A_{ij} *is an* $n \times n$ *matrix*, $i, j = 1, 2, \ldots, k$,

$$\det(A_{(k)}) = \det(\tilde{A}_{(k)})$$

if and only if $A_{(k)}$ *is block diagonal, i.e., if and only if*

$$A_{(k)} = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & A_{kk} \end{pmatrix}:$$

$$\det(A_{(k)}) = \det(\tilde{A}_{(k)}) = \det(A_{11}) \det(A_{22}) \cdots \det(A_{kk}).$$

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