Completing Verlinde algebras

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ABSTRACT

We compute the completion of the Verlinde algebra of a simply connected simple compact Lie group G at the augmentation ideal of the representation ring. By results of Freed, Hopkins, Teleman and Dwyer and Lahtinen, this gives a computation of (non-equivariant) twisted K-theory of the free loop space of BG.

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1. Introduction

The purpose of this note is to compute the completion of the Verlinde algebra of a simply connected simple compact Lie group G at the augmentation ideal of the representation ring $R(G)$. In the special case of the symplectic groups, this was previously done in [6]. Denote the weight lattice of G by $\Pi^* = Hom(T, S^1)$ where T is the maximal torus. There is a unique inner product on Π^* invariant under the action of the Weyl group such that all long roots u satisfy

$$\langle u, u \rangle = 2.$$

The fundamental Weyl chamber V is the set of all points in $\Pi^* \otimes \mathbb{R}$ with which all the positive roots have non-negative inner product. The *level m fundamental alcove A* is the subset of V of all points x satisfying the relation

$$\langle a, x \rangle \leqslant m \tag{1}$$

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where a is the highest root in V (cf. [8]). As usual, for weights c, b we write $c \leqslant b$ when $b - c$ is non-negative on V. A *level m regular weight* is a weight contained in the interior of A. Now the *level m Verlinde algebra* $V_m(G)$ is the quotient of $R(G)$ by the ideal J_m of all elements x such that for every level m regular weight c,

$$\phi_c(x) = 0 \tag{2}$$

where ϕ_c is the composition of the inclusion

$$\iota : R(G) \to \mathbb{Z}[\Pi^*] \tag{3}$$

with the map

$$\psi_c : \mathbb{Z}[\Pi^*] \to \mathbb{C} \tag{4}$$

given by sending

$$w \mapsto e^{2\pi i \langle w, c \rangle / m}. \tag{5}$$

In this paper, we compute (as an abelian group) the completion of the Verlinde algebra at the augmentation ideal I of the ring $R(G)$.

The definition of the Verlinde algebra, and our result, might at first appear artificial, but our purely algebraic computation actually solves a problem in topology. This is because of a theorem of Freed, Hopkins and Teleman [4], which identifies the Verlinde algebra $V_m(G)$ as

$$K^*_{G,\tau}(G), \tag{6}$$

the level m equivariant twisted K-theory of G with the conjugation action of G. The $*$ signifies that the non-trivial K-cohomology group is in the dimension of parity equal to $dim(G)$; the twisted K-group in the dimension of the other parity is 0. The twisting in K-theory actually coincides with m, i.e. it is m times the generator of the Borel cohomology groups $H^3_G(G) \cong \mathbb{Z}$. If one wants to view the twisting from the point of view of loop groups, i.e. count the multiple of the cocycle defining the universal central extension of LG, one must subtract from m the dual Coxeter number $h^\vee$ (cf. [4]).

Now it is well known (cf. [6]) that we have an equivalence

$$G \times_G EG \simeq LBG \tag{7}$$

where on the left-hand side of (7), the action of G on G is by conjugation, and $\times_G$ means the quotient of the product by the diagonal action of G, and the right-hand side denotes the free loop space on the bar construction of G. Combining this with the completion theorem for twisted K-theory by Dwyer and Lahtinen [3,7], we see that the object we compute is actually the level m (non-equivariant) twisted K-theory of LBG. It is certainly nice to have an explicit computation of these groups. As far as we know, this approach is the only known way of computing them. A purely non-equivariant computation is not known (see [6] for some comments on the difficulties of that approach). Therefore, the algebraic calculation performed in the present paper is the key step to getting an answer, which gives an instant source of examples of complete non-trivial computations of non-equivariant twisted K-groups.

The present note is organized as follows. In Section 2, we state our main results. In Section 3, we will prove Theorem 1. In Section 4, we will discuss applications of Theorem 1 to the individual Lie group types, and prove the remainder of the results of Section 2.

2. The main results

Theorem 1. *The completion* $(V_m(G))^\wedge_l$ *is isomorphic, as an abelian group, to the sum over primes* $p \in \mathbb{Z}$ *of* $N(G, m, p)$ *copies of* $\mathbb{Z}_p$ *where* $N(G, m, p)$ *is the number of regular weights* a *such that for every weight* w, *the denominator of the rational number* $\langle w, a \rangle / m$ *is a power of* p.

The values of the numbers $N(G, m, p)$ are as follows. Let

$$m = p^i m', \qquad n + 1 = p^\ell (n + 1)' \qquad p \text{ does not divide } m', \ (n + 1)'. \tag{8}$$

Type *A***:** The number $N(A_n, m, p)$ is the number of tuples

$$(b_1, \ldots, b_n) \in \mathbb{Z}^n$$

such that

$$p^i > b_1 > \cdots > b_n > 0, \tag{9}$$

$$(n + 1)' | (b_1 + \cdots + b_n). \tag{10}$$

Type *B***:** Assume $n > 1$. For $p = 2$, the number $N(B_n, m, 2)$ is the number of tuples

$$(b_1, \ldots, b_n) \in \mathbb{Z}^n \cup \left(\mathbb{Z} + \frac{1}{2} \right)^n \tag{11}$$

such that

$$b_1 > \cdots > b_n > 0, \tag{12}$$

$$p^i > (b_1 + b_2). \tag{13}$$

For $p > 2$, the number $N(B_n, m, p)$ is the number of tuples

$$(b_1, \ldots, b_n) \in \mathbb{Z}^n \tag{14}$$

which satisfy (12), (13) and

$$2 | (b_1 + \cdots + b_n). \tag{15}$$

Type *C***:** Assume $n > 1$. For $p = 2$, the number $N(C_n, m, 2)$ is the number of tuples

$$(b_1, \ldots, b_n) \in \mathbb{Z}^n \tag{16}$$

such that

$$p^i > b_1 > \cdots > b_n > 0. \tag{17}$$

Explicitly,

$$N(C_n, m, 2) = \binom{2^i - 1}{n}.$$

For $p > 2$, the number $N(C_n, m, p)$ is the number of tuples (16) which satisfy (17) and

$$2 | b_i. \tag{18}$$

Explicitly,

$$N(C_n, m, p) = \binom{\frac{p^i - 1}{2}}{n}.$$

Type D: Assume $n > 2$. For $p = 2$, the number $N(D_n, m, 2)$ is the number of tuples (11) such that

$$b_1 > \cdots > b_{n-1} > |b_n|, \tag{19}$$

$$p^i > b_1 + b_2. \tag{20}$$

For $p > 2$, $N(D_n, m, p)$ is the number of tuples (14) which satisfy (19), (20) and (15).

Type G: For $p = 3$, the number $N(G_2, m, 3)$ is the number of tuples

$$(b_1, b_2) \in \mathbb{Z}^2 \tag{21}$$

which satisfy

$$2b_2 > b_1 > b_2 > 0, \tag{22}$$

$$p^i > b_1. \tag{23}$$

For $p \neq 3$, $N(G_2, m, p)$ is the number of tuples (21) which satisfy (22), (23) and

$$3 | (b_1 + b_2). \tag{24}$$

Type F: For $p = 2$, the number $N(F_4, m, p)$ is the number of tuples (11) for $n = 4$ such that

$$b_2 > b_3 > b_4 > 0, \tag{25}$$

$$b_1 > b_2 + b_3 + b_4, \tag{26}$$

$$p^i > b_1 + b_2. \tag{27}$$

For $p > 2$, $N(F_4, m, p)$ is the number of tuples (14) for $n = 4$ which satisfy (25), (26), (27) and (15) for $n = 4$.

Type E_8: The number $N(E_8, m, p)$ is the number of tuples

$$(b_1, \ldots, b_8) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 \tag{28}$$

such that

$$b_2 > \cdots > b_7 > |b_8|, \tag{29}$$

$$b_1 > b_2 + \cdots + b_7 - b_8, \tag{30}$$

$$p^i > b_1 + b_2, \tag{31}$$

and (15) for $n = 8$.

Type E_7: For $p = 2$, the number $N(E_7, m, 2)$ is the number of tuples

$$(b_1, \ldots, b_7) \in \frac{1}{\sqrt{2}} \mathbb{Z} \times \left(\mathbb{Z}^6 \cup \left(\mathbb{Z} + \frac{1}{2} \right)^6 \right) \tag{32}$$

such that

$$b_2 > \cdots > b_6 > |b_7|, \tag{33}$$

$$b_1 \sqrt{2} > b_2 + \cdots + b_6 - b_7, \tag{34}$$

$$p^i > \sqrt{2} b_1, \tag{35}$$

$$2 | (\sqrt{2} b_1 + b_2 + \cdots + b_6 - b_7). \tag{36}$$

For $p > 2$, the number $N(E_7, m, 2)$ is the number of tuples (32) where

$$2b_i \equiv \sqrt{2} b_1 \mod 2 \quad \text{for } i = 2, \ldots, 7 \tag{37}$$

such that (33), (34), (35) and

$$2 | (b_2 + \cdots + b_7). \tag{38}$$

Type E_6: For $p = 3$, the number $N(E_6, m, 3)$ is the number of tuples

$$(b_1, \ldots, b_6) \in \left(\frac{1}{\sqrt{3}} \mathbb{Z} \times \mathbb{Z}^5 \right) \cup \left(\frac{1}{\sqrt{3}} \left(\mathbb{Z} + \frac{1}{2} \right) \times \left(\mathbb{Z} + \frac{1}{2} \right)^5 \right) \tag{39}$$

such that

$$b_2 > \cdots > b_5 > |b_6|, \tag{40}$$

$$\sqrt{3} b_1 > b_2 + \cdots + b_5 - b_6, \tag{41}$$

$$p^i > (\sqrt{3} b_1 + b_2 + \cdots + b_6)/2, \tag{42}$$

$$2 | (\sqrt{3} b_1 + b_2 + \cdots + b_6). \tag{43}$$

For $p \neq 3$, the number $N(E_6, m, 2)$ is the number of tuples

$$(b_1, \ldots, b_6) \in \left(\sqrt{3} \mathbb{Z} \times \mathbb{Z}^5 \right) \cup \left(\sqrt{3} \left(\mathbb{Z} + \frac{1}{2} \right) \times \left(\mathbb{Z} + \frac{1}{2} \right)^5 \right) \tag{44}$$

such that (40), (41), (42), (43).

Remark. As already remarked in [6] in the case of $Sp(n)$, it is interesting to note that completion at the augmentation ideal of the Verlinde algebra does not preserve the level-rank duality [2]. This indicates that twisted K-theory of LBG is perhaps not as natural an object in mathematical physics as the Verlinde algebra (which can be interpreted as the fusion ring of certain conformal field theories known as WZW models).

3. The fundamental lemmas

Lemma 2. *There exists a number N such that $N.1 \in J_m$ where J_m is in the augmentation ideal of $V_m(G)$ (i.e. the image $\bar{I}$ of the ideal I in $V_m(G)$).*

Proof. If an element $w \in J_m$ has non-zero augmentation N, then clearly $N \in \bar{I}$. Thus, it suffices to show that

$$J_m \nsubseteq I. \tag{45}$$

Suppose thus that (45) is false, i.e. that $J_m \subseteq I$. Then the augmentation $\epsilon : R(G) \to \mathbb{Z}$ must factor through $V_m(G)$. After tensoring with $\mathbb{C}$, the product of the maps ϕ_a over regular weights a induces an isomorphism of $V_m(G) \otimes \mathbb{C}$ with the product of rings

$$\prod_a \mathbb{C} \tag{46}$$

over regular weights a. Thus, the complexified augmentation must factor through a map of rings from (46) to $\mathbb{C}$. But one easily sees that the set of maps of rings from (46) to $\mathbb{C}$ is the set of projections (since such maps must preserve idempotents). So, such a factorization would mean that the augmentation is the map ϕ_a for some a. However, averaging over complex conjugates of roots of unity, we then get that the augmentation must coincide with the map which is 1 on any weight w satisfying $m | \langle w, a \rangle$, and 0 on other weights. This is clearly false, as any weight can occur in a representation. $\quad\square$

By Lemma 2, the completion of the Verlinde algebra is a finitely generated module over the ring of N-adic numbers $\mathbb{Z}_N$, and hence a finite sum of finitely generated $\mathbb{Z}_p$-modules for finitely many primes p. Our main tool is this: the canonical inclusion from $V_m(G)$ into the ring (46). Then this map factors through a finite integral extension $\mathbb{Z}'$ of $\mathbb{Z}$ (just adjoin all the necessary roots of unity). Then, we obtain a map of $\mathbb{Z}'$-algebras

$$V_m(G) \otimes \mathbb{Z}' \to \prod_a \mathbb{Z}' \tag{47}$$

where the product on the right-hand side is over level m regular weights. Denote the left-hand side of (47) by R, and the right-hand side by R'. Thus, we have a short exact sequence of R-modules

$$0 \to R \to R' \to F \to 0 \tag{48}$$

where F is finite (since it is finitely generated and vanishes after tensoring with $\mathbb{C}$). Our strategy is to compute the completion of R' instead of R, with the idea that the finite cokernel (47) will not make a difference in the completion. This is justified by the following result.

Theorem 3. *The sequence*

$$0 \to \varprojlim R/I^n \to \varprojlim R'/I^n \to \varprojlim F/I^n \to 0 \tag{49}$$

induced by (48) *is exact. Further, the first non-zero term of* (49) *is torsion free.*

Proof. First note that there exists a number N such that $NR' \subseteq R$. We conclude that

$$\left[I^n R' : I^n R \right] \leqslant \left[I^n R' : N I^n R' \right] \leqslant N^k$$

where k is the number of cyclic summands of R'. Thus, we obtain a short exact sequence

$$0 \to Q_n \to R/I^n R \to R/\big((I^n R') \cap R\big) \to 0 \tag{50}$$

where Q_n are finite of bounded order. It follows that the inverse limit of (50) is a short exact sequence of the form

$$0 \to Q \to \varprojlim R/I^n R \to \varprojlim R/\big((I^n R') \cap R\big) \to 0 \tag{51}$$

where Q is finite. If we know that the middle term (51) is torsion free, then $Q = 0$. Now by [1, Proposition 10.3], we have a short exact sequence

$$0 \to \varprojlim R/\big((I^n R') \cap R\big) \to \varprojlim R'/I^n R' \to \varprojlim T/I^n F. \tag{52}$$

This, together with (51) then implies the statement.

To prove that $\varprojlim R/I^n R$ is torsion free, let p_i be the composition of (47) with the projection of the right-hand side to the first i factors. Then p_i is a map of rings, so its kernel K_i is an ideal in R, and we have a finite filtration of R by ideals

$$R \supset K_1 \supset K_2 \supset \cdots \supset 0$$

where K_i/K_{i+1} is isomorphic to an ideal of $\mathbb{Z}'$. By Proposition 10.3 of [1], the completion of R with respect to the augmentation ideal I is filtered by the completions of K_i with respect to the filtrations $F(i)_n = K_i \cap I^n$, where the associated graded object is the sum of completions of K_i/K_{i+1} by the image of the filtration $F(i)$.

It suffices to prove then that the completion of K_i/K_{i+1} by the image of the filtration $F(i)$ is non-torsion. But since

$$I^n \cdot K_i \subseteq F(i)_n,$$

there is an onto map

$$(K_i/K_{i+1})_I^\wedge \to (K_i/K_{i+1})_{F(i)}^\wedge. \tag{53}$$

Therefore, the right-hand side of (53) can have non-zero torsion only when it is non-zero, which is when the left-hand side of (53) is non-zero. But $\mathbb{Z}'$ is a Dedekind domain, so a completion of any of its ideal by another non-zero ideal I is isomorphic to the completion of $\mathbb{Z}'$ by I. (The image of the ideal I is non-zero by Lemma 2.)

In other words, taking the direct sum over i, the left-hand side of (53) becomes the completion of the right-hand side of (47), the right-hand side of (53) is the associated graded of the completion of the left-hand side of (47). Therefore, if one of the right-hand sides of (53) contains torsion at a prime π in $\mathbb{Z}'$, then the $\mathbb{Z}'_\pi$-rank of the completion of the right-hand side of (47) is greater than the $\mathbb{Z}'_\pi$-rank of the completion of the left-hand side of (47), which contradicts (51) and (52). $\square$

Thus, it suffices to compute the completion of the R-module R' at the image of the ideal I. Since R' is a direct sum of R-modules, it suffices to compute the completion of each summand separately. We will denote the a-th summand by $\mathbb{Z}'_a$.

Proposition 4. $(\mathbb{Z}'_a)_I^\wedge$ *is the sum of* $\mathbb{Z}'_p$ *over all primes* $p \in \mathbb{Z}$ *such that the denominator of the rational number* $\langle w, a \rangle / m$ *is a power of* p.

This immediately implies Theorem 1. To prove Proposition 4, we will need the following lemma:

Lemma 5. *Let π be a prime of $\mathbb{Z}'$ which lies above a prime $p \in \mathbb{Z}$. Then a primitive m-th root of unity $\zeta_m \in \mathbb{Z}'$ satisfies*

$$\pi \,|\, (\zeta_m - 1) \tag{54}$$

if and only if m is a power of p.

Proof. If $m = p^j$, then $x = \zeta_m$ satisfies $x^{p^j} = 1$ in the residue field, which clearly implies $x = 1$. Conversely, when m is not a power of p, we may as well assume that m is relatively prime to p (since (54) implies the same statement with m replaced by any of its factors). But when m is relatively prime to p, then ζ_m is a root of the polynomial

$$x^{m-1} + x^{m-2} + \cdots + 1, \tag{55}$$

and hence the same is true in the residue field. But the polynomial (55) does not have the root 1 in $\overline{\mathbb{F}}_p$. $\square$

Proof of Proposition 4. It suffices to compute the completion of $\mathbb{Z}'_a$ at every ideal (I, π) where π is a prime of $\mathbb{Z}'$. This is $\mathbb{Z}_\pi$ when every element of I is divisible by π, and 0 else. The condition for the completion being non-zero then is

$$\pi \,|\, \big(u - dim(u)\big) \tag{56}$$

for the image u of every element of $R(G)$ in $\mathbb{Z}_a$. We claim that this is equivalent to (56) holding for every weight u! In effect, $\mathbb{Z}[\Pi^*]$ is an integral extension of $R(G) = \mathbb{Z}[\Pi^*]^{W(G)}$, so every weight $x = u$ is a root of a polynomial

$$x^n + w_1 x^{n-1} + \cdots + w_n \tag{57}$$

where $w_i \in R(G)$ (the other roots are $W(G)$-conjugate weights). Let $dim(w_i) = m_i$. When (56) holds for any weight, clearly it holds in particular for any element of the representation ring. When (56) holds for any element of the representation ring, the polynomial (57) is equal to

$$x^n + m_1 x^{n-1} + \cdots + m_n \tag{58}$$

over the residue field. But then all the roots of (58) are of the form $dim(u') = 1$ where u' are weights conjugate to u, since the augmentation is a ring homomorphism. Thus, (58) is in fact equal to $(x - 1)^n$ over the residue field, which shows that the image of the weight u (which was a root of (57)) in the residue field must be equal to 1.

Thus, we know that the completion of $\mathbb{Z}_a$ is non-zero if and only if

$$\pi \,|\, (u - 1) \tag{59}$$

for every weight u. Now u is the $\langle u, a \rangle$-th power of ζ_m, the m-th primitive root of unity. Thus, the statement follows from Lemma 5. $\square$

4. Discussion of the individual Lie group types

The purpose of this section is to deduce from Theorem 1 the computations of the numbers $N(G, m, p)$ in the individual types of simple simply connected compact Lie groups. Recall convention (8). For basic facts about compact Lie groups and their representation, see [5].

Type A**:** The weight lattice Π^* for A_n ($SU(n+1)$) can be identified with the quotient of $\mathbb{Z}^{n+1}$ by the subgroup $\langle 1, \ldots, 1 \rangle$. The inner product is such that the orthogonal projection to

$$\left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 0 \right\} \tag{60}$$

is an isometry (where we consider the induced inner product from $\mathbb{R}^{n+1}$ on (60)). The roots are vectors of the form

$$(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$$

(the numbers 1 and -1 are in two arbitrary different places). Thus, a fundamental Weyl chamber can be selected as the set of all weights

$$a = (a_1, \ldots, a_n, 0) \tag{61}$$

where

$$a_1 \geqslant \cdots \geqslant a_n \geqslant 0. \tag{62}$$

The highest root is

$$(1, 0, \ldots, 0, -1),$$

so the regular weights are those tuples (61) where $a_i \in \mathbb{Z}$, and

$$m > a_1 > \cdots > a_n > 0. \tag{63}$$

The weight lattice is generated by the weights

$$(0, \ldots, 0, 1, 0, \ldots, 0) = \bigl(-1/(1+n), \ldots, -1/(1+n), n/(1+n), -1/(1+n), \ldots\bigr),$$

so the condition of Theorem 1 is that the denominator of each of the numbers

$$\sum_{i=1}^{n} -a_i/(1+n), \tag{64}$$

$$\sum_{i=1}^{n} -a_i/(1+n) + a_j \tag{65}$$

for any fixed j is a power of p. Clearly, we may replace the numbers (65) with the a_i's. Now put

$$b_i := a_i/m'. \tag{66}$$

Translating our conditions from the a_i's to the b_i's gives the condition stated in Section 2.

Type *B*: The weight lattice of B_n ($Spin(2n+1)$ – we assume here $n > 1$, for $n = 1$, the group is $SU(2)$) is the set of all tuples

$$(a_1, \ldots, a_n) \quad \text{where all } a_i\text{'s are either integers or integers plus } 1/2. \tag{67}$$

The inner product is induced from $\mathbb{R}^n$. The long roots are all vectors of the form

$$(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

(the two non-zero entries are in arbitrary different places and the signs are not required to match), and the short roots are of the form

$$(0, \ldots, 0, \pm 1, 0, \ldots, 0).$$

Thus, the fundamental Weyl chamber can be chosen as the set of all weights (67) such that (62) holds. The highest root is then

$$(1, 1, \ldots, 0, 0),$$

so the regular weights are all those satisfying

$$a_1 > \cdots > a_n > 0 \quad \text{and} \quad m > a_1 + a_2. \tag{68}$$

The generating weights can be taken as

$$(0, \ldots, 0, 1, 0, \ldots, 0), \tag{69}$$

$$(1/2, \ldots, 1/2). \tag{70}$$

Putting again (66) gives the conditions of Section 2. Note that when $p = 2$, the denominator 2 does not matter in the condition, so the weight (70) does not contribute to the condition, and also (67) remains the same with a_i's replaced by b_i's. On the other hand, for $p > 2$, a denominator 2 violates the condition of Theorem 1, which is why the numbers b_i must be integers, and their sum must be even to get an integral inner product with the weight (70).

Type *C*: The weight lattice of C_n (usually denoted $Sp(n)$ although sometimes also $Sp(2n)$) is the set of all tuples

$$(a_1, \ldots, a_n) \quad \text{such that } a_i\sqrt{2} \in \mathbb{Z}. \tag{71}$$

The inner product is again induced from $\mathbb{R}^n$. The long roots are

$$(0, \ldots, 0, \sqrt{2}, 0, \ldots, 0),$$

the short roots are

$$(0, \ldots, 0, \pm 1/\sqrt{2}, 0, \ldots, 0, \pm 1/\sqrt{2}, 0, \ldots, 0)$$

(the non-zero entries are in arbitrary places and the signs are independent). Thus, the fundamental chamber can be selected as the set of all weights (71) where (62) holds. The highest root is

$$(\sqrt{2}, 0, \ldots, 0),$$

so the regular weights are those satisfying

$$m/\sqrt{2} > a_1 > \cdots > a_n > 0. \tag{72}$$

The weights are generated by

$$(0, \ldots, 0, 1/\sqrt{2}, 0, \ldots, 0).$$

To translate the condition of Theorem 1, we must now put

$$b_i := a_i \sqrt{2}/m'. \tag{73}$$

Clearly, we obtain the conditions listed in Section 2. Note in particular again that when the b_i's are integers, inner product with a generator weight can produce a denominator of 2, which can be neglected at $p = 2$, but violates the condition for $p > 2$. This is why for $p > 2$, the numbers b_i are required to be even.

Type D: For the group D_n ($Spin(2n)$), the weight lattice is the same as for B_n (with the same inner product), but only the long roots of B_n are roots of D_n (we assume here $n > 2$ – for $n = 2$ the group is $SU(2) \times SU(2)$). Thus, the fundamental Weyl chamber is the set of all weights (67) such that

$$a_1 \geqslant \cdots \geqslant a_{n-1} \geqslant |a_n|. \tag{74}$$

The highest root remains the same as for B_n, so the regular weights are those satisfying

$$a_1 > \cdots > a_{n-1} > |a_n|, \qquad m > a_1 + a_2. \tag{75}$$

Since the weights are the same as for B_n, clearly, using (66), the divisibility conditions for the b_i's are the same as in the B_n case, while (75) translates to (19), (20).

Type G: The weight lattice of G_2 is the same as for A_2. The long roots are the same as for A_2, but there are also short roots

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1).$$

Thus, the fundamental Weyl chamber can be chosen as the set of all weights (61) such that

$$2a_2 \geqslant a_1 \geqslant a_2 \geqslant 0.$$

The highest root is the same as for A_2, so the fundamental chamber is the set of all weights (61) such that

$$2a_2 > a_1 > a_2 > 0, \qquad m > a_1.$$

Using again (66), we obtain the conditions (22), (23), and the same divisibility conditions as for A_2.

Type F: The weight lattice of F_4 is the same as for B_4. The roots are those of B_4, plus the roots

$$(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$$

(the signs are independent). Thus, the fundamental Weyl chamber can be selected as the set of all weights (67) such that

$$a_2 \geqslant a_3 \geqslant a_4 \geqslant 0, \qquad a_1 \geqslant a_2 + a_3 + a_4.$$

The highest root is the same as for B_4. Thus, the regular weights are those weights (67) where

$$a_2 > a_3 > a_4 > 0, \qquad a_1 > a_2 + a_3 + a_4, \qquad m > a_1 + a_2. \tag{76}$$

Using (66), (76) translates to (25), (26), (27), and the divisibility conditions remain the same as for B_4.

Type E_8: The weight lattice of E_8 consists of all 8-tuples

$$(a_1, \ldots, a_8) \quad \text{such that } a_i \text{ are all integers or integers plus } 1/2, \text{ and their sum is even.} \tag{77}$$

(The evenness requirement comes from the fact that E_8 does not actually contain *Spin*(16), only *SO*(16).) The roots are all the elements of the form

$$(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

where the non-zero entries are in arbitrary places and the signs are independent, and elements of the form

$$(\pm 1/2, \ldots, \pm 1/2)$$

where the signs are arbitrary such that the number of minus signs is even. Thus, a fundamental Weyl chamber can be chosen as the set of all weights

$$a_2 \geqslant a_3 \geqslant \cdots \geqslant a_7 \geqslant |a_8|, \qquad a_1 \geqslant a_2 + \cdots + a_7 - a_8 \tag{78}$$

and the highest root is

$$(1, 1, \ldots, 0, 0),$$

so the regular weights are all weights satisfying

$$a_2 > \cdots > a_7 > |a_8|, \qquad a_1 > a_2 + \cdots + a_7 - a_8, \qquad m > a_1 + a_2. \tag{79}$$

The weights are generated by

$$(0, \ldots, 0, 1, -1, 0, \ldots, 0) \tag{80}$$

and

$$(1/2, \ldots, 1/2). \tag{81}$$

So, if we use (66), we obtain the conditions (29), (30), (31). Note that due to the unimodularity of the E_8 lattice, the divisibility condition becomes vacuous. (Note also that E_8 is the only case in which there are no exceptional primes.)

Type E_7: The weight lattice of E_7 can be identified with the orthogonal projection of the E_8 lattice (77) to the orthogonal complement of a fixed root of E_8. Choosing the E_8 root

$$(1, 1, 0, \ldots, 0),$$

the weights are all elements

$$(a_1/\sqrt{2}, -a_1/\sqrt{2}, a_2, \ldots, a_7) \tag{82}$$

where

$$a_1 \in \frac{1}{\sqrt{2}}\mathbb{Z}, a_2, \ldots, a_7 \text{ are all either all integers or integers plus } 1/2$$

$$\text{and } \sqrt{2}a_1 + a_2 + \cdots a_6 - a_7 \text{ is even.} \tag{83}$$

The roots are

$$(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

where the non-zero positions are in arbitrary places other than the first coordinate, and the signs are independent,

$$(\pm\sqrt{2}, 0, \ldots, 0),$$

and

$$(\pm 1/\sqrt{2}, \pm 1/2, \ldots, \pm 1/2)$$

where the signs are independent such that the number of minus signs is odd. Therefore, a fundamental Weyl chamber can be selected as the set of all weights (82), (83) such that

$$a_2 \geqslant \cdots \geqslant a_6 \geqslant |a_7|, \qquad a_1/\sqrt{2} \geqslant a_2 + \cdots + a_6 - a_7. \tag{84}$$

The highest root is

$$(\sqrt{2}, 0, \ldots, 0),$$

so the regular weights are subject to the conditions

$$a_2 > \cdots > a_6 > |a_7|, \qquad m > a_1\sqrt{2} > a_2 + \cdots + a_6 - a_7. \tag{85}$$

The weights are generated by

$$(\sqrt{2}, 0, \ldots, 0), \tag{86}$$

$$(1/\sqrt{2}, 0, \ldots, 0, 1, 0, \ldots, 0), \tag{87}$$

$$(0, 1/2, \ldots, 1/2). \tag{88}$$

Now using (66) leads to the divisibility conditions in Section 2. Note that for $p = 2$, a factor of 2 in the denominator of b_i, $i > 1$ or $b_1\sqrt{2}$ does not violate the condition of Theorem 1, so the condition of Theorem 1 with respect to the weights (86), (87), (88) are automatically satisfied by the conditions (32), (36), which in turn follow from (82), (83). For $p > 2$, on the other hand, a 2 in the denominator of $b_1/\sqrt{2} + b_i$, $i > 1$ and $b_2 + \cdots + b_7$ violates the condition of Theorem 1, which forces (37) and (38). The condition of Theorem 1 with (86) is automatic.

Type E_6**:** The weight lattice of E_6 can be identified with the orthogonal projection of the E_8 lattice to the orthogonal complement of the E_8 roots

$$(1, -1, 0, \ldots, 0), \quad (0, 1, -1, 0, \ldots, 0).$$

This gives weights

$$(a_1/\sqrt{3}, a_1/\sqrt{3}, a_1/\sqrt{3}, a_2, \ldots, a_6) \tag{89}$$

where

$$a_1\sqrt{3}, a_2, \ldots, a_6 \text{ are all either integers or integers plus } 1/2 \tag{90}$$

and

$$2 | (\sqrt{3}a_1 + a_2 + \cdots + a_6). \tag{91}$$

The roots are

$$(0, \ldots, 0, \pm 1, 0, \ldots, 0, \pm 1, 0, \ldots, 0)$$

where the non-zero positions are in arbitrary places other than the first coordinate, and the signs are independent, and

$$(\pm\sqrt{3}/2, \pm 1/2, \ldots, \pm 1/2)$$

where the signs are independent such that the number of minus signs is even. A fundamental Weyl chamber can be selected as the set of all weights (89), (90), (91) satisfying

$$a_2 \geqslant \cdots \geqslant a_5 \geqslant |a_6|, \qquad \sqrt{3}a_1 \geqslant a_2 + \cdots + a_5 - a_6. \tag{92}$$

The highest root is

$$(\sqrt{3}/2, 1/2, \ldots, 1/2),$$

so the regular weights are those satisfying the condition

$$a_2 > \cdots > a_5 > |a_6|, \qquad \sqrt{3}a_1 > a_2 + \cdots + a_5 - a_6, \qquad m > (\sqrt{3}a_1 + a_2 + \cdots + a_6)/2. \tag{93}$$

The weights are generated by

$$(2/\sqrt{3}, 0, \ldots, 0), \tag{94}$$

$$(1/\sqrt{3}, 0, \ldots, 0, 1, 0, \ldots, 0), \tag{95}$$

$$(\sqrt{3}/2, 1/2, \ldots, 1/2). \tag{96}$$

Using (66) gives the divisibility conditions in Section 2. Note that at $p = 3$, a factor of 3 in the denominator does not matter, so the defining conditions for weights (89), (90), (91) imply the divisibility conditions with respect to the weights (94), (95), (96). On the other hand, at $p \neq 3$, a denominator of 3 in $\sqrt{3}a_1$ immediately violates the condition of Theorem 1 with respect to the weight (94), which forces (44). This together with the defining conditions (90) and (91) forces the condition of Theorem 1 for the weights (95), (96).

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