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The reflexive and anti-reflexive solutions of the matrix equation $A^H XB = C$ ☆

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Abstract

In this paper, the necessary and sufficient conditions for the solvability of matrix equation $A^H XB = C$ over the reflexive or anti-reflexive matrices are given, and the general expression of the solution for a solvable case is obtained. Moreover, the relative optimal approximation problem is considered. The explicit expressions of the optimal approximation solution and the minimum norm solution are both provided.

MSC: 15A24

Keywords: Reflexive matrix; Anti-reflexive matrix; Matrix equation; Optimal approximation matrix; Minimum norm solution

1. Introduction

Throughout the paper, we denote the complex n-vector space by C^n, the set of $m \times n$ complex matrices by $C^{m\times n}$, the set of $n \times n$ nonsingular matrices by $C_n^{n\times n}$, and the conjugate transpose of a complex matrix A by A^H. We define an inner product $\langle A, B\rangle = \mathrm{trace}(B^H A)$ for all $A, B \in C^{m\times n}$. Then $C^{m\times n}$ is a Hilbert inner product space and the norm generated by this inner product is Frobenius norm. Let $\|A\|$ be the Frobenius norm of matrix A. The notation $A * B = (a_{ij} \cdot b_{ij})_{m\times n}$ represents the Hadamard product for any $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$.

Chen and Chen [1], Chen [2,3], and Chen and Sameh [4] introduce the following two special classes of subspaces in $C^{m\times n}$

$$C_r^{n\times n}(P) = \{X \in C^{n\times n} | X = PXP\},$$

$$C_a^{n\times n}(P) = \{X \in C^{n\times n} | X = -PXP\},$$

where P is a generalized reflection matrix of size n, that is $P^H = P$ and $P^2 = I$.

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Let $HOC^{n\times n}=\{P\in C^{n\times n}|P^H=P,\ P^2=I\}$. Matrix $X\in C_r^{n\times n}(P)$ (or $Y\in C_a^{n\times n}(P)$) is said to be a reflexive (or anti-reflexive) matrix with respect to the generalized reflection matrix P, respectively (abbreviated reflexive (or anti-reflexive) matrix in the paper). The reflexive and anti-reflexive matrices have many special properties and are widely used in engineering and scientific computation [1–4].

In this paper, we mainly consider the following two problems.

Problem I. Given matrices $A\in C^{n\times m}$, $B\in C^{n\times l}$, $C\in C^{m\times l}$, find $X\in C_r^{n\times n}(P)$ (or $C_a^{n\times n}(P)$) such that

$$A^HXB=C. \tag{1.1}$$

Denote the solution set of Problem I by S_{rE} (or S_{aE}). If S_{rE} (or S_{aE}) is nonempty, we consider the relative optimal approximation problem, namely

Problem II. Given $X^*\in C^{n\times n}$, find a $n\times n$ matrix $\hat{X}\in S_{\mathrm{rE}}$ (or S_{aE}) such that

$$\|\hat{X}-X^*\|=\min_{X\in S_{\mathrm{rE}}(\mathrm{or}\,S_{\mathrm{aE}})}\|X-X^*\|. \tag{1.2}$$

The matrix equation $A^{\mathrm{T}}XA=B$, one special form of $A^HXB=C$, was discussed in many papers with the unknown matrix X be symmetric, symmetric positive, central-symmetric, symmetric ortho-symmetric (see, for instance, Chu [5,6], Dai [8], He [10], Peng [12], Peng and Hu [14]) and the matrix equation $AX=B$, another special form of $A^HXB=C$, was discussed too with the unknown matrix X be reflexive, anti-reflexive, symmetric and re-positive define and etc., (see, for instance, [13,7,9,15]). Matrix equation $A^HXB=C$ with unknown X be only symmetric was studied in Dai [7]. In these papers, by using the structure properties of matrices in required subset of $C^{m\times n}$ and the generalized singular value decomposition (GSVD), the necessary and sufficient conditions for the existence and the expressions of the solution for the matrix equation were given. The reflexive and anti-reflexive matrices have extensive applications in engineering and scientific computation. However, the reflexive and anti-reflexive solutions of matrix equation $A^HXB=C$ have not been considered yet. This Problem I is to be settled.

The optimal approximation problem is initially proposed in the processes of test or recovery of linear systems due to incomplete dates or revising given dates. A preliminary estimate X^* of the unknown matrix X can be obtained by the experimental observation values and the information of statistical distribution and that is the Problem II we shall discuss in this paper. Finally, we will obtain the expression of the minimum norm solution which satisfies $\min_{X\in S_{\mathrm{rE}}}\|X\|$ (or $\min_{X\in S_{\mathrm{aE}}}\|X\|$).

The paper is organized as follows. In Section 2, the necessary and sufficient conditions for the existence of and the expressions for the reflexive and anti-reflexive solutions of Eq. (1.1) will be derived. In Section 3, the expressions for the optimal approximation solution of Problem II and the minimum norm solution of Problem I will be obtained. In Section 4, we have given a numerical example to illustrate our results.

2. Solutions of Problem I

In this section, we first introduce some structure properties of the generalized reflection matrix P and subsets $C_r^{n\times n}(P)$ and $C_a^{n\times n}(P)$ of $C^{n\times n}$. We then give the necessary and sufficient conditions for the existence of and the expressions for the reflexive and anti-reflexive solutions with respect to a generalized reflection matrix P of Eq. (1.1).

Lemma 2.1 (*Peng et al. [14]*). *If* $r=\mathrm{rank}(I_n+P)$, $n-r=\mathrm{rank}(I_n-P)$, *then there exist unit orthogonal column matrices* $U_1\in C^{n\times r}$ *and* $U_2\in C^{n\times(n-r)}$ *such that* $\frac{1}{2}(I_n+P)=U_1U_1^H$, $\frac{1}{2}(I_n-P)=U_2U_2^H$ *and* $U_1^HU_2=0$.

Let $U=(U_1,U_2)$, *it can be easily verified from Lemma* 2.1 *that* U *is an unitary matrix, and* P *can be expressed as*

$$P=U\begin{pmatrix} I_r & O\\ O & -I_{n-r}\end{pmatrix}U^H. \tag{2.1}$$

Lemma 2.2 (*Peng et al.* [14]). *$X \in C_r^{n\times n}(P)$ if and only if X can be expressed as*

$$X = U \begin{pmatrix} \bar{X}_{11} & O \\ O & \bar{X}_{22} \end{pmatrix} U^H, \tag{2.2}$$

where $\bar{X}_{11} \in C^{r\times r}$, $\bar{X}_{22} \in C^{(n-r)\times(n-r)}$, and U is the same as (2.1).

Lemma 2.3 (*Peng et al.* [14]). *$X \in C_a^{n\times n}(P)$ if and only if X can be expressed as*

$$X = U \begin{pmatrix} O & \bar{X}_{12} \\ \bar{X}_{21} & O \end{pmatrix} U^H, \tag{2.3}$$

where $\bar{X}_{12} \in C^{r\times(n-r)}$, $\bar{X}_{21} \in C^{(n-r)\times r}$, and U is the same as (2.1).

Denote

$$U^H A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad A_1 \in C^{r\times m}, \; A_2 \in C^{(n-r)\times m}, \tag{2.4}$$

$$U^H B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad B_1 \in C^{r\times l}, \; B_2 \in C^{(n-r)\times l}. \tag{2.5}$$

Using GSVD [11] on matrix pair $[A_1^H, A_2^H]$, we obtain

$$A_1^H = W_A \Sigma_{1A} U_A^H, \quad A_2^H = W_A \Sigma_{2A} V_A^H, \tag{2.6}$$

where $W_A \in C_m^{m\times m}$ is a nonsingular matrix, $U_A \in C^{r\times r}$ and $V_A \in C^{(n-r)\times(n-r)}$ are unitary matrices, and

$$\Sigma_{1A} = \begin{array}{l} \begin{pmatrix} I_{1A} & & \\ & D_{1A} & \\ & & O_{1A} \\ & O & \end{pmatrix} \begin{array}{c} k_1 \\ s_1 \\ r-k_1-s_1 \\ m-r \end{array} \\ \begin{array}{ccc} \;\; k_1 & s_1 & r-k_1-s_1 \end{array} \end{array}, \quad \Sigma_{2A} = \begin{array}{l} \begin{pmatrix} O_{2A} & & \\ & D_{2A} & \\ & & I_{2A} \\ & O & \end{pmatrix} \begin{array}{c} k_1 \\ s_1 \\ t_1-k_1-s_1 \\ m-t_1 \end{array} \\ \begin{array}{ccc} n-r+k_1-t_1 & s_1 & t_1-k_1-s_1 \end{array} \end{array}, \tag{2.7}$$

where $t_1 = \mathrm{rank}(A_1^H, A_2^H)$, $k_1 = t_1 - \mathrm{rank}(A_2^H)$, $s_1 = \mathrm{rank}(A_1^H) + \mathrm{rank}(A_2^H) - t_1$, I_{1A} and I_{2A} are identity matrices, O_{1A}, O_{2A} and O are zero matrices, $D_{1A} = \mathrm{diag}(\lambda_{1A}, \lambda_{2A}, \ldots, \lambda_{s_1 A}) > 0$, $D_{2A} = \mathrm{diag}(\mu_{1A}, \mu_{2A}, \ldots, \mu_{s_1 A}) > 0$ with $1 > \lambda_{1A} \geqslant \lambda_{2A} \geqslant \cdots \geqslant \lambda_{s_1 A} > 0$, $0 < \mu_{1A} \leqslant \mu_{2A} \leqslant \cdots \leqslant \mu_{s_1 A} < 1$ and $\lambda_{iA}^2 + \mu_{iA}^2 = 1$, $(i = 1, 2, \ldots, s_1)$.

Similarly, the GSVD of matrix pair $[B_1^H, B_2^H]$ is

$$B_1^H = W_B \Sigma_{1B} U_B^H, \quad B_2^H = W_B \Sigma_{2B} V_B^H, \tag{2.8}$$

where $W_B \in C_l^{l\times l}$ is a nonsingular matrix, $U_B \in C^{r\times r}$ and $V_B \in C^{(n-r)\times(n-r)}$ are unitary matrices,

$$\Sigma_{1B} = \begin{array}{l} \begin{pmatrix} I_{1B} & & \\ & D_{1B} & \\ & & O_{1B} \\ & O & \end{pmatrix} \begin{array}{c} k_2 \\ s_2 \\ r-k_2-s_2 \\ l-r \end{array} \\ \begin{array}{ccc} \;\; k_2 & s_2 & r-k_2-s_2 \end{array} \end{array}, \quad \Sigma_{2B} = \begin{array}{l} \begin{pmatrix} O_{2B} & & \\ & D_{2B} & \\ & & I_{2B} \\ & O & \end{pmatrix} \begin{array}{c} k_2 \\ s_2 \\ t_2-k_2-s_2 \\ l-t_2 \end{array} \\ \begin{array}{ccc} n-r+k_2-t_2 & s_2 & t_2-k_2-s_2 \end{array} \end{array}, \tag{2.9}$$

where $t_2 = \mathrm{rank}(B_1^H, B_2^H)$, $k_2 = t_2 - \mathrm{rank}(B_2^H)$, $s_2 = \mathrm{rank}(B_1^H) + \mathrm{rank}(B_2^H) - t_2$, I_{1B} and I_{2B} are identity matrices, O_{1B}, O_{2B} and O are zero matrices. $D_{1B} = \mathrm{diag}(\lambda_{1B}, \lambda_{2B}, \ldots, \lambda_{s_2 B}) > 0$, $D_{2B} = \mathrm{diag}(\mu_{1B}, \mu_{2B}, \ldots, \mu_{s_2 B}) > 0$ with $1 > \lambda_{1B} \geqslant \lambda_{2B} \geqslant \cdots \geqslant \lambda_{s_2 B} > 0$, $0 < \mu_{1B} \leqslant \mu_{2B} \leqslant \cdots \leqslant \mu_{s_2 B} < 1$, and $\lambda_{iB}^2 + \mu_{iB}^2 = 1$, $(i = 1, 2, \ldots, s_2)$.

We now state the following results. Let

$$W_A^{-1} C W_B^{-H} = \begin{array}{cc} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix} & \begin{array}{c} k_1 \\ s_1 \\ t_1 - k_1 - s_1 \\ m - t_1 \end{array} \\ \begin{array}{cccc} k_2 & s_2 & t_2 - k_2 - s_2 & l - t_2 \end{array} & \end{array}. \tag{2.10}$$

Theorem 2.1. *Given $A \in C^{n\times m}$, $B \in C^{n\times l}$, $C \in C^{m\times l}$ in Problem I, $P \in HOC^{n\times n}$ which can be expressed as* (2.1). *Partition $U^H A$ and $U^H B$ as* (2.4) *and* (2.5). *The GSVD of matrix pairs $[A_1^H, A_2^H]$ and $[B_1^H, B_2^H]$ are as in* (2.6) *and* (2.8), *respectively, the partition form of $W_A^{-1} C W_B^{-H}$ is* (2.10). *Then Eq.* (1.1) *has a solution $X \in C_r^{n\times n}(P)$, if and only if*

$$C_{14} = O,\ C_{24} = O,\ C_{34} = O,\ C_{13} = O,\ C_{31} = O,\ C_{41} = O,\ C_{42} = O,\ C_{43} = O,\ C_{44} = O. \tag{2.11}$$

In that case, the general solution can be expressed as

$$X = U \begin{pmatrix} U_A \begin{pmatrix} C_{11} & C_{12} D_{1B}^{-1} & X_{13} \\ D_{1A}^{-1} C_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} U_B^H & O \\ O & V_A \begin{pmatrix} X_{44} & X_{45} & X_{46} \\ X_{54} & D_{2A}^{-1}(C_{22} - D_{1A} X_{22} D_{1B}) D_{2B}^{-1} & D_{2A}^{-1} C_{23} \\ X_{64} & C_{32} D_{2B}^{-1} & C_{33} \end{pmatrix} V_B^H \end{pmatrix} U^H, \tag{2.12}$$

where $X_{13} \in C^{k_1\times(r-k_2-s_2)}$, $X_{22} \in C^{s_1\times s_2}$, $X_{23} \in C^{s_1\times(r-k_2-s_2)}$, $X_{31} \in C^{(r-k_1-s_1)\times k_2}$, $X_{32} \in C^{(r-k_1-s_1)\times s_2}$, $X_{33} \in C^{(r-k_1-s_1)\times(r-k_2-s_2)}$, $X_{44} \in C^{(n-r+k_1-t_1)\times(n-r+k_2-t_2)}$, $X_{45} \in C^{(n-r+k_1-t_1)\times s_2}$, $X_{46} \in C^{(n-r+k_1-t_1)\times(t_2+k_2-s_2)}$, $X_{54} \in C^{s_1\times(n-r+k_2-t_2)}$, and $X_{64} \in C^{(t_1-k_1-s_1)\times(n-r+k_2-t_2)}$ are arbitrary matrices.

Proof. We first prove the necessity of (2.11). If Eq. (1.1) has a reflexive solution X, then X satisfies (2.2), so Eq. (1.1) is equivalent to

$$(U^H A)^H \begin{pmatrix} \bar{X}_{11} & O \\ O & \bar{X}_{22} \end{pmatrix} U^H B = C. \tag{2.13}$$

Substituting (2.4) and (2.5) into (2.13), we obtain

$$A_1^H \bar{X}_{11} B_1 + A_2^H \bar{X}_{22} B_2 = C. \tag{2.14}$$

Note that W_A and W_B are nonsingular matrices. From (2.6) and (2.8), we get

$$\Sigma_{1A}(U_A^H \bar{X}_{11} U_B)\Sigma_{1B}^H + \Sigma_{2A}(V_A^H \bar{X}_{22} V_B)\Sigma_{2B}^H = W_A^{-1} C W_B^{-H}. \tag{2.15}$$

Let

$$U_A^H \bar{X}_{11} U_B = \begin{array}{cc} \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} & \begin{array}{c} k_1 \\ s_1 \\ r - k_1 - s_1 \end{array} \\ \begin{array}{ccc} k_2 & s_2 & r - k_2 - s_2 \end{array} & \end{array} \tag{2.16}$$

and

$$V_A^H \bar{X}_{22} V_B = \begin{array}{cc} \begin{pmatrix} X_{44} & X_{45} & X_{46} \\ X_{54} & X_{55} & X_{56} \\ X_{64} & X_{65} & X_{66} \end{pmatrix} & \begin{array}{c} n - r + k_1 - t_1 \\ s_1 \\ t_1 - k_1 - s_1 \end{array} \\ \begin{array}{ccc} n-r+k_2-t_2 & s_2 & t_2 - k_2 - s_2 \end{array} & \end{array}. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we obtain

$$\begin{pmatrix} X_{11} & X_{12}D_{1B} & O & O \\ D_{1A}X_{21} & D_{1A}X_{22}D_{1B}+D_{2A}X_{55}D_{2B} & D_{2A}X_{56} & O \\ O & X_{65}D_{2B} & X_{66} & O \\ O & O & O & O \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}. \tag{2.18}$$

Therefore (2.18) holds if and only if (2.13) holds and $X_{11}=C_{11}$, $X_{12}=C_{12}D_{1B}^{-1}$, $X_{21}=D_{1A}^{-1}C_{21}$, $X_{56}=D_{2A}^{-1}C_{23}$, $X_{65}=C_{32}D_{2B}^{-1}$, $X_{66}=C_{33}$, $X_{55}=D_{2A}^{-1}(C_{22}-D_{1A}X_{22}D_{1B})D_{2B}^{-1}$. Substituting the above into (2.16), (2.17) and (2.2), thus we have formulation (2.12).

Now we prove the sufficiency of (2.11). Let

$$X_{11}^{(0)} = U_A \begin{pmatrix} C_{11} & C_{12}D_{1B}^{-1} & O \\ D_{1A}^{-1}C_{21} & O & O \\ O & O & O \end{pmatrix} U_B^H,$$

$$X_{22}^{(0)} = V_A \begin{pmatrix} O & O & O \\ O & D_{2A}^{-1}C_{22}D_{2B}^{-1} & D_{2A}^{-1}C_{23} \\ O & C_{32}D_{2B}^{-1} & C_{33} \end{pmatrix} V_B^H.$$

Obviously, $X_{11}^{(0)} \in C^{r\times r}$, $X_{22}^{(0)} \in C^{(n-r)\times(n-r)}$. By Lemma 2.2 and

$$X_0 = U \begin{pmatrix} X_{11}^{(0)} & O \\ O & X_{22}^{(0)} \end{pmatrix} U^H,$$

we have $X_0 \in C_r^{n\times n}(P)$. Hence

$$A^H X_0 B = A^H U \begin{pmatrix} X_{11}^{(0)} & O \\ O & X_{22}^{(0)} \end{pmatrix} U^H B = A_1^H X_{11}^{(0)} B_1 + A_2^H X_{22}^{(0)} B_2 = C,$$

thus X_0 is the reflection solution with respect to generalized reflection matrix P of Eq. (1.1). The proof is completed. □

Similarly, according to Lemma 2.3 and formulations (2.4)–(2.10), we conclude the followings.

Theorem 2.2. *Given $A \in C^{n\times m}$, $B \in C^{n\times l}$, $C \in C^{m\times l}$ in Problem I, $P \in HOC^{n\times n}$ which can be expressed as* (2.1). *Partition $U^H A$ and $U^H B$ as* (2.4) *and* (2.5). *The GSVD of matrix pairs $[A_1^H, A_2^H]$ and $[B_1^H, B_2^H]$ are as in* (2.6) *and* (2.8), *respectively. The partition form of $W_A^{-1} C W_B^{-H}$ is* (2.10). *Then Eq.* (1.1) *has a solution $X \in C_a^{n\times n}(P)$, if and only if*

$$C_{11}=O,\ C_{14}=O,\ C_{24}=O,\ C_{34}=O,\ C_{33}=O,\ C_{41}=O,\ C_{42}=O,\ C_{43}=O,\ C_{44}=O. \tag{2.19}$$

In that case the general solution can be expressed as

$$X = U \begin{pmatrix} O & U_A \begin{pmatrix} X_{14} & C_{12}D_{2B}^{-1} & C_{13} \\ X_{24} & X_{25} & D_{1A}^{-1}C_{23} \\ X_{34} & X_{35} & X_{36} \end{pmatrix} V_B^{\mathrm{T}} \\ V_A \begin{pmatrix} X_{41} & X_{42} & X_{43} \\ D_{2A}^{-1}C_{21} & D_{2A}^{-1}(C_{22}-D_{1A}X_{25}D_{2B})D_{1B}^{-1} & X_{53} \\ C_{31} & C_{32}D_{1B}^{-1} & X_{63} \end{pmatrix} U_B^{\mathrm{T}} & O \end{pmatrix} U^{\mathrm{T}}, \tag{2.20}$$

where $X_{14} \in C^{k_1\times(n-r+k_2-t_2)}$, $X_{24} \in C^{s_1\times(n-r+k_2-t_2)}$, $X_{34} \in C^{(r-k_1-s_1)\times(n-r+k_2-t_2)}$, $X_{35} \in C^{(r-k_1-s_1)\times s_2}$, $X_{36} \in C^{(r-k_1-s_1)\times(t_2-k_2-s_2)}$, $X_{41} \in C^{(n-r+k_1-t_2)\times k_2}$, $X_{42}\in C^{(n-r+k_1-t_1)\times s_2}$, $X_{43}\in C^{(n-r+k_1-t_1)\times(r-k_2-s_2)}$, $X_{53}\in C^{s_1\times(r-k_2-s_2)}$, $X_{63} \in C^{(t_1-k_1-s_1)\times(r-k_2-s_2)}$ and $X_{25} \in C^{s_1\times s_2}$ are arbitrary matrices.

3. The solutions of Problem II

In this section, we first introduce a lemma, then get the general expression of the solutions for Problem II. Finally, we deduce the minimum norm solution X with $\min_X \|X\|$ subjected to $A^H XB = C$.

Lemma 3.1. *Given $E \in C^{m\times n}$, $F \in C^{m\times n}$, $D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) > 0$, $D_2 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n) > 0$. Let $h(G) = \|G - E\|^2 + \|D_1 G D_2 - F\|^2$. Then there exists an unique matrix $\hat{G} \in C^{m\times n}$ satisfy $h(\hat{G}) = \min_{G\in C^{m\times n}} h(G)$. Here*

$$\hat{G} = \Phi * (E + D_1 F D_2), \tag{3.1}$$

where $\Phi = (\varphi_{ij}) \in C^{m\times n}$, $\varphi_{ij} = 1/(1 + \lambda_i^2\mu_j^2)$, $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

Proof. Assume $G = (g_{ij}) \in C^{m\times n}$. Given $E = (e_{ij}) \in C^{m\times n}$, $F = (f_{ij}) \in C^{m\times n}$. We have

$$h(G) = \sum_{i,j=1}^{n} ((g_{ij} - e_{ij})^2 + (\lambda_i\mu_j g_{ij} - f_{ij})^2). \tag{3.2}$$

In Eq. (3.2), $h(G)$ is a continuously differentiable function of $m \times n$ variables g_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. According to the necessary condition of function which is minimizing at a point, we obtain the following expression

$$g_{ij} = \frac{1}{1 + \lambda_i^2\mu_j^2}(e_{ij} + \lambda_i f_{ij}\mu_j). \tag{3.3}$$

Then, Eq. (3.1) is obtained from Eq. (3.3). The proof is completed. $\square$

Theorem 3.1. *Suppose that the matrices A, B and C are those given in Problem I. Assume that the solution set $S_{\mathrm{rE}} \subseteq C_r^{n\times n}(P)$ of $A^H XB = C$ is nonempty and that $X^* \in C^{n\times n}$ is given. Let*

$$U^H X^* U = \begin{array}{cc} \begin{pmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{pmatrix} & \begin{array}{c} r \\ n-r \end{array} \end{array}, \tag{3.4}$$

$$U_A^H K_{11}^* U_B = \begin{array}{cc} \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix} & \begin{array}{c} k_1 \\ s_1 \\ r-k_1-s_1 \end{array} \\ \begin{array}{ccc} k_2 & s_2 & r-k_2-s_2 \end{array} & \end{array}, \tag{3.5}$$

$$V_A^H K_{22}^* V_B = \begin{array}{cc} \begin{pmatrix} X_{44}^* & X_{45}^* & X_{46}^* \\ X_{54}^* & X_{55}^* & X_{56}^* \\ X_{64}^* & X_{65}^* & X_{66}^* \end{pmatrix} & \begin{array}{c} n-r+k_1-t_1 \\ s_1 \\ t_1-k_1-s_1 \end{array} \\ \begin{array}{ccc} n-r+k_2-t_2 & s_2 & t_2-k_2-s_2 \end{array} & \end{array} \tag{3.6}$$

then $\|\hat{X}-X^*\|=\min_{X\in S_{\mathrm{rE}}}\|X-X^*\|$ *has an unique solution* $\hat{X}\in S_{\mathrm{rE}}$ *which can be expressed as*

$$\hat{X}=U\begin{pmatrix} U_A\begin{pmatrix} C_{11} & C_{12}D_{1B}^{-1} & X_{13}^* \\ D_{1A}^{-1}C_{21} & \hat{X}_{22} & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix}U_B^H & O \\ O & V_A\begin{pmatrix} X_{44}^* & X_{45}^* & X_{46}^* \\ X_{54}^* & D_{2A}^{-1}(C_{22}-D_{1A}\hat{X}_{22}D_{1B})D_{2B}^{-1} & D_{2A}^{-1}C_{23} \\ X_{64}^* & C_{32}D_{2B}^{-1} & C_{33} \end{pmatrix}V_B^H \end{pmatrix}U^H \tag{3.7}$$

and

$$\hat{X}_{22}=\Psi*(D_{1A}C_{22}D_{1B}+D_{2A}^2X_{22}^*D_{2B}^2-D_{1A}D_{2A}X_{55}^*D_{1B}D_{2B}), \tag{3.8}$$

where $\Psi=(\psi_{ij})\in C^{s_1\times s_2}$, $\psi_{ij}=1/(\lambda_{iA}^2\lambda_{jB}^2+\mu_{iA}^2\mu_{jB}^2)$, $(i=1,2,\ldots,s_1;\ j=1,2,\ldots,s_2)$.

Proof. Using the invariance of the Frobenius norm under the unitary transformations, from (2.12), (3.5) and (3.6), we have that $\|\hat{X}-X^*\|=\min_{X\in S_{\mathrm{rE}}}\|X-X^*\|$ is equivalent to

$$\begin{aligned}
&\|\hat{X}_{13}-X_{13}^*\|^2=\min\|X_{13}-X_{13}^*\|^2, \quad \|\hat{X}_{31}-X_{31}^*\|^2=\min\|X_{31}-X_{31}^*\|^2,\\
&\|\hat{X}_{32}-X_{32}^*\|^2=\min\|X_{32}-X_{32}^*\|^2, \quad \|\hat{X}_{23}-X_{23}^*\|^2=\min\|X_{23}-X_{23}^*\|^2,\\
&\|\hat{X}_{33}-X_{33}^*\|^2=\min\|X_{33}-X_{33}^*\|^2, \quad \|\hat{X}_{44}-X_{44}^*\|^2=\min\|X_{44}-X_{44}^*\|^2,\\
&\|\hat{X}_{45}-X_{45}^*\|^2=\min\|X_{45}-X_{45}^*\|^2, \quad \|\hat{X}_{46}-X_{46}^*\|^2=\min\|X_{46}-X_{46}^*\|^2,\\
&\|\hat{X}_{54}-X_{54}^*\|^2=\min\|X_{54}-X_{54}^*\|^2, \quad \|\hat{X}_{64}-X_{64}^*\|^2=\min\|X_{64}-X_{64}^*\|^2
\end{aligned} \tag{3.9}$$

and

$$h(\hat{X}_{22})=\min_{X_{22}\in C^{s_1\times s_2}}h(X_{22}), \tag{3.10}$$

where $h(X_{22})=\|D_{2A}^{-1}(C_{22}-D_{1A}\hat{X}_{22}D_{1B})D_{2B}^{-1}-X_{55}^*\|^2+\|\hat{X}_{22}-X_{22}^*\|^2$. From (3.9), we have

$$\hat{X}_{13}=X_{13}^*,\ \hat{X}_{31}=X_{31}^*,\ \hat{X}_{32}=X_{32}^*,\ \hat{X}_{33}=X_{33}^*,\ \hat{X}_{23}=X_{23}^*,$$

$$\hat{X}_{44}=X_{44}^*,\ \hat{X}_{45}=X_{45}^*,\ \hat{X}_{46}=X_{46}^*,\ \hat{X}_{54}=X_{54}^*,\ \hat{X}_{64}=X_{64}^*.$$

From (3.10), we have

$$h(X_{22})=\|D_{2A}^{-1}D_{1A}\hat{X}_{22}D_{1B}D_{2B}^{-1}-(D_{2A}^{-1}C_{22}D_{2B}^{-1}-X_{55}^*)\|^2+\|\hat{X}_{22}-X_{22}^*\|^2. \tag{3.11}$$

From Lemma 3.1, (3.10) and (3.11), (3.8) holds. Thus, (3.7) and (3.8) hold. □

When X^* is a zero matrix in Theorem 4, we can straightforward deduce

Corollary 3.1. *The expression of minimum norm solution* $\hat{X}_{\min}\in S_{\mathrm{rE}}\subseteq C_r^{n\times n}(P)$ *which satisfies* $\|\hat{X}_{\min}\|=\min_{X\in S_{\mathrm{rE}}}\|X\|$ $(A^H\hat{X}_{\min}B=C)$ *is*

$$\hat{X}_{\min}=U\begin{pmatrix} U_A\begin{pmatrix} C_{11} & C_{12}D_{1B}^{-1} & O \\ D_{1A}^{-1}C_{21} & \hat{X}_{22} & O \\ O & O & O \end{pmatrix}U_B^H & O \\ O & V_A\begin{pmatrix} O & O & O \\ O & D_{2A}^{-1}(C_{22}-D_{1A}\hat{X}_{22}D_{1B})D_{2B}^{-1} & D_{2A}^{-1}C_{23} \\ O & C_{32}D_{2B}^{-1} & C_{33} \end{pmatrix}V_B^H \end{pmatrix}U^H,$$

where $\hat{X}_{22}=\Psi^*(D_{1A}C_{22}D_{1B})$.

In a similar way, we can deduce the following.

Theorem 3.2. *Given $X^* \in C^{n\times n}$, let assume that the solution set $S_{\mathrm{aE}} \subseteq C_a^{n\times n}(P)$ of $A^H X B = C$ be nonempty. From* (3.4), *let*

$$U_A^{\mathrm{T}} K_{12}^* V_B = \begin{array}{c} \begin{pmatrix} X_{14}^* & X_{15}^* & X_{16}^* \\ X_{24}^* & X_{25}^* & X_{26}^* \\ X_{34}^* & X_{35}^* & X_{36}^* \end{pmatrix} \\ \begin{array}{ccc} n-r+k_2-t_2 & s_2 & t-k_2-s_2 \end{array} \end{array} \begin{array}{c} k_1 \\ s_1 \\ t_2-k_1-s_1 \\ \end{array}, \tag{3.12}$$

$$V_A^{H} K_{22}^* U_B = \begin{array}{c} \begin{pmatrix} X_{41}^* & X_{42}^* & X_{43}^* \\ X_{51}^* & X_{52}^* & X_{53}^* \\ X_{61}^* & X_{62}^* & X_{63}^* \end{pmatrix} \\ \begin{array}{ccc} k_2 & s_2 & r-k_2-s_2 \end{array} \end{array} \begin{array}{c} n-r+k_1-t_1 \\ s_1 \\ t_1-k_1-s_1 \\ \end{array} \tag{3.13}$$

then $\|X - X^*\| = \min$ *has an unique solution* $\hat{X} \in S_{\mathrm{aE}}$ *which can be expressed as*

$$\hat{X}_A = U \begin{pmatrix} O & U_A \begin{pmatrix} X_{14}^* & C_{12}D_{2B}^{-1} & C_{13} \\ X_{24}^* & \hat{X}_{25} & D_{1A}^{-1}C_{23} \\ X_{34}^* & X_{35}^* & X_{36}^* \end{pmatrix} V_B^H \\ V_A \begin{pmatrix} X_{41}^* & X_{42}^* & X_{43}^* \\ D_{2A}^{-1}C_{21} & D_{2A}^{-1}(C_{22} - D_{1A}\hat{X}_{25}D_{12B})D_{1B}^{-1} & X_{53}^* \\ C_{31} & C_{32}D_{1B}^{-1} & X_{63}^* \end{pmatrix} U_B^H & O \end{pmatrix} U^H \tag{3.14}$$

where $\hat{X}_{25} = \Omega * (D_{1A}C_{22}D_{2B} + D_{2A}^2 X_{25}^* D_{1B}^2 - D_{1A}D_{2A}X_{52}^* D_{1B}D_{2B})$, $\Omega = (\omega_{ij}) \in R^{s_1\times s_2}$, $\omega_{ij} = 1/(\lambda_{iA}^2\mu_{jB}^2 + \mu_{iA}^2\lambda_{jB}^2)$, $\lambda_{iA}^2 + \mu_{iA}^2 = 1$, $\lambda_{jB}^2 + \mu_{jB}^2 = 1$, $(i = 1, 2, \ldots, s_1;\ j = 1, 2, \ldots, s_2)$.

If X^* is a zero matrix in Theorem 3.2, we can straightforward obtain the following.

Corollary 3.2. *The expression of minimum norm solution* $\hat{X}_{\min} \in S_{\mathrm{aE}} \subseteq C_a^{n\times n}(P)$ *which satisfies* $\|\hat{X}_{\min}\| = \min_{X\in S_{\mathrm{aE}}} \|X\|$ $(A^H \hat{X}_{\min} B = C)$ *is*

$$\hat{X}_{A\,\min} = U \begin{pmatrix} O & U_A \begin{pmatrix} O & C_{12}D_{2B}^{-1} & C_{13} \\ O & \hat{X}_{25} & D_{1A}^{-1}C_{23} \\ O & O & O \end{pmatrix} V_B^H \\ V_A \begin{pmatrix} O & O & O \\ D_{2A}^{-1}C_{21} & D_{2A}^{-1}(C_{22} - D_{1A}\hat{X}_{25}D_{2B})D_{1B}^{-1} & O \\ C_{31} & C_{32}D_{1B}^{-1} & O \end{pmatrix} U_B^H & O \end{pmatrix} U^H$$

where $\hat{X}_{25} = \Omega * (D_{1A}C_{22}D_{2B})$, $\Omega = (\omega_{ij}) \in R^{s_1\times s_2}$, $\omega_{ij} = 1/(\lambda_{iA}^2\mu_{jB}^2 + \mu_{iA}^2\lambda_{jB}^2)$, $\lambda_{iA}^2 + \mu_{iA}^2 = 1$, $\lambda_{jB}^2 + \mu_{jB}^2 = 1$, $(i = 1, 2, \ldots, s_1;\ j = 1, 2, \ldots, s_2)$.

Based on Theorems 3.1 and 3.2, we propose the following algorithm for solving Problem II over sets S_{rE} or S_{aE}.

Algorithm.

1. Input $X^* \in C^{n\times n}$ and a generalized reflection matrix P of size n.
2. Compute r and U by (2.1).
3. Compute A_1, A_2, B_1 and B_2 by (2.4) and (2.5).
4. Compute W_A, Σ_{1A}, Σ_{2A}, U_A and V_A by (2.6).
5. Compute W_B, Σ_{1B}, Σ_{2B}, U_B and V_B by (2.8).
6. Compute X_{ij}^* $(i, j = 1, 2, \ldots, 6)$ by (3.4), (3.5), (3.6), (3.12), and (3.13).

7. Compute partition matrix $(C_{ij})_{4\times 4}$ by (2.10).
8. If $C_{i4}=0$ $(i=1,2,3,4)$ and $C_{4j}=0$ $(j=1,2,3)$ and $C_{13}=0$ and $C_{31}=0$ then go to step 9, else go to step 11.
9. Compute $\hat{X}\in S_{\mathrm{rE}}$ by (3.7) and $\hat{X}_{\min}\in S_{\mathrm{rE}}$ by Corollary 3.1.
10. Go to step 13.
11. If $C_{i4}=0(i=1,2,3,4)$ and $C_{4j}=0(j=1,2,3)$ and $C_{11}=0$ and $C_{33}=0$ then go to step 12, else go to step 13.
12. Compute $\hat{X}_A\in S_{\mathrm{aE}}$ by (3.14) and $\hat{X}_{A\min}\in S_{\mathrm{aE}}$ by Corollary 3.2.
13. Stop.

4. Numerical experiments

In this section, we will give a numerical example to illustrate our results. All the tests are performed by MATLAB6.5.

Example. The matrices P, A, B and X^* are given by following

$$P=\begin{pmatrix}
0.7025 & 0.3848 & 0.3132 & -0.3461 & -0.0204 & 0.0191 & -0.1502 & 0.2816 & -0.1910 & -0.0377\\
0.3848 & -0.2888 & -0.0096 & 0.3600 & -0.0409 & 0.1615 & 0.2182 & 0.2025 & 0.4281 & -0.5825\\
0.3132 & -0.0096 & 0.1648 & 0.0965 & 0.6190 & -0.0368 & 0.3884 & -0.5449 & 0.0136 & 0.1811\\
-0.3461 & 0.3600 & 0.0965 & -0.2728 & 0.5305 & -0.0155 & -0.1560 & 0.2062 & 0.5176 & -0.2250\\
-0.0204 & -0.0409 & 0.6190 & 0.5305 & -0.0408 & -0.1254 & -0.4564 & -0.0001 & 0.1944 & 0.2643\\
0.0191 & 0.1615 & -0.0368 & -0.0155 & -0.1254 & 0.9019 & -0.1135 & -0.2150 & 0.1980 & 0.2108\\
-0.1502 & 0.2182 & 0.3884 & -0.1560 & -0.4564 & -0.1135 & 0.6361 & 0.0051 & 0.3189 & 0.1649\\
0.2816 & 0.2025 & -0.5449 & 0.2062 & -0.0001 & -0.2150 & 0.0051 & 0.2353 & 0.3917 & 0.5341\\
-0.1910 & 0.4281 & 0.0136 & 0.5176 & 0.1944 & 0.1980 & 0.3189 & 0.3917 & -0.4231 & -0.0331\\
-0.0377 & -0.5825 & 0.1811 & -0.2250 & 0.2643 & 0.2108 & 0.1649 & 0.5341 & -0.0331 & 0.3849
\end{pmatrix},$$

$$A=\begin{pmatrix}
0.0312 & 0.0213 & 0.0571 & 0.0611 & 0.1344 & 0.1533 & 0.3473 & -0.3049\\
0.7876 & 0.9700 & 0.5988 & 0.3034 & 0.9823 & 0.9353 & 0.9274 & 0.6287\\
0.7229 & 0.7964 & 0.3359 & 0.2332 & 0.4507 & 0.4373 & 0.4613 & 0.6745\\
-0.0902 & -0.3433 & 0.0286 & -0.2658 & -0.1277 & 0.1147 & 0.0794 & -0.2484\\
0.7497 & 0.5493 & -0.0716 & -0.0361 & -0.1012 & 0.0533 & 0.0843 & 0.6990\\
-0.0426 & -0.0124 & -0.2885 & -0.0174 & -0.0008 & -0.0734 & 0.0162 & -0.1526\\
-0.0537 & 0.1343 & 0.6114 & 0.2362 & 0.4917 & 0.4328 & 0.4157 & -0.0893\\
0.4875 & 0.4862 & 0.4795 & 0.3056 & 0.0585 & 0.1130 & 0.3284 & 0.2782\\
0.7200 & 0.7496 & 0.3310 & 0.2220 & 0.2856 & 0.3032 & 0.3298 & 0.7120\\
0.2761 & 0.0334 & -0.2441 & -0.1824 & -0.1526 & 0.0240 & 0.1396 & -0.0024
\end{pmatrix},$$

$$B=\begin{pmatrix}
-0.0471 & 0.1624 & 0.2124 & 0.2108 & 0.3130 & 0.4500 & 0.6496\\
0.1144 & -0.2196 & 0.1297 & 0.0493 & -0.2194 & -0.0458 & -0.2282\\
0.2974 & 0.2981 & 0.2826 & 0.6621 & 0.5358 & 0.2719 & 0.5576\\
0.2612 & 0.0155 & 0.5931 & 0.0949 & -0.2127 & 0.1515 & 0.1661\\
0.3111 & 0.1312 & -0.2338 & 0.4387 & 0.2230 & -0.4279 & 0.0378\\
-0.2621 & -0.0076 & -0.4431 & -0.1140 & 0.1641 & -0.0339 & 0.0493\\
-0.0052 & -0.1073 & 0.3483 & -0.2266 & -0.3126 & 0.1489 & -0.1114\\
0.2240 & 0.1548 & 0.3010 & 0.3178 & 0.1408 & 0.1248 & 0.4628\\
0.5284 & 0.6772 & 0.6481 & 1.2460 & 1.1101 & 0.6787 & 1.2260\\
0.1323 & -0.0094 & 0.0803 & 0.2237 & 0.0479 & 0.0086 & 0.2303
\end{pmatrix},$$

$$X^* = \begin{pmatrix} 0.4486 & 0.2248 & 0.6724 & 0.2319 & 0.8133 & 0.8995 & 0.3974 & 0.6376 & 0.1633 & 0.9544 \\ 0.5244 & 0.9089 & 0.9383 & 0.4787 & 0.9238 & 0.6928 & 0.3333 & 0.2513 & 0.2110 & 0.1311 \\ 0.1715 & 0.0073 & 0.3431 & 0.5265 & 0.1990 & 0.4397 & 0.9442 & 0.1443 & 0.2168 & 0.0683 \\ 0.1307 & 0.5887 & 0.5630 & 0.7927 & 0.6743 & 0.7010 & 0.8386 & 0.6516 & 0.6518 & 0.1252 \\ 0.2188 & 0.5421 & 0.1189 & 0.1930 & 0.9271 & 0.6097 & 0.2584 & 0.9461 & 0.0528 & 0.1662 \\ 0.1055 & 0.6535 & 0.1690 & 0.9096 & 0.3438 & 0.2999 & 0.0429 & 0.8159 & 0.2293 & 0.9114 \\ 0.1414 & 0.3134 & 0.2789 & 0.9222 & 0.5945 & 0.8560 & 0.0059 & 0.9302 & 0.6674 & 0.1363 \\ 0.4570 & 0.2312 & 0.5568 & 0.0133 & 0.6155 & 0.1121 & 0.5744 & 0.3099 & 0.3109 & 0.6170 \\ 0.7881 & 0.4161 & 0.4856 & 0.7675 & 0.0034 & 0.2916 & 0.7439 & 0.2688 & 0.3066 & 0.2690 \\ 0.2811 & 0.2988 & 0.9522 & 0.9473 & 0.9820 & 0.0974 & 0.8068 & 0.5365 & 0.7207 & 0.2207 \end{pmatrix}.$$

If

$$C = \begin{pmatrix} 2.0461 & 2.0657 & 2.5944 & 4.5493 & 3.6081 & 2.4642 & 4.3491 \\ 2.1421 & 2.3033 & 2.6053 & 4.8920 & 4.0182 & 2.5875 & 4.6500 \\ 1.0014 & 1.2829 & 0.9332 & 2.5802 & 2.3282 & 1.1873 & 2.4239 \\ 0.5310 & 0.7236 & 0.6122 & 1.3912 & 1.2802 & 0.7601 & 1.3764 \\ 1.3838 & 1.5807 & 1.3866 & 3.2077 & 2.7395 & 1.4368 & 2.9053 \\ 1.4674 & 1.5807 & 1.5218 & 3.3285 & 2.7559 & 1.5077 & 3.0260 \\ 1.5336 & 1.6630 & 1.7668 & 3.4955 & 2.8836 & 1.7529 & 3.3107 \\ 1.7402 & 1.7919 & 2.0615 & 3.9187 & 3.1619 & 2.0045 & 3.6362 \end{pmatrix}$$

then partition matrix

$$(C_{ij}) = \left(\begin{array}{cc|cc|c|cc} 0.5275 & 0.1453 & 0.9636 & 0.3770 & -0.0000 & 0.0000 & 0.0000 \\ 0.5456 & 0.1715 & 0.1205 & 0.9073 & -0.0000 & 0.0000 & 0.0000 \\ \hline 0.2843 & 0.0680 & 0.0483 & 0.6702 & 0.5624 & 0.0000 & 0.0000 \\ 0.3708 & 0.8240 & 0.3802 & 0.9618 & 0.3723 & 0.0000 & -0.0000 \\ \hline 0.0000 & 0.0000 & 0.3683 & 0.5159 & 0.3771 & -0.0000 & -0.0000 \\ \hline 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \end{array}\right)$$

satisfies (2.11). Thus reflexive solution set S_{rE} of Eq. (1.1) is nonempty. The optimal approximation reflexive solution $\hat{X}$ to X^* in reflexive solution set S_{rE} is

$$\hat{X} = \begin{pmatrix} -0.1708 & 0.2920 & 0.0289 & -0.1515 & -0.2379 & 0.2882 & 0.7576 & -0.2009 & -0.6976 & 0.3899 \\ 0.0811 & -0.0118 & 0.7399 & -0.1921 & 0.2705 & 0.5920 & 0.1341 & -0.0691 & 1.2106 & -0.3242 \\ 0.0520 & -0.2825 & 0.6040 & 0.7229 & 0.1570 & 0.4963 & 0.5059 & -0.0689 & 1.5102 & -0.2747 \\ 0.2966 & 0.3405 & 0.5306 & 0.7151 & 0.7005 & 0.7371 & 0.6479 & 0.1230 & 0.0201 & -0.3001 \\ -0.0853 & 0.9973 & 0.5801 & 0.0917 & 0.4351 & 0.4240 & 0.6128 & 0.0722 & 0.1184 & 0.1386 \\ -0.1579 & 0.3183 & -0.0876 & 0.5205 & 0.1806 & -0.2893 & 0.6583 & 0.8600 & 0.6749 & 0.1466 \\ -0.2682 & 0.7709 & 0.4796 & 0.7188 & 0.8461 & 0.7430 & 0.2117 & 0.0721 & 0.2193 & -0.1238 \\ 0.6274 & -0.5459 & 0.4828 & -0.2074 & -0.1500 & 0.4530 & 0.4431 & -0.0958 & 0.5936 & -0.0006 \\ 0.1384 & 1.3239 & 0.6012 & 0.9141 & 0.2374 & 0.5154 & 0.3535 & -0.3351 & 0.2126 & 0.5733 \\ 0.0429 & 0.1902 & 0.1417 & -0.0961 & -0.0170 & -0.0607 & 0.5496 & 0.4133 & 0.2812 & 0.3036 \end{pmatrix}$$

and $\|\hat{X} - X^*\| = 2.9904$. The minimum norm reflexive solution is

$$\hat{X}_{\min}=\begin{pmatrix} 0.1673 & 0.6439 & -0.0841 & 0.2848 & 0.2147 & -0.0105 & 0.5614 & 0.2590 & -0.4956 & 0.0683 \\ 0.1960 & 0.0051 & 0.8171 & -0.1050 & 0.0922 & 0.1121 & -0.2080 & -0.1977 & 1.4038 & -0.1616 \\ 0.3042 & -0.1941 & 0.7475 & 0.3689 & 0.2617 & -0.0028 & 0.1740 & 0.1159 & 1.2573 & -0.2991 \\ 0.3218 & 0.2854 & 0.4500 & 0.5746 & 0.8792 & 0.5070 & 0.5815 & 0.0821 & -0.1511 & -0.3704 \\ 0.2888 & 0.8144 & 0.5136 & 0.4827 & 0.3553 & 0.1350 & 0.4244 & 0.0368 & 0.1864 & 0.0689 \\ -0.1168 & -0.1297 & -0.2882 & 0.1199 & -0.0936 & -0.6149 & 0.2840 & 0.5223 & 0.1346 & 0.0442 \\ 0.0739 & 0.2998 & 0.4193 & 0.4259 & 0.5921 & 0.4893 & 0.0809 & -0.2920 & -0.2414 & -0.1159 \\ 0.6910 & -0.3018 & 0.4377 & -0.0662 & -0.2350 & 0.5587 & 0.1815 & -0.1068 & 0.6833 & -0.3950 \\ 0.1331 & 1.2530 & 0.5441 & 0.5359 & 0.4301 & 0.3190 & 0.2208 & -0.3048 & -0.2984 & 0.3023 \\ 0.2702 & 0.3176 & -0.0405 & -0.3287 & -0.2526 & 0.0381 & 0.2109 & 0.1221 & 0.1376 & 0.0175 \end{pmatrix}$$

and $\|\hat{X}_{\min}\| = 2.7622$. If

$$C=\begin{pmatrix} 1.9249 & 2.2466 & 2.6371 & 4.4983 & 3.8222 & 2.6937 & 4.4531 \\ 2.1878 & 2.4620 & 2.7363 & 5.0205 & 4.2071 & 2.7376 & 4.8245 \\ 0.9658 & 1.3490 & 0.9534 & 2.5723 & 2.4072 & 1.2683 & 2.4682 \\ 0.6565 & 0.7580 & 0.7117 & -2.5pt507 & 1.3219 & 0.7537 & 1.4773 \\ 1.4154 & 1.6099 & 1.4247 & 3.2581 & 2.7746 & 1.4566 & 2.9506 \\ 1.3398 & 1.6372 & 1.4794 & 3.2108 & 2.8225 & 1.6095 & 3.0101 \\ 1.3847 & 1.7388 & 1.7244 & 3.3622 & 2.9726 & 1.8822 & 3.3002 \\ 1.7765 & 1.9458 & 2.1828 & 4.0348 & 3.3455 & 2.1526 & 3.8021 \end{pmatrix}$$

then partition matrix

$$(C_{ij})=\left(\begin{array}{cc|cc|c|cc} 0.0000 & 0.0000 & 0.9636 & 0.3770 & 0.5275 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.1205 & 0.9073 & 0.5456 & 0.0000 & 0.0000 \\ \hline 0.2843 & 0.0680 & 0.0483 & 0.6702 & 0.5624 & 0.0000 & 0.0000 \\ 0.3708 & 0.8240 & 0.3802 & 0.9618 & 0.3723 & 0.0000 & -0.0000 \\ \hline 0.5456 & 0.3456 & 0.3683 & 0.5159 & -0.0000 & 0.0000 & 0.0000 \\ \hline 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \end{array}\right)$$

satisfies (2.19). Thus anti-reflexive solution set S_{aE} of Eq. (1.1) is nonempty. The optimal approximation anti-reflexive solution $\hat{X}_A$ to X^* in anti-reflexive solution set S_{aE} is

$$\hat{X}_A=\begin{pmatrix} -0.2228 & -0.3041 & -0.2794 & -0.2526 & -0.7881 & -0.0434 & -0.2659 & -0.1360 & 0.1599 & -0.0109 \\ 0.1856 & 0.4455 & 0.9399 & 0.2493 & 0.5279 & -0.3095 & -0.2856 & 0.0975 & 1.0299 & 0.1418 \\ 0.2648 & -0.1941 & 0.8154 & -0.1789 & -0.1013 & -0.1413 & -0.0060 & -0.0813 & 0.8646 & 0.0777 \\ -0.4910 & -0.3821 & -0.5514 & -0.7453 & -1.0112 & 0.0011 & -0.5471 & -0.0343 & 0.2643 & 0.4753 \\ 0.0834 & 0.0277 & 0.1012 & 0.1306 & -0.3428 & 0.0075 & -0.7536 & -0.0034 & 0.7128 & 0.1386 \\ 0.0567 & 0.0202 & 0.0401 & 0.3770 & -0.4286 & 0.0479 & -0.2424 & -0.0327 & -0.1406 & 0.3348 \\ -0.1241 & -0.4756 & -0.2349 & -0.3844 & -0.2898 & 0.1759 & -0.4834 & 0.2520 & 0.4131 & -0.0871 \\ 0.3378 & -0.0026 & 0.3715 & 0.3768 & -0.2456 & 0.0531 & -0.2810 & -0.1013 & 0.3051 & 0.5273 \\ -0.0537 & 0.1922 & 0.4803 & -0.2657 & -0.7664 & -0.4569 & 0.1762 & -0.6476 & 0.5416 & -0.4264 \\ 0.1150 & -0.2293 & 0.1604 & 0.8032 & -0.4078 & -0.2192 & 0.0831 & -0.1569 & 0.0483 & 0.0451 \end{pmatrix}$$

and $\|\hat{X}_A - X^*\| = 5.9564$. The minimum norm anti-reflexive solution is

$$\hat{X}_{A,\min}=\begin{pmatrix} -0.0707 & -0.3454 & -0.1580 & -0.0781 & -0.8335 & -0.1439 & -0.3267 & -0.3274 & 0.0705 & 0.0832 \\ 0.0673 & 0.2762 & 0.6802 & 0.2379 & 0.3752 & -0.2639 & -0.0438 & 0.2056 & 1.1734 & 0.1365 \\ 0.2265 & 0.1405 & 0.5150 & -0.0733 & -0.0545 & -0.0360 & -0.3525 & 0.0412 & 1.0014 & 0.3120 \\ -0.1431 & -0.5272 & -0.4346 & -0.7876 & -0.8073 & -0.2264 & -0.7265 & -0.0656 & 0.2076 & 0.3327 \\ -0.0134 & 0.0300 & 0.2985 & -0.0117 & -0.5940 & -0.1441 & -0.2704 & -0.3696 & 0.6478 & 0.0730 \\ 0.0814 & -0.0289 & 0.0067 & 0.0971 & -0.2396 & 0.0157 & -0.1335 & -0.0694 & -0.0130 & 0.1601 \\ -0.1246 & -0.2452 & -0.1324 & -0.7302 & -0.2051 & -0.0413 & -0.2752 & -0.0618 & 0.3525 & -0.1074 \\ 0.3693 & -0.0563 & 0.3068 & 0.4228 & -0.2597 & 0.1068 & -0.2187 & -0.0046 & 0.3368 & 0.3418 \\ -0.1539 & 0.0651 & 0.3342 & -0.2090 & -0.4602 & -0.1223 & -0.0662 & -0.4914 & 0.7254 & -0.2983 \\ 0.0031 & -0.0420 & 0.0923 & 0.6476 & -0.3285 & -0.2364 & -0.0146 & -0.0684 & 0.0080 & 0.1999 \end{pmatrix}$$

and $\|\hat{X}_{A,\min}\| = 2.3220$.

5. Conclusions

In this paper, we obtained the equation's solution sets of reflexive and anti-reflexive matrices with respect to a given generalized reflection matrix P. We also obtained, in corresponding solution set, the nearest solution to a given matrix in Frobenius norm. The solvability conditions and the explicit formula for the solution are given. Given matrix, we have obtained some of the nearest matrices to the matrix and minimum norm matrices in the given solution set by computing.

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