On the Wiener integral with respect to the fractional Brownian motion on an interval☆

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Abstract

We characterize the domain of the Wiener integral with respect to the fractional Brownian motion of any Hurst parameter $H \in (0, 1)$ on an interval $[0, T]$. The domain is the set of restrictions to $D((0, T))$ of the distributions of $W^{1/2-H, 2}(\mathbb{R})$ with support contained in $[0, T]$. In the case $H \leq 1/2$ any element of the domain is given by a function, but in the case $H > 1/2$ this space contains distributions that are not given by functions. The techniques used in the proofs involve distribution theory and Fourier analysis, and allow to study simultaneously both cases $H < 1/2$ and $H > 1/2.$

Keywords: Fractional Brownian motion; Wiener integral; Fractional Sobolev spaces

1. Introduction

In this work we characterize the domain of the Wiener integral on $[0, T]$ with respect to the fractional Brownian motion, with any Hurst parameter $H$, in terms of the ordinary fractional Sobolev spaces. By the domain of the Wiener integral $I$ with respect to some second order process we mean the completion of the space of simple functions, for which the integral is defined in the natural way, with respect to the inner product given by

$$(f, g) = E[I(f)I(g)],$$

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for \( f \) and \( g \) simple functions. This domain is already known in the case where the Hurst parameter is \( H < 1/2 \) and given in terms of fractional derivatives (see, for instance, Decreusefond and Üstünel [2], Alòs et al. [1] or Pipiras and Taqqu [6]). In the case \( H > 1/2 \) there are known some spaces of functions that are strict subsets of the domain of the integral because they are not complete (see Pipiras and Taqqu [6]). These last authors also propose the question of determining if the domain of the integral is or not a space of functions.

We will prove (see Theorem 3.3) that for any Hurst parameter \( H \in (0, 1/2) \cup (1/2, 1) \), the domain of \( I \) is given by the space

\[
\{ f \in D'((0, T)) : \exists f^* \in W^{1/2-H,2}([0, T]) \text{ s.t. } f = f^*|_{[0,T]} \}. \tag{1}
\]

In the case \( H < 1/2 \), this space is a space of functions, and the notation \( f^*|_{[0,T]} \) means, as usual, the restriction of \( f^* \) to the interval \([0, T]\). When \( H > 1/2 \), \( W^{1/2-H,2}([0, T]) \) is a space of distributions and we must interpret the notation \( f^*|_{[0,T]} \) as the restriction of \( f^* \) to the test functions space \( D((0, T)) \). Observe that in the case \( H = 1/2 \), the fractional Brownian motion is a standard Wiener process and expression (1) is an unusual way to describe the domain of the integral, \( L^2((0, T)) \).

The proof of our result, is based on some properties of the Wiener integral that are not particular of the fractional Brownian motion but valid for any centered second order process with continuous paths and continuous covariance function.

So, along the first part of the paper we will study the Wiener integral with respect to a general process \( X \) with these properties. The main result of this part is that if we start the definition of the Wiener integral with respect to \( X \) taking as elementary functions the \( C^\infty \) functions with compact support contained in \((0, T)\), and for a \( \varphi \) of this type define

\[
I(\varphi) = - \int_0^T X_s \varphi'(s) \, ds,
\]

the extension of this integral gives exactly the same domain that if we started with the simple functions (see Theorem 2.3). This is not at all a surprising result, but the interest here is that this fact allows to use the powerful tools of distribution theory.

In the second part (Section 3) of the paper we find the domain for the Wiener integral with respect to the fractional Brownian motion (see Theorem 3.3). We have also added Appendix A with the proof of Theorem 2.3 and a technical lemma used in the proof of Theorem 3.3.

2. Equivalent constructions of the Wiener integral

Along this section we will study the Wiener integral with respect to a process \( X = \{X_t, \ t \in [0, T]\} \) that will be a centered second order process with covariance function

\[
R(s, t) = E[X_s X_t],
\]

defined on a certain probability space \((\Omega, \mathcal{F}, P)\). We will suppose in addition that \( X \) has continuous paths and also that \( R \) is a continuous function.

Consider the set \( S_T \) of all step functions on \([0, T]\) of the form

\[
f = \sum_{j=0}^{N-1} f_j 1_{[t_j, t_{j+1})},
\]
where \( \{0 = t_0 < t_1 < \cdots < t_N = T\} \) is a partition of \([0, T]\) and \( f_j \in \mathbb{R} \). Clearly \( S_T \) is a vectorial space. For a function of the above type, define its Wiener integral with respect to \( X \) in the natural way as follows:

\[
I(f) = \sum_{j=0}^{N-1} f_j (X_{t_{j+1}} - X_{t_j}).
\]

\( (*) \)

It is easily verified that this is a good definition, being \( I \) a linear operator from \( S_T \) into a subspace of \( L^2(\Omega) \).

Then, we introduce in \( S_T \), the bilinear and symmetric form given by

\[
(f, g) = E[I(f)I(g)],
\]

which is a scalar product if we identify two functions \( f \) and \( g \) when the quantity \( (f - g, f - g) \) equals to zero.

The linear space generated by the increments of \( X \) is given by

\[
L(X) = \left\{ Z \in L^2(\Omega): Z = L^2(\Omega) - \lim_{n \to \infty} I(f_n), \text{ for some } (f_n) \subset S_T \right\}.
\]

It is a closed subspace of \( L^2(\Omega) \) with respect to the usual topology.

The following proposition gives sufficient conditions for a space of functions to be a set of integrators with respect to \( X \). This result can be proved in the same way that Proposition 2.1 of [5].

**Proposition 2.1.** Suppose that \( C \) is a space of functions defined on \([0, T]\) verifying that

1. \( C \) is an inner product space with inner product \((f, g)_C\), for \( f, g \in C \).
2. \( S_T \subset C \) and \((f, g)_C = (f, g)\) for \( f, g \in S_T \).
3. The set \( S_T \) is dense in \( C \).

Then

- There is an isometry between the space \( C \) and a linear subspace of \( L(X) \) which is an extension of the map \( f \to I(f) \) for \( f \in S_T \).
- \( C \) is isometric to \( L(X) \) if and only if \( C \) is complete.

**Remark 2.2.** Suppose that \( C \) is a space verifying the hypotheses of the above proposition. In the case in which \( C \) is complete, it can be identified with the completion of \( S_T \) with respect to the inner product \((\cdot, \cdot)_C\), that is, the domain of \( I \) obtained by the standard extension procedure. On the other hand, if we denote by \( L_T \) the domain of \( I \), we have also that, in any case,

\[
L_T = \overline{C((\cdot, \cdot)_C}}.
\]

Let \( \mathcal{V}_T \) be the set of bounded variation functions on \([0, T]\). Let also \( \mathcal{D}_T \) be the set of \( C^\infty \) functions defined on \([0, T]\) with compact support contained in \((0, T)\). This space can be identified with the space of test functions \( \mathcal{D}((0, T)) \) of the theory of distributions. Any element of \( \mathcal{D}_T \) is the extension by 0 on the boundary of \((0, T)\) of a unique function of \( \mathcal{D}((0, T)) \).

The main result of this section is the following theorem proved in Appendix A.
Theorem 2.3. The domain $\mathcal{L}_T$ of the Wiener integral, $I$, contains as dense subsets the spaces $S_T$, $V_T$, $C^\infty(\lbrack 0, T \rbrack)$ and $\mathcal{D}_T$. If $\mathcal{C}$ denotes any of these spaces, we have that, for $f \in \mathcal{C}$,

$$I(f) = -\int_0^T X_t \, d\mu_f(t).$$

Here, $\mu_f$ is the restriction to $\lbrack 0, T \rbrack, \mathcal{B}(\lbrack 0, T \rbrack)$ of the Lebesgue–Stieltjes signed measure associated with $f^0$ defined as

$$f^0(x) = \begin{cases} f(x) & \text{if } x \in \lbrack 0, T \rbrack, \\ 0 & \text{otherwise}. \end{cases} \quad (3)$$

Moreover

$$\mathcal{L}_T = \mathcal{C}^{(\cdot, \cdot)},$$

with

$$(f, g)_\mathcal{C} = \int_0^T \int_0^T R(s, t) \, d(\mu_f \otimes \mu_g)(s, t),$$

for $f, g \in \mathcal{C}$.

3. Application to fractional Brownian motion

Let $B^H = \{ B^H_t, t \in \lbrack 0, T \rbrack \}$ be a continuous fractional Brownian motion with Hurst parameter $H \in (0, 1)$. That is, $B^H$ is a continuous centered Gaussian process with covariance function given by

$$R^H(s, t) = E(B^H_s B^H_t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).$$

It is well known that any Gaussian process with the above covariance function has a version with continuous paths and that $B^{1/2}$ is a standard Wiener process.

The fractional Brownian motion can be defined on all $\mathbb{R}$ and, then, its covariance is given by

$$R^H(s, t) = E(B^H_s B^H_t) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right),$$

for any $s, t \in \mathbb{R}$. We will apply the results of the preceding section in order to describe the domain of the Wiener integral with respect to $B^H$, that will be denoted by $\mathcal{L}^H_T$, when $H \neq 1/2$.

Along this section we will make use of Fourier transforms. As usual, $\hat{f}$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R})$, that is:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{it\xi} f(x) \, dx.$$ 

We also denote by $\mathcal{F}$ the Fourier transform when defined on the space $\mathcal{S}'$ of tempered distributions.

The following lemma will be very useful in what follows. It is inspired in equality (3.4) of [5], proved for $f$ and $g$ elementary functions on $\mathbb{R}$ with compact support.

We denote by $\mathcal{S}$ the Schwartz space of real-valued $C^\infty$-functions with rapid decrease.
Lemma 3.1. Let \( f, g \in S \) and \( H \in (0, 1/2) \cup (1/2, 1) \). Then
\[
\int_{\mathbb{R}^2} R_H(s, t)f'(s)g'(t) \, ds \, dt = \frac{\sin(H\pi)\Gamma(2H + 1)}{2\pi} \int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)|x|^{1-2H} \, dx.
\]

Proof. We have that
\[
\int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)|x|^{1-2H} \, dx = \int_{\mathbb{R}} (\hat{f} \ast \hat{g})(x)|x|^{1-2H} \, dx,
\]
where, as usual, \( \hat{g} \) denotes the function defined by \( \hat{g}(x) = g(-x) \).

So, the last expression can be written as
\[
\langle |x|^{1-2H}, \mathcal{F}(f \ast \hat{g}) \rangle,
\]
that is, the action of the tempered distribution given by the locally integrable function \( h(x) = |x|^{1-2H} \) on the Fourier transform of the function of rapid decrease \( f \ast \hat{g} \). Therefore, we obtain
\[
\int_{\mathbb{R}} \hat{f}(x)\hat{g}(x)|x|^{1-2H} \, dx = 2\pi \langle \mathcal{F}^{-1}(|x|^{1-2H}), f \ast \hat{g} \rangle. \tag{4}
\]

From the table of Fourier transforms of [3, p. 359], we obtain that \( \mathcal{F}^{-1}(|x|^{1-2H}) \) is given by the tempered distribution
\[
\frac{1}{2\sin(H\pi)\Gamma(2H - 1)} |x|^{2H-2}.
\]

This distribution is a locally integrable function, when \( H \in (1/2, 1) \). When \( H \in (0, 1/2) \), the function \( \rho(x) = |x|^{2H-2} \) is not locally integrable and the distribution \( |x|^{2H-2} \) is defined as a regularization of \( \rho \) (see, for instance, [3, Chapter 1, Section 3]). This regularization satisfies that the action over a test function \( \varphi \) null at 0 (the singularity of \( \rho \)) is equal to
\[
\int_{\mathbb{R}} |x|^{2H-2} \varphi(x) \, dx,
\]
and also that
\[
|x|^{2H-2} = \left( \frac{1}{2H(2H - 1)} |x|^{2H} \right)'' ,
\]
here \( |x|^{2H} \) is a locally integrable function and the second derivative is taken in the distributional sense. So, it can be seen that the action of the distribution \( |x|^{2H-2} \) (with \( H < 1/2 \)) over an element \( \varphi \in S \) is given by
\[
\langle |x|^{2H-2}, \varphi \rangle = \int_{\mathbb{R}} (\varphi(x) - \varphi(0)) |x|^{2H-2} \, dx.
\]

On the other hand, when \(-1 < \lambda < 0\), the definition of \( \Gamma(\lambda) \) is
\[
\Gamma(\lambda) = \int_{0}^{\infty} x^{\lambda-1}[e^{-x} - 1] \, dx,
\]
see [3, Chapter 1, Section 3.3, Example 1].
Then, identity (4) writes as
\[
\int_{\mathbb{R}} \hat{f}(x) \overline{\hat{g}(x)} |x|^{1-2H} \, dx = \frac{\pi}{\sin(H\pi)\Gamma(2H-1)} |x|^{2H-2} \langle f \ast \hat{g}, \hat{f} \rangle
\]
\[
= \frac{\pi}{\sin(H\pi)\Gamma(2H-1)2H(2H-1)} \langle |x|^{2H}''', f \ast \hat{g} \rangle
\]
\[
= \frac{\pi}{\sin(H\pi)\Gamma(2H+1)} \int_{\mathbb{R}} |x|^{2H} (f \ast \hat{g})''(x) \, dx
\]
\[
= \frac{\pi}{\sin(H\pi)\Gamma(2H+1)} \int_{\mathbb{R}} |x|^{2H} \left( \int_{\mathbb{R}} f(y) g'''(y-x) \, dy \right) \, dx.
\]

And applying integration by parts in this last expression we obtain
\[
\int_{\mathbb{R}} \hat{f}(x) \overline{\hat{g}(x)} |x|^{1-2H} \, dx = -\frac{\pi}{\sin(H\pi)\Gamma(2H+1)} \int_{\mathbb{R}} |x|^{2H} \left( \int_{\mathbb{R}} f'(y) g'(y-x) \, dy \right) \, dx
\]
\[
= -\frac{\pi}{\sin(H\pi)\Gamma(2H+1)} \iint_{\mathbb{R}^2} |s-t|^{2H} f'(s) g'(t) \, ds \, dt
\]
\[
= \frac{2\pi}{\sin(H\pi)\Gamma(2H+1)} \iint_{\mathbb{R}^2} R_H(s,t) f'(s) g'(t) \, ds \, dt. \quad \square
\]

In order to obtain the main result of this section, we will discuss some properties of the Sobolev spaces with order \( s \in (-1/2, \infty) \).

Recall that one can give the following representation of the Sobolev space \( W^{s,2}(\mathbb{R}) \), for any \( s \in \mathbb{R} \):
\[
W^{s,2}(\mathbb{R}) = \left\{ f \in S' : (1 + |x|^2)^{s/2} \mathcal{F} f(x) \in L^2(\mathbb{R}) \right\},
\]
that is a Hilbert space with respect to the scalar product
\[
\langle f, g \rangle = \int_{\mathbb{R}} \mathcal{F} f(x) \overline{\mathcal{F} g(x)} (1 + |x|^2)^s \, dx.
\]

Moreover, if \( f \in S' \) has compact support, it is known that \( \mathcal{F} f \in C^\infty(\mathbb{R}) \), and then \( \mathcal{F} f \) is bounded on any bounded neighborhood of 0. Therefore, for \(-1/2 < s < \infty\) and \( f \in S' \) with compact support, it is equivalent to say that
\[
(1 + |x|^2)^{s/2} \mathcal{F} f(x) \in L^2(\mathbb{R})
\]
and that
\[
|x|^s \mathcal{F} f(x) \in L^2(\mathbb{R}).
\]

We will also need to restrict some distributions of \( S' \) to \( \mathcal{D}((0, T)) \).
Definition 1. Given $f \in S'$ its restriction to $\mathcal{D}((0, T))$, denoted by $f|_{[0,T]}$, is the distribution of $\mathcal{D}((0, T))'$ given by

$$\langle f|_{[0,T]}, \varphi \rangle := \langle f, \varphi^0 \rangle,$$

for $\varphi \in \mathcal{D}((0, T))$.

It is easily seen that this is a good definition, $f|_{[0,T]}$ is really a distribution of $\mathcal{D}((0, T))'$.

The use of the notation $f|_{[0,T]}$ is justified by the fact that when $f \in S'$ is given by a function with slow growth, then $f|_{[0,T]}$ is given by the usual restriction of $f$ to the interval $[0, T]$.

If $f \in \mathcal{D}((0, T))'$ and there exists $f^* \in S'$ with support contained in $[0, T]$ such that $f = f^*|_{[0,T]}$, we can see this $f^*$ as an extension of $f$ by 0 to all $\mathbb{R}$. This extension is not necessarily unique. Nevertheless, we can prove the following result.

Proposition 3.2. Let $-1/2 \leq s < \infty$ and let $f^*, g^* \in W^{s, 2}(\mathbb{R})$ with support contained in $[0, T]$ such that $f^*|_{[0,T]} = g^*|_{[0,T]}$. Then, $f^* = g^*$.

Proof. Under our hypotheses, $f^* - g^* \in W^{s, 2}(\mathbb{R})$ and its support is contained in $\{0\} \cup \{T\}$. This implies that $f^* - g^* = 0$. Indeed, on one hand the distributions having support on a finite number of points are finite linear combinations of the $\delta$'s distributions at these points and their derivatives. On the other hand, a finite linear combination of this kind does not belong to any $W^{s, 2}(\mathbb{R})$ with $s \geq -1/2$, except in the case in which all the coefficients are zero.

By Theorem 2.3 we have that for $f, g \in \mathcal{D}_T$

$$E[I(f)I(g)] = \int_0^T \int_0^T R^H(s,t) \, d(\mu_f \otimes \mu_g)(s,t)$$

$$= \int_0^T \int_0^T R^H(s,t) \, f'(s)g'(t) \, ds \, dt = \int_{\mathbb{R}^2} R^H(s,t) \, (f^0)'(s)(g^0)'(t) \, ds \, dt,$$

with $f^0$ defined in (3).

Then, due to Lemma 3.1, we have that for $f, g \in \mathcal{D}_T$

$$E[I(f)I(g)] = C \int_{\mathbb{R}} \widehat{f^0}(x) \widehat{g^0}(x) \, |x|^{1-2H} \, dx,$$

where $C$ is some constant. This fact can suggest that $\mathcal{L}^H_T$ is the set of distributions $f \in \mathcal{D}((0, T))'$ which have an (unique) extension by 0 to all $\mathbb{R}$ belonging to $W^{1/2-H, 2}(\mathbb{R})$. That is, $f = f^*|_{[0,T]}$ with $f^* \in W^{1/2-H, 2}(\mathbb{R})$ and $\text{supp}(f^*) \subset [0, T]$. Moreover, the inner product in the domain must be given by

$$(f, g) = C \int_{\mathbb{R}} \mathcal{F}f^*(x)\mathcal{F}g^*(x) \, |x|^{1-2H} \, dx.$$
This result is proved in the next theorem. On the other hand, this last expression was obtained by Hu [4] (see equality (2.8) in Proposition 2.1 of this last paper) for \( f \) and \( g \) such that \( f^0, g^0 \in L^2_{H-1/2}(\mathbb{R}) \), where this space is defined by
\[
L_{H-1/2}(\mathbb{R}) = \{ f : \mathbb{R} \rightarrow \mathbb{R} \ \text{measurable}; \ |x|^{1-2H} \mathcal{F}(f) \in L^2(\mathbb{R}) \}.
\]

**Theorem 3.3.** The domain of the Wiener integral with respect to the fractional Brownian motion with parameter \( H \in (0, 1/2) \cup (1/2, 1) \) is given by
\[
\mathcal{W}^H = \{ f = f^*|_{[0,T]} : f^* \in W^{1/2-H,2}(\mathbb{R}) \text{ with supp}(f^*) \subset [0, T] \}
\]
endowed with the inner product
\[
(f, g) = \frac{\Gamma(2H + 1) \sin(H\pi)}{2\pi} \int_{\mathbb{R}} \mathcal{F}f^*(x)\mathcal{F}g^*(x)|x|^{1-2H} \, dx.
\]

**Proof.** By Theorem 2.3, expression (5) and Lemma 3.1, \( \mathcal{W}^H \) is the completion of \( \mathcal{D}_T \) with respect to the inner product given by
\[
(f, g) = E[I(f)I(g)] = \frac{\Gamma(2H + 1) \sin(H\pi)}{2\pi} \int_{\mathbb{R}} \mathcal{F}f^*(x)\overline{\mathcal{F}g^*(x)}|x|^{1-2H} \, dx.
\]

Define \( \mathcal{W}^H = \{ f = f^*|_{[0,T]} : f^* \in W^{1/2-H,2}(\mathbb{R}) \text{ and supp } f^* \subset [0, T] \} \), endowed with the inner product given by (6). So, to prove the theorem, we must see that \( \mathcal{W}^H \) is complete and that \( \mathcal{D}_T \) is dense therein.

To see completeness, take \( (f_n)_n \), a Cauchy sequence in \( \mathcal{W}^H \). Then, the sequence \( (f_n^*)_n \), with \( f_n = f_n^*|_{[0,T]} \), is a Cauchy sequence of \( W^{1/2-H,2}(\mathbb{R}) \) (with respect to the usual topology of \( W^{1/2-H,2}(\mathbb{R}) \)). Since \( W^{1/2-H,2}(\mathbb{R}) \) is complete, there exists \( f^* \in W^{1/2-H,2}(\mathbb{R}) \) such that
\[
\int_{\mathbb{R}} |\mathcal{F}(f_n^* - f^*)(x)|^2(1 + |x|^2)^{1/2-H} \, dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

To prove that \( f = f^*|_{[0,T]} \) is an element of \( \mathcal{W}^H \), we must show that \( \text{supp}(f^*) \subset [0, T] \). But this is a consequence of the fact that the convergence in \( W^{1/2-H,2}(\mathbb{R}) \) implies the convergence in \( \mathcal{D}(\mathbb{R})' \) and then, for any \( \varphi \in \mathcal{D}(\mathbb{R})' \), with support contained in \( [0, T]^c \), we have
\[
\langle f^*, \varphi \rangle = \lim_{n \rightarrow \infty} \langle f_n^*, \varphi \rangle = 0.
\]

To see now that
\[
\int_{\mathbb{R}} |\mathcal{F}(f_n^* - f^*)(x)|^2|x|^{1-2H} \, dx \rightarrow 0, \quad \text{as } n \rightarrow \infty,
\]
we will show that there is a subsequence of \( (f_n)_n \) tending, in \( \mathcal{W}^H \), to \( f = f^*|_{[0,T]} \). By (7), we can take \( (f_{n_k})_k \) such that \( \mathcal{F}f_{n_k}^* \rightarrow \mathcal{F}f^* \) almost everywhere. Then, given \( \varepsilon > 0 \), we have that
\[
\int_{\mathbb{R}} |\mathcal{F}(f_{n_k}^* - f^*)(x)|^2|x|^{1-2H} \, dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}} |\mathcal{F}(f_{n_k}^* - f_{n_k}^*)(x)|^2|x|^{1-2H} \, dx < \varepsilon,
\]
the latter inequality following from the fact that the sequence \( (f_{n_k}^*)_k \) is Cauchy in \( \mathcal{W}^H \).
for \( k \) big enough, because \((f_n)\) is a Cauchy sequence. Therefore, \( f_{n_k} \to f \) in \( \mathcal{W}^H \), as \( k \to \infty \).

Next, we will show that \( \mathcal{D}_T \) is densely included in \( \mathcal{W}^H \). Take \( f \in \mathcal{W}^H \), that is, \( f = f^*|_{[0,T]} \) with \( f^* \in W^{1/2-H,2}(\mathbb{R}) \) and \( \text{supp}(f^*) \subset [0,T] \). In Lemma A.1 of Appendix A, it is proved that there exist \( f_\varepsilon \in W^{1/2-H,2}(\mathbb{R}) \) with support in \([\varepsilon, T-\varepsilon]\) such that \( f_\varepsilon \to f \) in \( \mathcal{W}^H \), as \( \varepsilon \to 0 \).

So, the proof will be finished if we show that for \( f \in \mathcal{W}^H \) with support contained in an interval \([a, b] \subset (0, T)\), there exist functions \( \varphi_n \in \mathcal{D}_T \) such that \( \varphi_n \to f \) in \( \mathcal{W}^H \), as \( n \to \infty \).

Take \( h \in C^\infty \) with support contained in \([-1, 1]\) such that \( \int_{\mathbb{R}} h(x) \, dx = 1 \) and define \( h_n(x) = nh(nx) \). We have that \( \varphi_n^* = f^* \ast h_n \in \mathcal{D}(\mathbb{R}) \) and these functions have support in \([a - \frac{1}{n}, b + \frac{1}{n}]\).

So, if \( n \) is big enough, the support of the \( \varphi_n^* \) will be contained in \((0, T)\). To conclude, we will see now that \( \varphi_n = \varphi_n^*|_{[0,T]} \) tends to \( f \) in \( \mathcal{W}^H \), that is:

\[
\int_{\mathbb{R}} \left| \mathcal{F}(\varphi_n^* - f^*)(x) \right|^2 |x|^{1-2H} \, dx \to 0, \quad \text{as} \ n \to \infty.
\]

In fact,

\[
\int_{\mathbb{R}} \left| \mathcal{F}(f^* \ast h_n - f^*)(x) \right|^2 |x|^{1-2H} \, dx = \int_{\mathbb{R}} \left| \mathcal{F} f^*(x) \right|^2 |h_n(x)| \, dx \to 0, \quad \text{as} \ n \to \infty.
\]

By dominated convergence, this last integral goes to zero, as \( n \to \infty \), if \( f \in \mathcal{W}^H \). Indeed, \( \mathcal{F} h_n(x) \) tends pointwise to 1, and moreover, \( \| \mathcal{F} h_n(x) \| \leq 1 \). □

**Remark 3.4.** Observe that in the case \( H < 1/2 \), \( \mathcal{L}^H_T \) is a space of functions and equals to the already known domains given by Pipiras and Taqqu [6] and other authors (see, for instance, Decreusefond and Üstünel [2] and Alòs et al. [1]). Nevertheless, for \( H > 1/2 \), \( \mathcal{L}^H_T \) actually is a space of distributions.

**Appendix A**

In this section we give the proof of Theorem 2.3 and a technical result used in the proof of Theorem 3.3.

**Proof of Theorem 2.3.** Observe that by summation by parts formula, if \( f \in \mathcal{S}_T \)

\[
I(f) = f_{N-1}X_T - f_0X_0 - \sum_{j=1}^{N-1} X_{t_j}(f_j - f_{j-1}) = -\int_{[0,T]} X_t \, d\mu_f(t), \tag{A.1}
\]

where \( \mu_f \) is the following signed measure:

\[
\mu_f = \sum_{j=1}^{N-1} (f_j - f_{j-1})\delta_{t_j} + f(0+)\delta_0 - f(T)\delta_T.
\]

Notice that, for \( f \in \mathcal{S}_T \), \( \mu_f \) is the restriction to \((0, T], \mathcal{B}([0, T]))\) of the signed measure associated with \( f^0 \), where \( f^0 \) is given by (3).

Then, from (A.1), it is easily verified that for \( f \) and \( g \) in \( \mathcal{S}_T \),

\[
E[I(f)I(g)] = \int_0^T \int_0^T R(s, t) \, d(\mu_f \otimes \mu_g)(s, t), \tag{A.2}
\]
where we are denoting the integrals on \([0, T]\) by \(\int_0^T\).

Consider now \(\mathcal{V}_T\) the space of bounded variation functions defined on \([0, T]\), and define for \(f\) and \(g\) in \(\mathcal{V}_T\)

\[
(f, g)_{\mathcal{V}_T} = \int_0^T \int_0^T R(s, t) d(\mu_f \otimes \mu_g)(s, t),
\]

where \(\mu_f\) is again the signed measure associated to the function \(f^0\) given by (3), restricted to \(([0, T], \mathcal{B}([0, T]))\).

The form \((\cdot, \cdot)_{\mathcal{V}_T}\) is obviously bilinear and symmetric. Since \(R\) is continuous and non-negative defined, we have also that

\[
(f, f)_{\mathcal{V}_T} = \lim_{|\pi| \to 0} \sum_{i,j} \mu_f((s_i, s_{i+1}]) R(s_i, s_j) \mu_f((s_j, s_{j+1}]) \geq 0,
\]

where we have denoted by \(\pi\) the elements of a family of partitions of \([0, T]\). Then, if we identify two functions \(f, g \in \mathcal{V}_T\) if \((f - g, f - g)_{\mathcal{V}_T} = 0\), \(\mathcal{V}_T\) endowed with \((\cdot, \cdot)_{\mathcal{V}_T}\) is an inner product space. Then, we can prove the following claim:

(I) The linear space \(\mathcal{V}_T\) equipped with \((\cdot, \cdot)_{\mathcal{V}_T}\) verifies conditions (1)–(3) of Proposition 2.1. Moreover, for \(f \in \mathcal{V}_T\),

\[
I(f) = -\int_0^T X_t \, d\mu_f(t).
\]

Indeed, properties (1) and (2) of Proposition 2.1 are clearly satisfied. To see (3), given \(f \in \mathcal{V}_T\), and a sequence of partitions \(\pi^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{k_n} = T\}\) such that \(|\pi^n| \to 0\) as \(n \to \infty\), consider

\[
f_n = \sum_{i=0}^{k_n-1} f(t^n_i) 1_{[t^n_i, t^n_{i+1})} \in S_T.
\]

It can be seen that \((f_n)_n\) converges to \(f\) in \(\mathcal{V}_T\), as \(n \to \infty\). That is, one can check that

\[
\lim_{n \to \infty} (f_n, f_n)_{\mathcal{V}_T} = (f, f)_{\mathcal{V}_T} \quad \text{(A.3)}
\]

and

\[
\lim_{n \to \infty} (f_n, f)_{\mathcal{V}_T} = (f, f)_{\mathcal{V}_T}. \quad \text{(A.4)}
\]

For instance, to see (A.3), one can show that for every continuous function \(K : [0, T]^2 \to \mathbb{R}\), it follows that

\[
\lim_{n \to \infty} \int_{[0,T]^2} K(s, t)(d\mu_{f_n} \otimes d\mu_{f_n})(s, t) = \int_{[0,T]^2} K(s, t)(d\mu_f \otimes d\mu_f)(s, t). \quad \text{(A.5)}
\]

This fact is easily verified when \(K = K_1 \otimes K_2\), with \(K_i : [0, T] \to \mathbb{R} C^1\)-functions, \(i = 1, 2\). From this, the result follows by a density argument, taking into account that

\[
|\mu_{f_n} \otimes \mu_{f_n}|([0, T]^2) \leq 2|\mu_f \otimes \mu_f|([0, T]^2),
\]

where \(|\mu|\) is denoting the total variation of the measure \(\mu\).
Finally, to show that for \( f \in V_T \)
\[
I(f) = - \int_0^T X_t d\mu f(t),
\]
it suffices to observe that, with the \( f_n \) considered above,
\[
I(f) = L^2(\Omega) - \lim_{n \to \infty} I(f_n) = L^2(\Omega) - \lim_{n \to \infty} \left(- \int_0^T X_t d\mu f_n(t)\right)
\]
and that, as it is easily checked, \( \int_0^T X_t d\mu f_n(t) \) converges pointwise in \( \Omega \) to \( \int_0^T X_t d\mu f(t) \), due to the continuity of \( X \).

From claim (I), we have that for any \( f \in C^\infty([0, T]) \), \( I(f) \) can be defined and equals to
\[
f(T)X_T - f(0)X_0 - \int_0^T X_t f'(t) dt.
\]
Now, it can be proved this second claim:

(II) The space \( D_T \) is a dense subset of \( V_T \), with respect to the topology induced by \( (\cdot, \cdot)_{V_T} \).

In fact, it suffices to see that we can approximate in \( V_T \) the indicator functions \( 1_{[a,b)} \), with \( [a, b) \subset [0, T] \), by functions belonging to \( D_T \). Consider, for each \( n \in \mathbb{N} \), \( \varphi_n \), an element of \( D_T \) with support in \( [a + \frac{1}{n}, b - \frac{1}{n}] \), such that it is equal to 1 in \( [a + 2\frac{1}{n}, b - 2\frac{1}{n}] \), and it is increasing on the interval \( [a + \frac{1}{n}, a + \frac{2}{n}] \) and decreasing on the interval \( [b - \frac{2}{n}, b - \frac{1}{n}] \).

One can check the analogous of convergence (A.5), that is:
\[
\lim_{n \to \infty} \int_0^T \int_0^T K(s, t)\varphi_n'(s)\varphi_n'(t) ds dt = K(a, a) + K(b, b) - K(a, b) - K(b, a).
\]
(Notice that \( \mu_1_{[a,b]} = \delta_a - \delta_b \).

It is easy to see convergence (A.6) in the case where \( K \) is a polynomial. From this, the claim is proved also by a density argument.

From claims (I) and (II), the conclusions of the theorem follow easily. \( \square \)

The following lemma involves computations with distributions and is used in the proof of Theorem 3.3.

**Lemma A.1.** Let \( H \in (0, 1/2) \cup (1/2, 1) \) and let \( f \in W^{1/2-H, 2}(\mathbb{R}) \) such that \( \text{supp}(f) \subset [0, T] \). Then, there exist \( f_\varepsilon \in W^{1/2-H, 2}(\mathbb{R}) \) with \( \text{supp}(f_\varepsilon) \subset [\varepsilon, T - \varepsilon] \) such that
\[
\int_{\mathbb{R}} |\mathcal{F}(f - f_\varepsilon)(x)|^2 |x|^{1-2H} dx \to 0, \quad \text{as } \varepsilon \to 0+.
\]

**Proof.** Suppose \( 0 < \varepsilon < \frac{T}{2} \). The affine transformation
\[
\rho_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad x \to \frac{T - 2\varepsilon}{T}x + \varepsilon
\]
maps the interval $[0, T]$ on $[\varepsilon, T - \varepsilon]$. Therefore, the distribution $f_\varepsilon$ given by
\[
(f_\varepsilon, \varphi) = (f, \varphi \circ \rho_\varepsilon),
\]
for $\varphi \in \mathcal{S}$, has support contained in $[\varepsilon, T - \varepsilon]$. On the other hand, it is easily checked that if $f$ is a function, then $f_\varepsilon = \frac{T}{T - 2\varepsilon} f \circ \rho_\varepsilon^{-1}$.

We can also verify that, for any $f \in W^{1/2 - H, 2}(\mathbb{R})$, $\mathcal{F}f_\varepsilon$ is a function and that $\mathcal{F}f_\varepsilon(x) = \mathcal{F}f(\rho_\varepsilon(x))$. Indeed,
\[
\langle \mathcal{F}f_\varepsilon, \varphi \rangle = \langle f_\varepsilon, \mathcal{F}\varphi \rangle = \langle f, \mathcal{F}(\varphi \circ \rho_\varepsilon^{-1}) \rangle = \frac{T - 2\varepsilon}{T} \int_{\mathbb{R}} \mathcal{F}f(x)\varphi(\rho_\varepsilon^{-1}(x)) \, dx = \int_{\mathbb{R}} \mathcal{F}f(\rho_\varepsilon(x))\varphi(x) \, dx.
\]

Next, we will check (A.7). Suppose that $\varepsilon < T/4$. For any $A > 0$ big enough, we have that
\[
\int_{\mathbb{R}} \left| \mathcal{F}(f - f_\varepsilon)(x) \right|^2 |x|^{1-2H} \, dx
\leq \int_{-A}^{A} \left| \mathcal{F}(f - f_\varepsilon)(x) \right|^2 |x|^{1-2H} \, dx
+ 2 \int_{[-A, A]^c} \left| \mathcal{F}f(x) \right|^2 |x|^{1-2H} \, dx + 2 \int_{[-A, A]^c} \left| \mathcal{F}f_\varepsilon(x) \right|^2 |x|^{1-2H} \, dx
\leq \sup_{\eta \in [-A, A]} \left( (\mathcal{F}f)'(\eta) \right)^2 \int_{-A}^{A} \left| \rho_\varepsilon(x) - x \right|^2 |x|^{1-2H} \, dx
+ 2 \int_{[-A, A]^c} \left| \mathcal{F}f(x) \right|^2 |x|^{1-2H} \, dx
+ 2 \left( \frac{T}{T - 2\varepsilon} \right)^{2 - 2H} \int_{[-A, A]^c} \left| \mathcal{F}f(x) \right|^2 |x - \varepsilon|^{1-2H} \, dx.
\]

Taking into account that $\frac{T}{T - 2\varepsilon} \leq 2$ and that, for $\varepsilon < A/2 \leq |x|/2$, we have that
\[
\left| x \right|^2 \leq |x - \varepsilon| \leq \frac{3|x|}{2},
\]
and, then, there exists a constant $C_1$, only depending on $H$ such that
\[
|x - \varepsilon|^{1-2H} \leq C_1 |x|^{1-2H},
\]
we can bound the right-hand side of (A.8) by
\[
\sup_{\eta \in [-A, A]} \left( (\mathcal{F}f)'(\eta) \right)^2 \int_{-A}^{A} \left| \rho_\varepsilon(x) - x \right|^2 |x|^{1-2H} \, dx + C_2 \int_{[-A, A]^c} \left| \mathcal{F}f(x) \right|^2 |x|^{1-2H} \, dx,
\]
with $C_2$ only depending on $H$.

The proof concludes by using that $\rho_\varepsilon(x) \to x$ uniformly on the compacts and that for $f \in W^{1/2-H,2}(\mathbb{R})$ with compact support, $\mathcal{F} f \in \mathcal{C}^\infty$ and $\int_\mathbb{R} |\mathcal{F} f(x)|^2 |x|^{1-2H} dx < \infty$. □

References