A tight lower bound on the maximum genus of a simplicial graph

Jianer Chena,1, Saroja P. Kanchib,2, Jonathan L. Grossc,*,3

a Department of Computer Science, Texas A&M University, College Station, TX 77843-3112, USA
b Science and Mathematics Department, GMI Engineering and Management Institutes, Flint, MI 48504, USA
c Department of Computer Science, Columbia University, New York, NY 10027, USA

Received 6 July 1993; revised 20 September 1994

Abstract

It is proved that every connected simplicial graph with minimum valence at least three has maximum genus at least one-quarter of its cycle rank. This follows from the technical result that every 3-regular simplicial graph except $K_4$ has a Xuong co-tree whose odd components have only one edge each. It is proved, furthermore, that this lower bound is tight. However, examples are used to illustrate that it does not apply to non-simplicial graphs. This result on maximum genus leads to several immediate consequences for average genus.

1. Introduction

A graph may have multiple adjacencies or self-loops. It is said to be simplicial if it contains neither multiple adjacencies nor self-loops. It is always assumed to be connected unless the context requires otherwise.

It is expected that the reader is somewhat familiar with topological graph theory. For general background, see Gross and Tucker [10] or White [26].

The maximum genus $\gamma_M(G)$ of a connected graph $G$ is defined to be the maximum integer $k$ such that there exists a cellular imbedding of $G$ into the orientable surface of genus $k$. From the Euler polyhedral equation, we see that the maximum genus has the obvious upper bound

$$\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor,$$

where $\beta(G)$ denotes the cycle rank.

* Corresponding author.
1 Supported by National Science Foundation under Grant CCR-9110824 and by the Engineering Excellence Award from Texas A&M University.
2 Partially supported by the Engineering Excellence Award from Texas A&M University.
3 Supported by ONR contract N00014-85-0768.

0012-365X/96/$15.00 © 1996 Elsevier Science B.V. All rights reserved
SSDI 0012-365X(95)00070-4
Our present concern is to derive a lower bound. In particular, Theorem 5.5 — our main result — asserts that for any simplicial graph $G$ with minimum valence at least 3, the maximum genus is at least $\beta(G)/4$.

2. A review of maximum genus

Since the introductory investigation of maximum genus by Nordhaus et al. [17] and a rapid follow-up [16] on forbidden subgraphs, nearly all studies of maximum genus have fallen into one of five classes. The brief summary here provides a context for our present work.

Class 1: Upper imbeddability. A graph $G$ is said to be upper imbeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$ exactly. Therefore, to show that a graph is upper imbeddable is equivalent to deriving a lower bound of $\lfloor \beta(G)/2 \rfloor$ for the maximum genus of the graph. Nordhaus et al. [17], Nebeský [15], Ringeisen [19,20], Zaks [29], Xuong [28], and more recently, Skoviera [22] have shown that various classes of graphs are upper-imbeddable. In particular, every 4-edge connected graph is upper-imbeddable [14]. However, there are examples of 3-edge connected graphs that are not upper imbeddable [12].

Class 2: Characterization. Xuong [27] characterized a maximum genus imbedding in terms of components of the complements of spanning trees.

The edge complement $G - T$ of a spanning tree $T$ in a graph $G$ is called a co-tree. Clearly, the number of edges in any co-tree of a connected graph $G$ is equal to the cycle rank $\beta(G)$. A co-tree is not necessarily connected. A connected component $H$ of a co-tree is called a $k$-edge-component if it contains exactly $k$ edges. If $k$ is even (odd) then we also call $H$ an even (odd) component.

The deficiency $\xi(G,T)$ of a spanning tree $T$ for a connected graph $G$ is defined to be the number of odd components of the co-tree $G - T$. The deficiency $\xi(G)$ of the graph $G$ is defined to be the minimum of $\xi(G,T)$ over all spanning trees $T$. A spanning tree $T$ of $G$ is called a Xuong tree if the deficiency $\xi(G,T)$ of $T$ is equal to the deficiency $\xi(G)$ of the graph $G$. A co-tree $G - T$ is called a Xuong co-tree if $T$ is a Xuong tree.

**Theorem 2.1** (Xuong [27]). Let $G$ be a connected graph. The maximum genus of $G$ is given by the equation

$$\gamma_M(G) = \frac{\beta(G) - \xi(G)}{2}$$


Class 4: Distributions. The systematic study of imbedding distributions was inaugurated by Gross and Furst [8] and has been strongly pursued by Stahl (e.g. [23,24]). Of particular interest here, Stahl [25] provides an asymptotic lower bound for the number of maximum-genus imbeddings.
Class 5: Lower bounds. Recent investigations have focused on deriving a lower bound on the maximum genus of graphs that are not upper imbeddable. Gross, et al. [9] (see also [4]) have constructed a class of graphs whose cycle rank can be arbitrarily large but whose maximum genus is bounded by 1. Skoviera [22] showed that the maximum genus of a 2-connected graph of diameter 2 is at least \(\lceil \beta(G)/2 \rceil - 2\). Chen and Gross [2] proved that the maximum genus of a 2-connected simplicial graph or of a 3-connected graph is at least \(\Omega(\log \beta(G))\). The last result was further improved by Chen et al. [5], who proved that the maximum genus of a 2-connected simplicial graph \(G\) is at least \(\beta(G)/8\).

The main result of this paper, Theorem 5.5, is that the maximum genus of a simplicial graph is greater than \(\frac{1}{4}\) of its cycle rank. Our present methods differ from those used in [5]. Elementary examples indicate that this lower bound on maximum genus is tight and that it does not apply to certain non-simplicial graphs.

The main theorem has several interesting consequences. Using techniques of Chen and Gross [2], we show that the average genus of a simplicial graph is at least \(\frac{1}{8}\) of its cycle rank. This improves a result in [5] that the average genus of a 2-connected simplicial graph is at least \(\frac{1}{16}\) of its cycle rank. Moreover, our result implies that the average genus of a simplicial graph is at least \(\frac{1}{4}\) of its maximum genus, thereby complementing another result in [5] that for a 3-regular graph the average genus is at least half its maximum genus.

3. On odd components in a Xuong co-tree

The objective of this section is to prove that every 3-regular simplicial graph \(G\) has a Xuong tree \(T\) such that every odd component in the co-tree \(G - T\) is a 1-edge-component. This result will be used in the next section to compare the number of odd components to the number of even components and thereby arrive at a lower bound for the maximum genus of a 3-regular simplicial graph.

**Lemma 3.1.** Let \(T\) be a spanning tree of a 3-regular simplicial graph. Then every component of the co-tree \(G - T\) is either a simple cycle or a simple path.

**Proof.** Since every vertex of \(G\) has valence 3 in \(G\) and valence 1, 2, or 3 in the spanning tree \(T\), it follows that every vertex has valence 2, 1, or 0 in \(G - T\), from which the conclusion follows immediately. \(\square\)

Let \(T\) be a spanning tree of a graph \(G\). Denote by \(\mathcal{P}_k(G - T)\) the set of \(k\)-edge-components that are simple paths in the co-tree \(G - T\). Similarly, denote by \(\mathcal{C}_k(G - T)\) the set of \(k\)-edge-components that are simple cycles in \(G - T\). As usual, we denote by \(|S|\) the cardinality of the set \(S\).

Let \(T\) be a spanning tree of a connected graph \(G\), and let \(e_1\) be an edge in the co-tree \(G - T\). Adding the edge \(e_1\) to the spanning tree \(T\) creates a unique cycle, which
is called the fundamental cycle of $e_1$ with respect to $T$. Therefore, if we then delete an edge $e_2$ from the fundamental cycle of $e_1$ in $T \cup e_1$, we obtain a new spanning tree $T' = (T \cup e_1) - e_2$ for the graph $G$. We say that the new spanning tree $T'$ is obtained from the spanning tree $T$ by swapping the edges $e_1$ and $e_2$.

**Lemma 3.2.** Let $T$ be a Xuong tree of a 3-regular simplicial graph $G$ other than the complete graph $K_4$, such that $|P_{2l+1}(G - T)| > 0$ for some $l \geq 1$. Then there exists another Xuong tree $T'$ in $G$ satisfying one of these two properties.

C1. $|P_1(G - T')| = |P_1(G - T)| + 1$, or

C2. $|P_1(G - T')| = |P_1(G - T)|$ and $|P_{2l-1}(G - T')| = |P_{2l-1}(G - T)| + 1$.

**Proof.** Let the simple path $P = (e_1, e_2, \ldots, e_{2l+1})$ be a component of the co-tree $G - T$, where edge $e_j$ runs from vertex $v_j$ to vertex $v_{j+1}$, for $j = 1, \ldots, 2l+1$. Also, let $F = (f_1, f_2, \ldots, f_k)$ be the unique path in tree $T$ from vertex $v_2$ to vertex $v_3$, where vertex $u_i$ lies at the juncture of edges $f_i$ and $f_{i+1}$, for $i = 1, \ldots, k-1$.

Since the graph $G$ has no multiple edges, it follows that $k \geq 2$, so that path $F$ has an interior vertex $u_1$ where the two edges $f_1$ and $f_2$ meet. Since the graph $G$ is 3-regular, some third edge $g$ must also be incident on vertex $u_1$. We designate $w_1$ as the other endpoint of $g$. There are three different cases of construction of the Xuong tree $T'$, depending on properties of the edge $g$.

**Case 1:** The edge $g$ is a tree edge in $T$ as illustrated in Fig. 1(a). In this case, we define $T' = (T - f_1) \cup e_2$, as illustrated in Fig. 1(b). Such swapping of edge $e_2$ into the Xuong tree in place of edge $f_1$ replaces the odd co-tree component $P = (e_1, e_2, \ldots, e_{2l+1})$ by the even component with edgeset $\{e_1, f_1\}$ and the odd component $P' = (e_3, e_4, \ldots, e_{2l+1})$. Thus, the conclusion holds.

**Case 2:** The edge $g$ is a co-tree edge in $G - T$, but not an edge of the path $P$.

Since the endpoint $u_1$ of the edge $g$ is 1-valent in co-tree $G - T$, it follows from Lemma 3.1 that the co-tree component containing the edge $g$ must be a path $P_g$; moreover, the path $P_g$ must terminate at $u_1$, as illustrated in Fig. 2(a).

---

\* We will use solid lines for tree edges and broken lines for co-tree edges. A solid curve represents a path of tree edges and a broken curve represents a path of co-tree edges.
We claim that the path $P_g$ is an even component of $G - T$. Otherwise, consider the subgraph $T^* = (T - f_1) \cup e_1$. Since the vertex $v_2$ is 1-valent in $T$, it follows that $(T - f_1)$ is a tree incident on every vertex of $G$ except $v_2$, and accordingly, that $T^*$ is a spanning tree. The effect on the co-tree would be to replace the two odd components $P_g$ and $P = \langle e_1, e_2, \ldots, e_{2l+1} \rangle$ by the one even component $P_g \circ \langle f_1, e_2, \ldots, e_{2l+1} \rangle$, as illustrated in Fig. 2(b), thereby contradicting the designation of $T$ as a Xuong tree.

We now observe that $T' = (T - f_1) \cup e_2$ is a spanning tree, and that the result of swapping the edge $e_2$ into the Xuong tree in place of edge $f_1$ replaces the odd co-tree component $P = \langle e_1, e_2, \ldots, e_{2l+1} \rangle$ by the odd component $P' = \langle e_3, e_4, \ldots, e_{2l+1} \rangle$ and extends the even co-tree path $P_g$ to a longer even path $P_g \circ \langle f_1, e_1 \rangle$, as illustrated in Fig. 2(c). Thus, the conclusion holds.

**Case 3:** The edge $g$ is a co-tree edge in the path $P$.

Since the end vertex $u_1$ of the edge $g$ is 1-valent in $G - T$, the edge $g$ must be either $e_1$ or $e_{2l+1}$. The edge $g$ cannot be $e_1$ since then $u_1 = v_1$ and the edges $e_1$ and $f_1$ would have the same two endpoints (remember that the graph $G$ is simplicial). So we must have $g = e_{2l+1}$, as illustrated in Fig. 3(a).

Now let us consider the vertex $u_2$. The vertex $u_2$ cannot be identical with the vertex $v_3$, since otherwise the tree edges $f_1$ and $f_2$ would not be adjacent to any other tree edges. (Note that the graph is 3-regular, so if $u_2 = v_3$, no tree edges except $f_1$ and $f_2$ would be incident on the vertices $v_2$, $u_1$ and $u_2$.) Consequently, the graph $G$ would contain only 3 vertices. But then the graph $G$ would not be a 3-regular simplicial graph.
Therefore, there must be at least a third tree edge $f_3$ in the fundamental cycle $F$ of the edge $e_2$ with respect to $T$. Let $h$ be the edge other than $f_2$ and $f_3$, incident on vertex $u_2$. We designate $w_2$ as the other endpoint of $h$. We again have three possible cases for the edge $h$.

**Subcase 3.1:** The edge $h$ is a tree edge, as illustrated in Fig. 3(b).

We let $T' = (T - f_2) \cup e_2$, as illustrated in Fig. 3(c), and the conclusion holds.

**Subcase 3.2:** The edge $h$ is a co-tree edge in $G - T$ but is not an edge of the path $P$.

Since the endpoint $u_2$ of the edge $h$ is 1-valent in the co-tree $G - T$, it follows that the co-tree component containing the edge $h$ must be a path $P_h$, as illustrated...
in Fig. 3(d). We claim that $P_k$ is an even component of $G - T$, since otherwise the subgraph $T^* = (T - f_2) \cup e_1$ would be a spanning tree such that the co-tree $G - T^*$ contains fewer odd components than the Xuong co-tree $G - T$, contradicting the designation of $T$ as a Xuong tree. Now we set $T' = (T - f_2) \cup e_2$, as illustrated in Fig. 3(e), and the conclusion holds.

**Subcase 3.3:** The edge $h$ is a co-tree edge in the path $P$.

It follows that the edge $h$ must be identical to the edge $e_1$, as illustrated in Fig. 3(f).

The vertex $u_3$ cannot be identical with the vertex $v_3$, for otherwise, the tree edges $f_1$, $f_2$ and $f_3$ would not be adjacent to any other tree edges. Consequently, the graph $G$ would contain exactly 4 vertices. This would contradict the assumption that $G$ is 3-regular, simplicial and not the complete graph $K_4$. Therefore, there must be at least a fourth tree edge $f_4$ on $F$. Let $k$ be the edge other than $f_3$ and $f_4$ incident on vertex $u_3$. We designate $w_3$ as the other endpoint of $k$.

If $k$ is a tree edge as illustrated in Fig. 3(g), then we set $T' = (T - f_3) \cup e_2$, as illustrated in Fig. 3(h), and the conclusion holds.

On the other hand, if $k$ is a co-tree edge in $G$, first note that the edge $k$ cannot be in the path $P$, since the endpoint $u_3$ of $k$ is 1-valent in $G - T$ and the edge $k$ is not identical to either $e_1$ or $e_{2l+1}$. Thus, the edge $k$ must be contained in another simple path $P_k$ in $G - T$, as illustrated in Fig. 3(i). We claim that $P_k$ is an even component of $G - T$, since otherwise the subgraph $T^* = (T - f_3) \cup e_{2l+1}$ is a spanning tree of $G$ such that the co-tree $G - T^*$ contains fewer odd components than the Xuong co-tree.
$G - T$, contradicting that $T$ is a Xuong tree. We now set $T' = (T - f_3) \cup e_2$, as illustrated in Fig. 3(j), and the conclusion holds.

This completes the proof of the Lemma 3.2. \[ \]

Fix a 3-regular simplicial graph $G$. Consider the collection $X(G)$ of all Xuong trees of $G$. Each of these trees has the property that its co-tree contains the minimum number $\xi(G)$ of odd components. Let $X_{1 \text{max}}(G)$ be the subset of $X(G)$ that contains every Xuong tree whose co-tree has the maximum number of 1-edge-components. We have the following two lemmas.

**Lemma 3.3.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$. Then for any Xuong tree $T$ in the set $X_{1 \text{max}}(G)$, we have $\mathcal{P}_{2l+1}(G - T) = \emptyset$ for $l \geq 1$.

**Proof.** Assume the contrary. Let $T^*$ be a Xuong tree such that $\mathcal{P}_{2m+1}(G - T^*) \neq \emptyset$ with the smallest $m \geq 1$, among all Xuong trees in the set $X_{1 \text{max}}(G)$.

If $m = 1$, by Lemma 3.2, there is a Xuong tree $T'$ such that $|\mathcal{P}_1(G - T')| = |\mathcal{P}_1(G - T^*)| + 1$. But this is impossible since $T^*$ is a Xuong tree in $X_{1 \text{max}}(G)$.

Now suppose that $m > 1$. Again by Lemma 3.2 there is a Xuong tree $T'$ such that either C1 or C2 of Lemma 3.2 holds.

Case C1 is impossible since $T^*$ is a Xuong tree in $X_{1 \text{max}}(G)$. Case C2 is also impossible since if $|\mathcal{P}_1(G - T')| = |\mathcal{P}_1(G - T^*)|$ then $T' \in X_{1 \text{max}}(G)$. However, $\mathcal{P}_{2m-1}(G - T') \neq \emptyset$ contradicts the choice of $T^*$.

Hence the proof. \[ \]

Now we consider the odd component that is a simple cycle in the co-tree $G - T$ for a Xuong tree $T \in X_{1 \text{max}}(G)$.

**Lemma 3.4.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$ and let $T$ be a Xuong tree in the set $X_{1 \text{max}}(G)$. Then $\mathcal{C}_{2l+1}(G - T) = \emptyset$ for $l \geq 0$.

**Proof.** Clearly, $\mathcal{C}_1(G - T) = \emptyset$, since the graph $G$ does not contain self-loops.

Suppose, to the contrary, that $\mathcal{C}_{2m+1}(G - T) \neq \emptyset$ for some $m \geq 1$. Let $C = (e_1, e_2, \ldots, e_{2m+1}) \in \mathcal{C}_{2m+1}(G - T)$, where $e_i$ runs from $v_i$ to $v_{i+1}$ for $i = 1, \ldots, 2m$ and $e_{2m+1}$ runs from $v_{2m+1}$ to $v_1$. Let $F = (f_1, f_2, \ldots, f_k)$ be the unique tree path from $v_2$ to $v_3$, where vertex $u_i$ lies at the juncture of edges $f_i$ and $f_{i+1}$ for $i = 1, \ldots, k - 1$.

Note that $k \geq 2$ since $G$ is simplicial.

Since $G$ is 3-regular, there exists another edge $g$, incident on $u_1$ other than the edges $f_1$ and $f_2$. Let $w_1$ be the other endpoint of $g$.

\[ ^5 \text{In this case } 2m - 1 = 1. \text{ Hence, the case } |\mathcal{P}_1(G - T')| = |\mathcal{P}_1(G - T^*)| \text{ and } |\mathcal{P}_{2m-1}(G - T')| = |\mathcal{P}_{2m-1}(G - T^*)| + 1 \text{ does not apply.} \]
If $g$ is a tree edge as illustrated in Fig. 4(a), we set $T' = (T - f_1) \cup e_2$, as illustrated in Fig. 4(b). It is easy to see that $T' \in X_{1\text{max}}(G)$. However, the path $\langle e_3, e_4, \ldots, e_{2m+1}, e_1, f_1 \rangle \in \mathcal{P}_{2m+1}(G - T')$ contradicting Lemma 3.3.

On the other hand, if $g$ is a co-tree edge then, since the vertex $u_1$ is 1-valent in $G - T$, the component of the co-tree containing the edge $g$ is a path $P_g$, as illustrated in Fig. 5(a). Thus, $g$ is not one of the edges in $C$. Then we set $T' = (T - f_1) \cup e_1$, as illustrated in Fig. 5(b), and observe that $T'$ is a spanning tree, such that $G - T'$ contains a component $P_g \subseteq \{f_1, e_2, e_3, \ldots, e_{2m+1}\}$ that is a simple path. However, this is impossible, since if $P_g$ is an even component, then $T' \in X_{1\text{max}}(G)$ and $\mathcal{P}_{2l+1}(G - T') \neq \emptyset$ for $2l + 1 = (2m + 1 + \text{the number of edges in } P_g)$. This contradicts Lemma 3.3. On the other hand, if $P_g$ is an odd component, then the co-tree $G - T'$ contains two fewer odd components than $G - T$, contradicting the assumption that $T$ is a Xuong tree. 

Lemma 3.3 and Lemma 3.4 immediately imply the following theorem.

**Theorem 3.5.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$. Then $G$ has a Xuong tree such that every odd component of the edge-complement $G - T$ has only one edge.
4. An edge-vertex association for odd-minimum Xuong co-trees

Given a graph $G$, a Xuong tree $T$ is said to be odd-minimum if, among all Xuong trees, the cardinality of the union of edge sets for all odd components of $G - T$ is as small as possible. Under this circumstance we also say that the co-tree $G - T$ is odd-minimum. Of course, if every odd component of $G - T$ has only one edge, then $T$ and $G - T$ are odd-minimum.

Our approach to a lower bound for maximum genus is now easy to describe. We shall construct an upper bound for the number of odd components. In the 3-regular case, Theorem 3.5 implies this means an upper bound for the number of 1-edge-components for an odd-minimum co-tree. As a technical device, we now introduce an association from the set of such 1-edge-components into the set of interior vertices of the Xuong tree.

Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$, let $T$ be an odd-minimum Xuong tree, and let the edge $e = [v, v']$ be a 1-edge-component of $G - T$. (We recall the convention of topological graph theory that every edge has an orientation, which implies that a proper edge has a well-defined originating and terminating endpoint.) We define the associated $T$-vertex $u_T(e)$ to be the first interior vertex on the path in $T$ from $v$ to $v'$. Lemmas 4.1–4.3 describe some useful properties of this association. In particular, they lead to the upper bound asserted by Theorem 4.4.

**Lemma 4.1.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$, let $T$ be an odd-minimum Xuong tree, and let the edge $e = [v, v']$ be a 1-edge-component of $G - T$. Then the associated $T$-vertex $u_T(e)$ is either 3-valent in the tree $T$ or is an end vertex of an even-length path-component of $G - T$.

**Proof.** Since $u_T(e)$ is associated with $e = [v, v'] \in P_1(G - T)$, $u_T(e)$ is the first interior vertex on the tree path $P = (f_1, f_2, \ldots, f_k)$, from $v$ to $v'$. Let $u_i$ be the vertex at the juncture of $f_i$ and $f_{i+1}$ for $i = 1, \ldots, k - 1$, where $u_T(e) = u_1$. Since $G$ is simplicial, $k \geq 2$, as illustrated in Fig. 6(a).

---

**Fig. 6.**
Suppose \( u_1 \) is not a 3-valent vertex in the tree \( T \), then \( u_1 \) is 2-valent in \( T \), and hence \( u_1 \) is an endpoint of a simple path in \( G - T \). By Theorem 3.5, \( u_1 \) is either an endpoint of a 1-edge-component \( e' \in \mathcal{P}_1(G - T) \) or \( u_1 \) is an end vertex of an even-length path-component of \( G - T \).

If \( u_1 \) is an endpoint of \( e' \) as illustrated in Fig. 6(a), the subgraph \( T^* = (T - f_1) \cup e \) is a spanning tree that has fewer odd components in the co-tree than \( G - T \), as illustrated in Fig. 6(b). This contradicts the premise that \( T \) is a Xuong tree. Thus, \( u_1 \) is an end vertex of an even-length path-component of \( G - T \).

**Lemma 4.2.** Let \( G \) be a 3-regular simplicial graph other than the complete graph \( K_4 \), and let \( T \) be an odd-minimum Xuong tree. Then the association \( e \to u_T(e) \) is one-to-one.

**Proof.** Assume the contrary. Let \((d = [v,v']) \to u \) and \((e = [x,x']) \to u \). Let \( P_1 = (f_1, f_2, \ldots, f_k) \) be the tree path from \( v \) to \( v' \) and \( P_2 = (g_1, g_2, \ldots, g_m) \) be the tree path from \( x \) to \( x' \). Let \( u_i \) be the vertex at the juncture of \( f_i \) and \( f_{i+1} \) for \( i = 1, \ldots, k - 1 \) and let \( w_i \) be the vertex at the juncture of \( g_i \) and \( g_{i+1} \) for \( i = 1, \ldots, m - 1 \). Since \( G \) is simplicial, \( k, m \geq 2 \), as illustrated in Fig. 7. Then \( u = u_1 = w_1 \).

Since the graph \( G \) is 3-regular, the edge \( f_1 \) must be one of the three edges incident on the vertex \( w_1 \). The edge \( f_1 \) cannot be the edge \( g_1 \), since \( d \) and \( e \) are two different 1-edge-components in \( \mathcal{P}_1(G - T) \). If the edge \( f_1 \) were edge \( g_2 \), as illustrated in Fig. 8(a), then \( T^* = (T - g_2) \cup e \) results in a spanning tree whose co-tree contains fewer odd components than the Xuong co-tree \( G - T \), as illustrated in Fig. 8(b), contradicting that \( T \) is a Xuong tree.

Finally, suppose that the edge \( f_1 \) is the third edge other than \( g_1 \) and \( g_2 \), incident on vertex \( u_1 \) as illustrated in Fig. 9(a). Then, \( f_1 \) is not on the path \( P_2 \), and therefore, we can set \( T^* = (T - f_1 - g_1) \cup d \cup e \), as illustrated in Fig. 9(b). Then \( T^* \) is a spanning tree whose co-tree contains fewer odd components than the Xuong co-tree \( G - T \). This again contradicts the premise that \( T \) is a Xuong tree. \( \square \)

**Lemma 4.3.** Let \( G \) be a 3-regular simplicial graph other than the complete graph \( K_4 \), let \( T \) be an odd-minimum Xuong tree, and let \( P \) be an even-length path-component of \( G - T \). Then at most one of the end vertices of \( P \) is in the image of the association \( e \to u_T(e) \).

![Fig. 7.](image-url)
Proof. Suppose the contrary. Let $P = (u_1, \ldots, w_1)$ and let $(e_1 = [v, v']) \rightarrow u_1$ and $(e_2 = [w, w']) \rightarrow w_1$, for some $e_1, e_2 \in \mathcal{P}(G - T)$.

Let $P_1 = \langle f_1, f_2, \ldots, f_k \rangle$ and $P_2 = \langle g_1, g_2, \ldots, g_m \rangle$ be the tree paths from $v$ to $v'$ and $w$ to $w'$, respectively. Then $u_1$ and $w_1$ are the first interior vertices on $P_1$ and $P_2$, respectively. We let $u_i$ be the vertex at the juncture of $f_i$ and $f_{i+1}$ for $i = 1, \ldots, k - 1$ and let $w_i$ be the vertex at the juncture of $g_i$ and $g_{i+1}$ for $i = 1, \ldots, m - 1$. Since $G$ is simplicial, $k, m \geq 2$, as illustrated in Fig. 10(a).

If $f_1 \notin P_2$, then we set $T^* = (T - f_1 - g_1) \cup e_1 \cup e_2$, as illustrated in Fig. 10(b), and if $f_1 \in P_2$, as illustrated in Fig. 11(a), we set $T^* = (T - f_1) \cup e_2$, as illustrated in Fig. 11(b). In both cases the co-tree $G - T^*$ has fewer odd components than the Xuong co-tree $G - T$. This is a contradiction. \qed

**Theorem 4.4.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$, and let $T$ be an odd-minimum Xuong tree. Then the number of 1-edge-components of $G - T$ is less than or equal to the sum of the number of interior vertices of $T$ that are 3-valent in $T$ plus the number of even-length path-components of $G - T$.

Proof. The theorem follows immediately from Lemmas 4.1–4.3, and the observation that the number of even-length path-components of $G - T$ is precisely half the number of end vertices of even-length path-components of $G - T$. \qed
5. A lower bound on maximum genus

We now derive our main result for 3-regular simplicial graphs by counting the number of 1-edge-components using the association defined in the previous section. The main result is then extended to general simplicial graphs by observing that corresponding to a simplicial graph that is not 3-regular, there exists a 3-regular simplicial graph with the same maximum genus and cycle rank.
Let $T$ be any subgraph of a graph $G$ (in the present context $T$ is a spanning tree). It is convenient to introduce notation $V_{T,j}$ for the subset of vertices of $G$ that are $j$-valent in $T$, for $j = 0, 1, 2, \ldots$.

**Lemma 5.1.** Let $T$ be any spanning tree of a 3-regular simplicial graph $G$. Then $|V_{T,1}| - |V_{T,3}| = 2$.

**Proof.** This follows immediately from these simultaneous equations:

\[
|V_{T,1}| + |V_{T,2}| + |V_{T,3}| = |V_G|,
\]

\[
|V_{T,1}| + 2|V_{T,2}| + 3|V_{T,3}| = 2|E_T| = 2|V_G| - 2. \quad \square
\]

We let $\mathcal{P}(G - T)$ denote the set of even-length path-components of $G - T$.

**Lemma 5.2.** Let $G$ be a 3-regular simplicial graph other than the complete graph $K_4$, and let $T$ be an odd-minimum Xuong tree. Then the tree $T$ has the following three properties:

P1. $|V_{T,2}| = 2|\mathcal{P}(G - T)| + 2|\mathcal{P}_1(G - T)|$,

P2. $|V_G| = 2|V_{T,1}| - 2 + 2|\mathcal{P}(G - T)| + 2|\mathcal{P}_1(G - T)|$,

P3. $|\mathcal{P}_1(G - T)| \leq |V_{T,3}| + 2|\mathcal{P}(G - T)|$.

**Proof.** (P1). The vertices of valence 2 in the tree $T$ are obviously the end vertices of the path-components.

(P2). We first observe that

\[
|V_G| = |V_{T,1}| + |V_{T,2}| + |V_{T,3}|
\]

and then apply Lemma 5.1 and Property P1.

(P3) This is a restatement of Theorem 4.4. \quad \square

**Theorem 5.3.** Let $G$ be a 3-regular simplicial graph. Then

$$\gamma_M(G) > \frac{\beta(G)}{4}.$$  

**Proof.** In the special case where $G$ is isomorphic to the complete graph $K_4$, we have $\gamma_M(G) = 1$ and $\beta(G) = 3$, from which it is immediate that the conclusion is true. In what follows, we assume that the graph $G$ is not isomorphic to $K_4$ and that $T$ is an odd-minimum Xuong tree. Then we can calculate an upper bound on the deficiency.

\[
\beta(G) = |E_G| - |V_G| + 1
\]

\[
= \frac{|V_G|}{2} + 1 \quad \text{since $G$ is 3-regular}
\]

\[
= |V_{T,1}| + |\mathcal{P}(G - T)| + |\mathcal{P}_1(G - T)| \quad \text{by P2 of Lemma 5.2}
\]

\[
> |V_{T,3}| + |\mathcal{P}(G - T)| + |\mathcal{P}_1(G - T)| \quad \{\text{by Lemma 5.1}\}
\]
\[ \geq 2|\mathcal{P}_1(G - T)| \{\text{by P3 of Lemma 5.2}\} \]
\[ = 2\xi(G) \{\text{by Theorem 3.5, since } T \text{ is an odd-minimum Xuong tree} \}. \]

Hence,
\[ \xi(G) < \frac{\beta(G)}{2}. \]

By substituting this upper bound into Xuong's equation, we conclude
\[ \gamma_M(G) = \frac{\beta(G) - \xi(G)}{2} > \frac{\beta(G)}{4}. \]

Lemma 5.4. Let \( G \) be a simplicial graph with minimum valence at least 3. Then there exists a 3-regular simplicial graph \( H \) with the same cycle rank and maximum genus as \( G \).

Proof. If the graph \( G \) itself is 3-regular, then take \( H = G \). Accordingly, we now suppose that there is a vertex \( v \) of valence \( k > 3 \), and we fix a maximum genus imbedding \( I(G) \), in which we suppose that the rotation at \( v \) is \((v_1, \ldots, v_k)\).

We now perform topological surgery on the graph \( G \) within its imbedding \( I(G) \) in which we split the vertex \( v \) into a pair of adjacent vertices \( u_1 \) and \( u_2 \), so that the resulting imbedding \( I(G') \) of the resulting graph \( G' \) has the rotations
\[ u_1 \cdot u_2, v_1, v_2 \] and \[ u_2 \cdot u_1, v_3, v_4, \ldots, v_k \]
as illustrated in Fig. 12 and rotations elsewhere identical to those of the imbedding \( I(G) \).

We now observe the following properties of \( G' \) and \( I(G') \):
(1) \( \beta(G') = \beta(G) \),
(2) \( G' \) is simplicial.

Moreover, since \( I(G) \) was a maximum genus imbedding, we know that \( \gamma_M(G') \geq \gamma_M(G) \). However, since any imbedding of \( G' \) can be contracted to an imbedding of \( G \), we also have \( \gamma_M(G') \leq \gamma_M(G) \), so we may infer the property
(3) \( \gamma_M(G') = \gamma_M(G) \).

To obtain the graph \( H \) specified in the conclusion of the lemma, we simply reiterate this splitting operation until every vertex has valence 3. \( \square \)

\[ \text{Fig. 12.} \]
Theorem 5.5. Let $G$ be any simplicial graph of minimum valence at least 3. Then

$$
\gamma_M(G) > \frac{\beta(G)}{4}.
$$

Proof. This follows easily from Theorem 5.3 and Lemma 5.4. □

Remark 1. The premise that $G$ is simplicial cannot be ignored. To see this, we define the necklace $N_d$ to be the graph obtained from a $2d$-cycle by doubling every other edge, as in [9,4]. Fig. 13 illustrates the necklace $N_4$.

It is obvious that $\beta(N_d) = d + 1$. If we removed one of the undoubled edges, then Xuong's theorem would imply that the resulting graph has maximum genus 0, from which it follows that $\gamma_M(N_d) = 1$.

Remark 2. Nor may we ignore the premise that the graph $G$ has minimum valence at least 3. In particular, we could simplicialize a necklace by subdividing the doubled edges. This would not change the maximum genus or the cycle rank.

Remark 3. One referee has sketched an alternative derivation of Theorems 5.3 and 5.5 for the special case of cubic graphs from [11] and [18].

6. Tightness of the lower bound

In this section, we prove that our lower bound on maximum genus is unimprovable for simplicial graphs. Toward this objective, we recall that a graph obtained from two graphs $G_1$ and $G_2$ by running a new edge from a vertex of one to a vertex of the other is called a bar-amalgamation of $G_1$ and $G_2$.

Theorem 6.1. The maximum genus of a bar amalgamation of $G_1$ and $G_2$ is equal to the sum of the maximum genera of $G_1$ and $G_2$.

Proof. This is a direct consequence of Theorem 5 of [8]. □
Theorem 6.2. For every positive real number \( \varepsilon \), there exists a simplicial graph \( G \) such that

\[
\gamma_M(G) < (\frac{1}{4} + \varepsilon)\beta(G).
\]

Proof. Subdivide one edge of each doubled pair of necklace \( N_d \) and in each of \( d \) copies of the complete graph \( K_4 \), subdivide a single edge. Then from new vertex in the necklace, run a bar to the new vertex in a copy of \( K_4 \). We call the resulting graph the flower \( F_d \). Fig. 14 illustrates the flower \( F_4 \). We observe that for \( d \geq 3 \), the flower \( F_d \) is simplicial.

We can easily calculate the cycle rank and maximum genus of a flower.

\[
\beta(F_d) = d\beta(K_4) + \beta(N_d) = d \cdot 3 + (d + 1) = 4d + 1,
\]

\[
\gamma_M(F_d) = d\gamma_M(K_4) + \gamma_M(N_d) = d \cdot 1 + 1 = d + 1.
\]

From these values, we have

\[
\gamma_M(F_d) = \left(\frac{1}{4} + \frac{3}{16d + 4}\right)(4d + 1) = \left(\frac{1}{4} + \frac{3}{16d + 4}\right)\beta(F_d).
\]

The conclusion now follows, since

\[
\lim_{d \to \infty} \frac{3}{16d + 4} = 0. \quad \square
\]

7. Applications to average genus and to algorithms

For any graph \( G \), if the number of imbeddings in the surface \( S_k \) of genus \( k \) is denoted \( g_k \), then the sequence

\[
g_0, g_1, g_2, \ldots
\]

is called the genus distribution of \( G \). The average genus of \( G \) is defined to be the value

\[
\gamma_{avg}(G) = \frac{\sum_{i=0}^{\infty} i \cdot g_i}{\sum_{i=0}^{\infty} g_i}.
\]
Theorem 7.1. Let \( c_0 \) be the infimum of the ratio \( \gamma_M(G)/\beta(G) \) over all 3-regular simplicial graphs. Then the ratio \( \gamma_{avg}(G)/\beta(G) \) is at least \( c_0/2 \) for all simplicial graphs in which all vertices have valence at least 3.

**Proof.** Let \( G \) be a simplicial graph in which all vertices have valence at least 3 and at most \( d \). We prove the theorem by induction on \( d \) and on the number of vertices of valence \( d \) in \( G \).

If \( d = 3 \), then the graph \( G \) is 3-regular. By a result of Chen et al. [5], \( \gamma_{avg}(G) \geq \gamma_M(G)/2 \). Combining this with the inequality \( \gamma_M(G)/\beta(G) \geq c_0 \) gives

\[
\frac{\gamma_{avg}(G)}{\beta(G)} \geq \frac{c_0}{2}.
\]

If \( d > 3 \), let \( v \) be a vertex of valence \( d \) in \( G \) and let \( v_1, \ldots, v_d \) be the \( d \) neighbors of \( v \). For \( i = 2, \ldots, d \), we define the graph \( G_i \) to be the supergraph obtained from \( G - v \) by adjoining \( v_i \) and \( v_i \) to a new vertex \( x \), adjoining all the other ex-neighbors of \( v \) to a new vertex \( y \), and adjoining \( x \) and \( y \). It is easy to verify that the graph \( G_i \) is simplicial and that \( \beta(G_i) = \beta(G) \). Moreover, \( G_i \) has one less vertex of valence \( d \) than \( G \).

By Chen and Gross [2] (the proof of Theorem 4.3, Eqs. (1)-(6)), we have

\[
\gamma_{avg}(G) = \frac{1}{d-1} \sum_{i=2}^{d} \gamma_{avg}(G_i).
\]

By our inductive hypothesis,

\[
\frac{\gamma_{avg}(G_i)}{\beta(G)} = \frac{\gamma_{avg}(G_i)}{\beta(G_i)} \geq \frac{c_0}{2}.
\]

Therefore,

\[
\frac{\gamma_{avg}(G)}{\beta(G)} = \frac{1}{d-1} \sum_{i=2}^{d} \frac{\gamma_{avg}(G_i)}{\beta(G)} \geq \frac{c_0}{2}.
\]

Theorem 7.1 together with our Theorem 5.3 immediately gives the following.

Theorem 7.2. The average genus of a simplicial graph \( G \) in which all vertices are of valence at least 3, is larger than \( \frac{1}{8} \) of its cycle rank \( \beta(G) \).

Theorem 7.2 is an improvement of a result of Chen and Gross [2] that the average genus \( \gamma_{avg}(G) \) is in \( \Omega(\log \beta(G)) \) for a 2-connected simplicial graph with all vertices of valence at least 3, as well as an improvement of a result of Chen et al. [5] that the average genus \( \gamma_{avg}(G) \) is at least \( \beta(G)/16 \) for a 2-connected simplicial graph \( G \) with all vertices of valence at least 3.

Chen et al. [5] have proved that for a graph \( G \) with all vertices of valence at most 3, the average genus of \( G \) is at least half its maximum genus. Our Theorem 5.5 generalizes this result in the following sense.
Theorem 7.3. For a simplicial graph $G$ with all its vertices of valence at least 3, the average genus of $G$ is larger than $\frac{1}{2}$ of its maximum genus.

Proof. By Theorem 7.2, we have

$$\gamma_{\text{avg}}(G) > \frac{1}{2} \beta(G).$$

Moreover, it is well known that $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$. Combining these two inequalities, we immediately get

$$\gamma_{\text{avg}}(G) > \frac{\gamma_M(G)}{4}. \quad \square$$

In conclusion we point out that the theorems developed here are also useful for designing efficient graph imbedding algorithms. In particular, the technique of proofs in Section 3 is used to obtain a polynomial-time algorithm for imbedding a graph $G$ into a surface of genus $\gamma_M(G) - 1$ [6], thereby answering a question posed by Furst et al. [7]. The technique of counting 1-edge-components is also used to arrive at a tight lower bound on the maximum genus of a 2-connected simplicial graph [13].

References