Numerical study of heat convective mass transfer in a fully developed laminar flow with constant wall temperature

CrossMark

A. Belhocine [a,*], W.Z. Wan Omar [b,c]

[a] Faculty of Mechanical Engineering, University of Sciences and the Technology of Oran, L.P 1505 El–MNAOUER, USTO 31000 ORAN Algeria
[b] Faculty of Mechanical Engineering
[c] Centre of Electrical Energy Systems, Universiti Teknologi Malaysia, 81310 UTM Johor, Malaysia

ARTICLE INFO

ABSTRACT

This numerical study is aimed at investigating the convective heat transfer and flow fluid inside a horizontal circular tube in the fully-developed laminar flow regime under the constant wall temperature boundary condition,, is commonly called the Graetz Problem that our goal is to get the steady temperature distribution in the fluid. The complexity of the partial differential equation that describes the temperature field with the associated linear or nonlinear boundary conditions is simplified by means of numerical methods using current computational tools. The simplified energy equation is solved numerically by the orthogonal collocation method followed by the finite difference method (Crank–Nicholson method).The calculations were effected through a FORTRAN computer program and the results show that orthogonal collocation method giving better results than Crank–Nicholson method.

1. Introduction

The Graetz problem is known as the heat transfer problem of laminar fluid flow in ducts. It has many applications and has been studied extensively since Graetz (1885). The classical Graetz problem considers the forced convection heat transfer of fluids flowing in ducts while neglecting axial conduction in the fluid as cited in [1].

The mathematical and approximations solutions of the Graetz problem have been obtained in numerous previous contributions. Braga et al. [2] considered an extended version of a Graetz problem with axial diffusion in an infinite domain. Two heating variations were studied, firstly step change in wall temperatures, and step change in wall heat flux. The solution is expressed in terms of an eigenseries expansion using orthogonal Sturm–Liouville eigenfunction expansion of the sought solutions. The resulting problem was then solved analytically to yield the eigenvalues and eigenvectors.

Hsu [3] studied the Graetz problem with axial diffusion in circular tube, using a semi- infinite domain formulation with a specified inlet condition, while Michelsen and Villadsen [4] analyzed the effects of axial diffusion in a infinite domain formed by an insulated preparation region followed by an isothermal wall. Both studies used numerical schemes to complete the solution of the problem. Barros and Sphaier [5] proposed analytical approximations for solving an extended version of the Graetz problem with axial diffusion in an infinite domain. The adopted methodology consists of transforming the original convection-diffusion partial-differential equations into a simpler one-dimensional form, using approximation

---

* Corresponding author.
  E-mail address: al.belhocine@yahoo.fr (A. Belhocine).

rules provided by the Coupled Integral Equations Approach (CIEA).

In classical Gratez problems, the dimensionless parameter well-known Peclet number is usually assumed large such that the heat transport equation is simplified, as the axial diffusion becomes negligible. When axial diffusion is taken into account, an extended Graetz problem is obtained. An exact solution of this problem was presented by Acrivos [6], Vick and Özisik [7] and Ebadian and Zhang [8] for low Péclet numbers and different boundary conditions. Approximate solutions were also used (Villadsen and Michelson, [9]).

Oliveira et al. [10] presented the Graetz problem as a solution of the thermal entry flow in pipes for the FENE-P fluid, for imposed constant wall temperature and wall heat flux and in the presence of viscous dissipation.

Papoutsakis et al. [11] presented an analytical solution to the extended Graetz problem with prescribed wall flux, based on a self adjoint formalism resulting from the decomposition of the convective diffusion equation into a pair of first-order differential equations. The Graetz problem for non-Newtonian fluids with constant wall temperature and heat flux was studied numerically by Shih and Tsou [12].

The major objective of this paper is the study of convective heat transfer in a circular tube carrying fluid in laminar flow called commonly as the Graetz problem. A mathematical model was developed was then solved by orthogonal collocation method.

The second objective of study is the development of a FORTRAN program to solve the problem using the orthogonal collocation method followed by the Crank–Nicholson method. This latter is based on the Lagrange polynomial approximation. The program is able to generate highly accurate results with a less instability. The results are discussed and a general conclusion, which is the main thrust of this work, is summarized.

2. Mathematical modeling

2.1. Flow in the conduits in laminar flow

Laminar flow in a circular conduit could be modeled as thin layers of concentric fluid hollow cylinders forming the boundary layer, each traveling at different speeds, and a homogenous single cylinder out of the boundary layer traveling at free stream velocity. Based on this model, the heat transfer between adjacent fluids cylinders are:

1. In directions perpendicular to the cylinder surface, heat transfer occurs only through conduction.
2. In the direction of the flow, the heat transfer occurs through convection and conduction simultaneously. In most cases the hollow fluid cylinder is thin enough such that conduction could be neglected.

The flow taking place in the z direction, it has the cylindrical symmetry compared to the radius vector r:

Figure 1 Consider the element of the cylindrical volume of radius r, between the planes of abscissa x and x+dx. The balance of the amount of steady motion is:

$$-d\left(\frac{F_t}{\Omega}\right) = -\frac{dF_t}{\pi\, r^2} = -p \tag{1}$$

With

$$dF_t = \tau.\, dS = \tau.2.\,\pi.\, r.\, dx = -\mu.\frac{du}{dr}.2.\,\pi.\, r.\, dx \tag{2}$$

Where

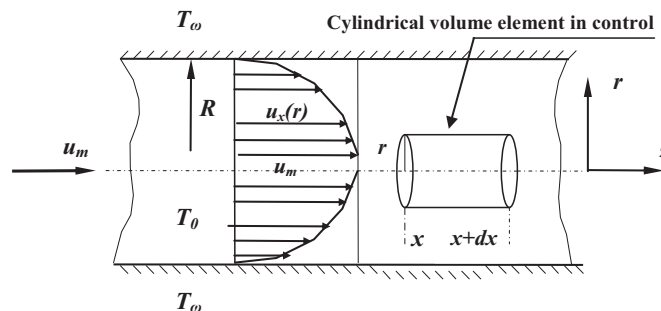$$-\frac{dp}{dx} = -\frac{2.\,\mu.\, du}{r.\, dr}$$

that is

Fig. 1. Geometry of a laminar flow in control.

$$du = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)rdr \tag{3}$$

For unidirectional flow, the total pressure p remains constant. Under these conditions U depends only on r and the integration of the above equation leads to as demonstrated in [13]:

$$du = -\frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\frac{r^2}{2} + Cte \tag{4}$$

However $u=0$ for $r=R$

$$u = -\frac{1}{4\mu}\left(-\frac{dp}{dx}\right)\left[R^2 - r^2\right] \tag{5}$$

The speed is maximum at the axis of the control volume:

$$u_{max} = -\frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right) \tag{6}$$

Hence

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{7}$$

The average speed of the control volume could be calculated based on the volume flow:

$$\dot{V} = \bar{u}\pi R^2 = \int_0^R u(r)2\pi rdr = 2\pi u_{max}\int_0^R\left[1 - \left(\frac{r}{R}\right)^2\right]rdr \tag{8}$$

Where

$$\frac{\bar{u}}{u_{max}} = 2.\int_0^R\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{r}{R}.\frac{dr}{R} = 2.\int_0^1\left[t - t^3\right]dt = 2.\left[\frac{t^2}{2} - \frac{t^4}{4}\right]_0^1 = \frac{1}{2} \tag{9}$$

$$u_{max} = 2\bar{u} \tag{10}$$

This shows that the speed profile is parabolic according to the normal variable space of flow and the maximum speed is equal twice the average speed.

2.2. The heat equation in cylindrical coordinates

The general equation for heat transfer in cylindrical coordinates developed by Bird, Stewart and Lightfoot [14] is as follows;

$$u_Z\frac{\partial T}{\partial z} = \frac{k}{\rho\,C_p}\nabla^2 T \tag{11}$$

$$\rho\,C_p\left(\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial\theta} + u_Z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$+ 2\,\mu\left\{\left(\frac{\partial u_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right)\right]^2 + \left(\frac{\partial u_Z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_Z}{\partial\theta}\right)^2 + \left(\frac{\partial u_Z}{\partial r} + \frac{\partial u_r}{\partial z}\right)^2\right\}$$

$$\mu\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]^2 \tag{12}$$

Considering that the flow is steady, laminar and fully developed flow (Re < 2400), and if the thermal equilibrium had already been established in the flow, then $\frac{\partial T}{\partial t} = 0$. The dissipation of energy would also be negligible. Other physical properties would also be constant and would not vary with temperature.

After applying the above assumptions, Eq. (12) reduces to the following form:

$$u_Z\frac{\partial T}{\partial z} = \frac{k}{\rho\,C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] \tag{13}$$

Given that the flow is fully developed laminar flow, then the velocity profile would have followed the parabolic distribution across the pipe section, represented by
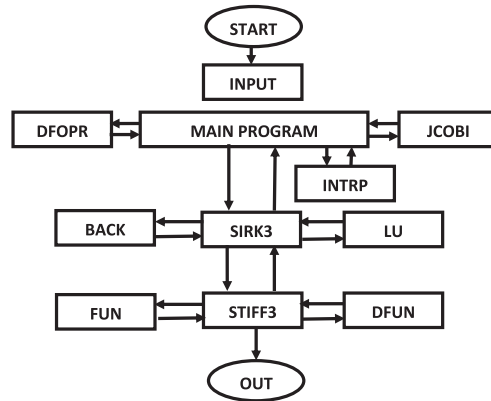
Fig. 2. Digital simulation Flowchart.

$$u_Z = 2\bar{u}\left[1 - \left(\frac{r}{R} \right)^2 \right]$$

(14)

By replacing the speed term in Eq. (13), we get:

$$2\,\bar{u}\left[1 - \left(\frac{r}{R} \right)^2 \right]\frac{\partial T}{\partial z} = \frac{k}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) \right]$$

(15)

Solving the equations requires the boundary conditions as set in Fig. 2, thus

$$\left. \begin{array}{ll} @\,z = 0 & T = T_0 \\ @\,r = 0 & \dfrac{\partial T}{\partial r} = 0 \\ @\,r = R & T = T_\omega \end{array} \right\}$$

(16)

It is more practical to study the problem with standardized variables from 0 to 1. To do this, new variables without dimension (known as adimensional) are introduced, defined as

$$\theta = \frac{T_\omega - T}{T_\omega - T_0}, \ \ x = \frac{r}{R} \ and \ y = \frac{z}{L}$$

The substitution of the adimensional variables in Eq. (15) gives

$$2\bar{u}\left[1 - \frac{x^2 R^2}{R^2} \right]\frac{(T_0 - T_\omega)}{L}\frac{\partial \theta}{\partial y} = \frac{k}{\rho c_p}\left[\frac{1}{xR}\frac{(T_0 - T_\omega)}{R}\frac{\partial \theta}{\partial x} + \frac{(T_0 - T_\omega)}{R^2}\ \frac{\partial^2 \theta}{\partial x^2} \right]$$

(17)

After making the necessary arrangements and simplifications, the following simplified equation is obtained.

$$(1 - x^2)\frac{\partial \theta}{\partial y} = \frac{kL}{\rho C_P 2\,\bar{u}R^2}\left[\frac{1}{x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 \theta}{\partial x^2} \right]$$

(18)

where the term $\frac{2\bar{u}\,R\,\rho\,C_p}{k}$ is the adimensional number known as Peclet number (*Pe*), which in fact is Reynolds number divided by Prandtl number. In steady state condition, the partial differential equation resulting from this, in the adimensional form can be written as follows:

$$(1 - x^2)\frac{\partial \theta}{\partial y} = \frac{L}{RPe}\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x} \right) \right]$$

(19)

This equation, if subjected to the new boundary conditions, would be transformed to the followings:

$$@z = 0, T = T_0 \Rightarrow @y = 0, \ \theta = 1$$
$$@r = 0, \frac{\partial T}{\partial r} = 0 \Rightarrow @x = 0, \ \frac{\partial \theta}{\partial x} = 0 \Rightarrow \theta(0, y) \neq 0$$
$$@r = R, T = T_\omega \Rightarrow @x = 1, \ \theta = 0 \Rightarrow \theta(1, y) = 0$$

To solve Eq. (19), it is hereby proposed that the orthogonal collocation method could be applied, based on orthogonal polynomials.

2.3. Application of the orthogonal collocation method

For this problem, the governing Eq. (19) is subject to the following conditions:

CI: à $y = 0$, $\theta = 1$

CL1: à $x = 0$, $\dfrac{\partial \theta}{\partial x} = 0$

CL2: à $x = 1$, $\theta = 0$

Operators gradient and Laplacian of the functions are given by [15,16]:

$$\left(\frac{\partial \theta}{\partial x}\bigg|_{x=x_i} \right) = \sum_{j=1}^{n+1} A_{i,j}\, \theta\,(x_j, y) \tag{20}$$

$$\nabla^2 \theta\big|_{x=xi} = \left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x} \right)\bigg|_{x=xi} \right) = \sum_{j=1}^{n+1} B_{ij}\theta(x_j, y) \tag{21}$$

To evaluate the points of the interior collocation and the two points at the end, the substitution of Eq. (19) in the governing partial differential equation with initial and boundary conditions, leads us to:

$$\left| \frac{\partial \theta}{\partial y} \right|_{xi} = \frac{L}{PeR(1 - x_i^2)}\left[\sum_{j=1}^{n+1} B_{i,j}\theta(x_j, y\,) \right] \tag{22}$$

$$\sum_{j=1}^{n+1} A_{1,j}\theta(x_j, y) = 0 \; and \; \theta(1, y) = 0 \tag{23}$$

A partial expansion of the Eq. (22) gives:

$$\left| \frac{\partial \theta}{\partial y} \right|_{xi} = \frac{L}{PeR(1 - x_i^2)}\left[B_{i,1}\theta(0, y) + \sum_{j=2}^{n} B_{i,j}\theta(x_j, y) + B_{i,n+1}\theta(1, y) \right] \tag{24}$$

Note that in the above expression as mentioned in [17,18] we must use some other means to eliminate. That can be done by developing the Eq. (23) of boundary condition.

$$A_{1,1}\theta\,(0, y) + \sum_{j=2}^{n} A_{1,j}\theta\,(x_j, y) + A_{1,n+1}\theta(1, y) = 0$$

Solving for, we get [19]:

$$\theta\,(0, y) = -\frac{1}{A_{1,1}}\left[\sum_{j=2}^{n} A_{1,j}\theta\,(x_j, y\,) \right] \tag{25}$$

And finally, substitute back in the governing equation and after rearranging, the following equation is obtained:

$$\left| \frac{\partial \theta}{\partial y} \right|_{xi} = \frac{L}{PeR(1 - x_i^2)}\left\{ \sum_{j=2}^{n} B_{i,j}\theta(\,x_j\,, y) - \left[\frac{B_{i,1}}{A_{1,1}} \right]\sum_{j=2}^{n} A_{1,j}\theta(x_j, y) \right\} \tag{26}$$

It is noticed that orthogonal collocation for infinitesimal x and y for the PDE, i.e. formation of a PDE to a series of ODE (ordinary differential equation). To solve this series of ODE the use of the traditional Runge–Kutta method of order 6. With this intention, two subroutines FUN and DFUN are needed to be able to apply the algorithm STIFF3, which are detailed in the next part.

3. Numerical solution of the system of differential equations

A numerical solution procedure is proposed using the orthogonal collocation method, which is based on the approximation of Lagrange polynomials. It is able to deliver highly accurate results with stiffness and reduced instability. This method reduces the degree of the equation and can be solved step by step manner.

In the following, subroutines that is used in the simulation, which requires the determination of several input and output parameters are presented. The language used in the main program is FORTRAN which itself contains other subroutines.

3.1. The procedure of the numerical solution

By applying the equations developed to the problem, a more general equation could be written as follows;

$$\left[\frac{d\theta}{dy}\right]_{xi} = \frac{L}{PeR\,(1-x_i^2)}\left\{\sum_{j=2}^{n} B_{i,j}\theta(x_j, y) - \left[\frac{B_{i,1}}{A_{1,1}}\right]\sum_{j=2}^{n} A_{1,j}\theta(x_j, y)\right\} \tag{27}$$

Now the problem is reduced to solving a system of simultaneous equations in view of obtaining θ at each point of the collocation point. It starts from the normal form at this point to fully develop the program needed to solve this basic example, such that for the latter example only the discretization of the equation is given. In the example, the Lagrange polynomial of the order six were used (that is to say that we have six inner roots or collocation points), and both the temperature boundary conditions for θ variable were taken into account. For economy of computation time and programming, it is recommended to resolve the six internal temperatures θ variable, instead of all the eight points (eight equations). Thus, the optimal procedure was to write three short routines FUN, DFUN and OUT as cited in [20].

3.1.1. Subroutine FUN
In this example, the FUN subroutine is to define the ordinary partial differential equations and write them in the form F (I) for $I=1$, 6 for θ variable. Therefore, for the 6 points for internal nodes, if one chooses a polynomial of 8th order, it can be written as:

$$F(1) = \left[\frac{d\theta}{dy}\right]_{x1} = \frac{L}{PeR\,(1-x_1^2)}\left\{\sum_{j=1}^{6} B_{1,j+1}\theta(x_j, y) - \left[\frac{B_{1,1}}{A_{1,1}}\right]\sum_{j=1}^{6} A_{1,j+1}\theta(x_j, y)\right\} \tag{28}$$

$$F(2) = \left[\frac{d\theta}{dy}\right]_{x2} = \frac{L}{PeR\,(1-x_2^2)}\left\{\sum_{j=1}^{6} B_{2,j+1}\theta(x_j, y) - \left[\frac{B_{2,1}}{A_{1,1}}\right]\sum_{j=1}^{6} A_{1,j+1}\theta(x_j, y)\right\} \tag{29}$$

$$F(3) = \left[\frac{d\theta}{dy}\right]_{x3} = \frac{L}{PeR\,(1-x_3^2)}\left\{\sum_{j=1}^{6} B_{3,j+1}\theta(x_j, y) - \left[\frac{B_{3,1}}{A_{1,1}}\right]\sum_{j=1}^{6} A_{1,j+1}\theta(x_j, y)\right\} \tag{30}$$

Thus, generally a global expression for F(I) in Fortran allowed the equation to be written as

$$F(I) = \frac{L}{PeR(1-X(I)^2)}\left\{\sum_{j=1}^{6} B(I, J+1)*TETA(J) - \frac{B(I, 1)}{A(1, 1)}\sum_{j=1}^{6} A(1, J+1)*TETA(J)\right\} \tag{31}$$

And to eliminate the summation term a DO loop in Fortran is used.

3.1.2. Subroutine DFUN
DFUN is a subroutine used for the sub 1in the integration of STIFF3 program. DFUN must provide the first derivative of F (I) in each collocation point. The differentiation of F(I) is simple and straightforward:

$$F(I) = \frac{L}{PeR(1-x_i^2)}\left\{\sum_{j=1}^{6} B_{i,j+1}\,\theta(x_j, y) - \left[\frac{B_{i,1}}{A_{1,1}}\right]\sum_{j=1}^{6} A_{1,j+1}\theta(x_j, y)\right\} \tag{32}$$

Thus for $J=1$ to 6, the results would be as follows;

$$dF(1, 1) = \frac{L}{PeR\,(1-x_1^2)}\left\{B_{1,2} - \left[\frac{B_{1,1}}{A_{1,1}}\right]A_{1,2}\right\}$$

$$dF(1, 2) = \frac{L}{PeR\,(1-x_1^2)}\left\{B_{1,3} - \left[\frac{B_{1,1}}{A_{1,1}}\right]A_{1,3}\right\} \tag{33}$$

and so on. Thus, the general equation could be written as;

$$dF(I, J) = \frac{dF(I)}{d\theta(x_j, y)} = \frac{L}{PeR(1-X(I)^2)}\left\{B(I, j+1) - \left[\frac{B(I, 1)}{A(1, 1)}\right]A(1, J+1)\right\} \tag{34}$$

3.1.3. Subroutine OUT
OUT subroutine is called at a predetermined time from the STIFF3 integration subroutine.
When OUT is called, the temperature profile values are available for printing with X: the current value of time.
Y: the present value of temperatures at each collocation point.
IH: number of bisection that occurs before a satisfactory integration.

Q: passing factor of integration acceleration.

3.2. Definitions of the various sub-programs used in the simulation

3.2.1. JCOBI subroutine
This is a subroutine used for the calculation of derivatives and the roots of the polynomial as mentioned in [18]. It uses: call JCOBI (n, n, n0, n1, al, be, DIFL, diF2, dif3, root).

3.2.2. DFOPR subroutine
This is a sub-program for the calculation of interpolation of elements Aij and Bij and IQ quadrature moments. It uses: Call DFOPR (n, n, n0, D, I, id, DIFL, diF2, dif3, root).

3.2.3. STIFF3 Subroutine
This subroutine is used in the operation of the integration of a single or several pairs of differential equation systems (STIFF) for ISIS3 method that is inspired by the Runge–Kutta method. The STIFF subroutine communicates with three sub SIRCK3 program, LU and BACK.

3.2.4. LU subroutine
This subroutine is the decomposition of the matrix A using the method of "Crout" ND: row dimension of A in the called program, with.
N: present size of A. INDX: exchange pivot row vector during the factorization. A: matrix factorization.

3.2.5. Subroutine BACK
BACK subroutine allows the substitution of Crout and is written in the following form:
ND, INDX, N: as showed in LU subroutine.
A: matrix factorization.
V: vector right term of the linear equation solution.

3.2.6. Subroutine SIRK3
The subroutine uses SIRK3 Runge–Kutta technique of 6th order for the solution of one or more ordinary differential equation system. The program calls sub programs FUN and DFUN that were detailed above.

3.2.7. Subroutine INTRP
This is a subroutine that is used to compute the interpolation coefficients. The non-dimensional temperatures y (i) interpolation points calculated by the program under INTRP $x=0$ to $x=1$.

4. Main program

The main program is written in FORTRAN to solve the above equations. This program communicates with other sub-routines to perform the calculations needed to solve different parameters. This program calculates the non-dimensional temperature profiles in the tube with fully developed and laminar flow.

5. Application of the Crank–Nicholson method

Therefore for this problem there is a non-linear PDEs with the following boundary conditions:

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} = (1 - R^2)\frac{\partial U}{\partial X}$$

$$B.\,C.1\;U = \;1,\;\;X = 0\;\;B.\,C.2\;\;\partial U/\partial R = 0,\;\;R = 0\;\;B.\,C.3\;U = 0,\;\;R = 1$$

(35)

A numerical method can be used to solve Eq. (35). The analog finite difference corrected second order, Crank–Nicholson method as cited in [21] is used in the following manner;

$$\left.\frac{\partial^2 U}{\partial R^2}\right|_{m,n+1/2} = \frac{1}{2}\left(\frac{U_{m+1,n+1} - 2U_{m,n+1} + U_{m-1,n+1}}{h^2} + \frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{h2}\right)$$

$$\left.\frac{1}{R}\frac{\partial U}{\partial R}\right|_{m,n+1/2} = \frac{1}{2R_m}\left(\frac{U_{m+1,n+1} - U_{m-1,n+1}}{2h} + \frac{U_{m+1,n} - U_{m-1,n}}{2h}\right)$$

$$(1 - R^2)\left.\frac{\partial U}{\partial X}\right|_{m,n+1/2} = (1 - R_m^2)\left(\frac{U_{m+1,n} - U_{m,n}}{2(k/2)}\right) \tag{36}$$

Where h and k are the mesh sizes of the R and X respectively. The Crank–Nicholson equation is the result of successively application of forward and backward difference equations. This is stable for all values of the ratio h/k. After the derivatives are replaced, Eq. (35) becomes:

$$A_m U_{m-1,n+1} + B_m U_{m,n+1} + C_m U_{m+1,n+1} = D_m \quad , \ 2 \le m \le M \tag{37}$$

where

$$A_m = \frac{1}{2h^2} - \frac{1}{4R_m h}$$

$$B_m = -\frac{1}{h^2} - (1 - R_m^2)\frac{1}{k}$$

$$C_m = \frac{1}{2h^2} + \frac{1}{4R_m h}$$

$$D_m = \left(\frac{1}{2h^2} + \frac{1}{4R_m h}\right)U_{m-1,n} + \left(\frac{1}{h^2} - \frac{(1 - R_m^2)}{k}\right)U_{m,n} - \left(\frac{1}{2h^2} + \frac{1}{4R_m h}\right)U_{m+1,n}$$

Eq. (37) contains three unknowns: $U_{m-1,n+1}$, $U_{m,n+1}$, $U_{m+1,n+1}$. Analogs of finite difference B.C2 and B.C3 are:

CL2 $\quad B_1 U_{1,n+1} + C_1 U_{2,n+1} = D_1$ (37)

Where

$B_1 = 1/k + 2/h^2, \ C_1 = -2/h^2 D_1 = (1/k - 2/h^2)U_{1,n} + (2/h^2)U_{2,n}$ and

CL3 $\quad A_M U_{M-1,n+1} + B_M U_{M,n+1} = D_M$ (38)

Where

$A_M = 1/2h^2 - 1/4h, \ B_M = -1/h^2 D_M = (1/4h - 1/2h^2)U_{M-1,n} + (1/h^2)U_{M,n}$

The Eqs. (36), (37) and (38) are an $M+1$ system of algebraic equations with $M+1$ unknowns. Using a known auxiliary condition, which is CL1: $U_{m,1} = 1$ à $X = 0$.

The system of equations above can be solved to yield $U_{1,2}$, $U_{2,2}$, $U_{3,2}$, …, $U_{M,2}$. These values are in turn used to calculate $U_{1,3}$, $U_{2,3}$, $U_{3,3}$, …, $U_{M,3}$. This system of simultaneous equations can be represented by a coefficient of tridiagonal matrix and
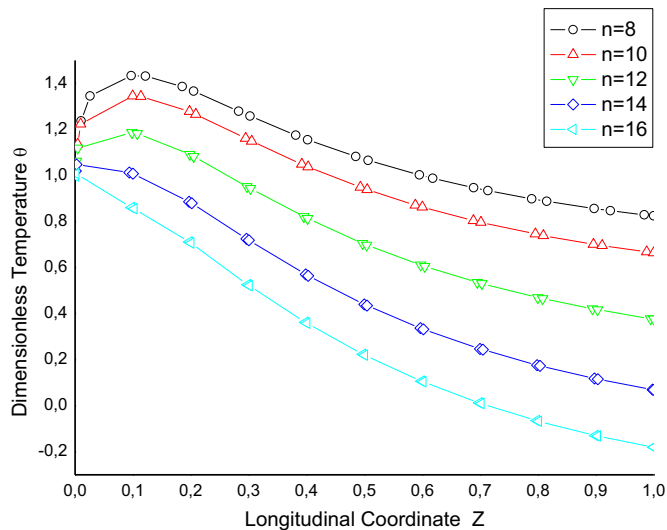

Fig. 3. Dimensionless temperature, θ profile as a function of Z at various collocation points n.

can be solved more effectively by the method of Thomas as demonstated in [21]. The algorithm of the method first calculates:

$$\beta_m = B_m - A_m C_{m-1}/\beta_{m-1} \text{ with } \beta_1 = B_1 \text{ and}$$

$$\gamma_m = \frac{D_m - A_m \gamma_{m-1}}{\beta_m} \qquad \text{with } \gamma_1 = \frac{D_1}{B_1}$$

U values are then calculated back from U_M to U_1

$$U_{M,n+1} = \gamma_M$$

$$U_{m,n+1} = \gamma_m - \frac{C_m U_{m+1,n+1}}{\beta m}$$

6. Results and discussion

Fig. 3 illustrates the results obtained by the method of orthogonal collocation in a cylindrical model. These results are expressed in different points of internal collocation n. Therefore, the more number of nodes used, the acuity of results is better. We can see from this figure that at most n, the higher the normalized dimensionless temperature evolution of the unit decreases exponentially to zero. It can be seen that the temperature trend as the fluid cools from its inlet temperature to near the constant wall temperature. The value of $\theta=1$ corresponds to the fluid temperature at the inlet, while $\theta=0$ corresponds to the wall temperature.

Fig. 4 presents the non-dimensional θ distribution as a function of Z at different values of Peclet number, Pe. The figure shows the influence of the Peclet number in the flow regime. As it is seen from the graphs, when Peclet number gets smaller the temperature of the fluid upto the entrance increases. It must be remembered that the Peclet number is defined as a measure of the ratio of the power transmission rate by convection to energy transport rate by conduction. It is observed that the variations is large when the Pe value increases.

Fig. 5 shows the temperature variations at specific distances R/L from the intake. The numerical solution shows that the iso-temperature lines are almost linear, ie the temperature remains almost constant along a streamline far from the intake. We see that the results for $R/L=0.1$ are above results when $R/L=0.01$ and 0.001. In other words, as the distance from the entrance of the tube (R/L reducing), the wall temperature value gradually becomes predominant in the liquid stream. For laminar fully developed conditions and large Peclet numbers, dimensionless temperature goes to a stable value for constant wall temperature.

Fig. 6 shows the distribution of the non-dimensional temperature U with Z. The curves were obtained by the implicit Crank–Nicholson method with different number of mesh. It should be noted that for low values of the number of mesh N in the discretization, the influence of N is more significant to the distribution of temperature, compared to when N is high. The graphs show that the dimensionless temperature is very high at the beginning of the entrance region of tube and there after decreases exponentially to zero, in other words at a contant wall temperature.

Fig. 7 compares the computation results by the orthogonal collocation method and that of the implicit Crank–Nicholson method. It can be seen that the two curves deviate significantly when the number of collocations or the number of point of mesh are large.
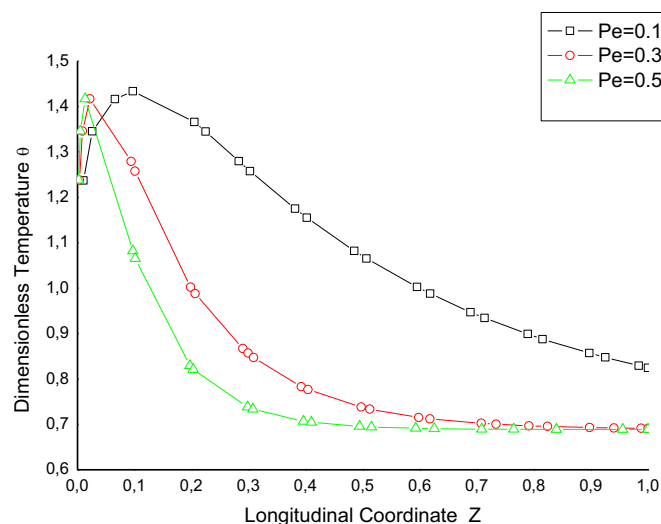

Fig. 4. The distribution of the non-dimensional temperature θ with Z at different Peclet number.
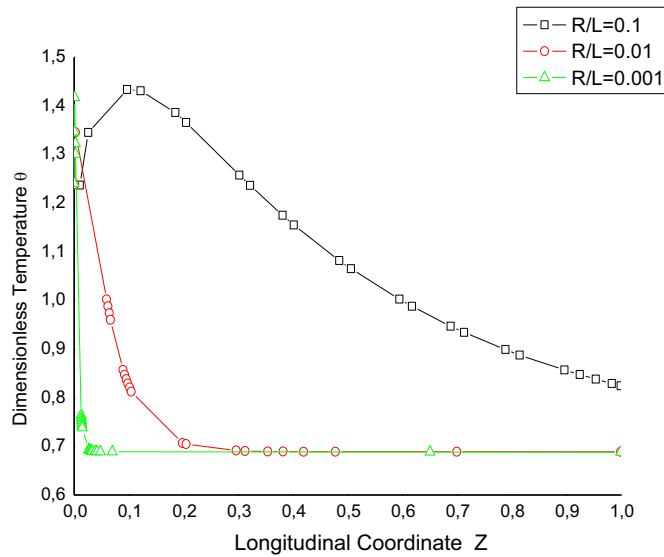
Fig. 5. The distribution of the non-dimensional temperatures θ with Z, taking account the Influence of the ratio R/L.
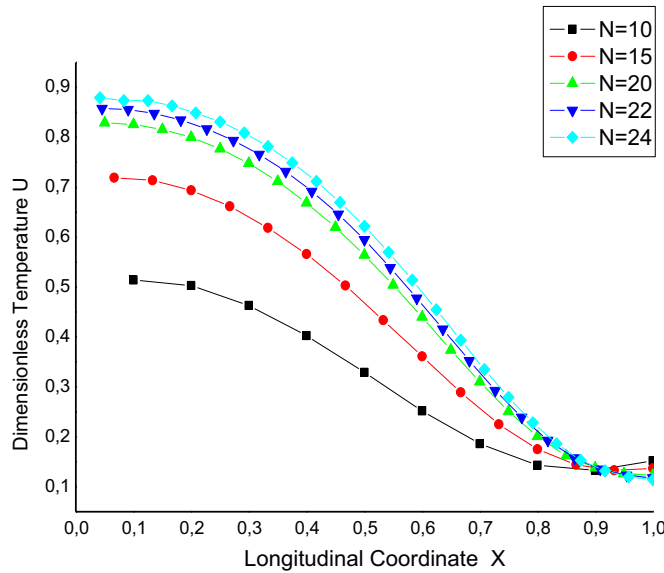

Fig. 6. Variation in the non-dimensional temperature U in X according to the diagram of Crank–Nicholson.

We could also note that the orthogonal collocation method combines at a large number of collocation points the rapid convergence than finite element method of locating grid point where the temperature has large gradients.

This means that the technique of orthogonal collocation allowed fast simulations whereas the implicit Crank–Nicholson method, would be more reliable even if it requires more intensive computer time.

7. Conclusions

This paper shows how temperature profile of a fully developed laminar flow in a hot wall tube could be determined. Firstly the equation was simplified. Assuming the flow is incompressible, it is possible to decouple the thermal and dynamic equations. Then the dimensionless number, the Peclet number was introduced. Its influence was in the case of flow in a tube (Graetz problem) was examined. This is the basics of all internal thermal flow problems. This work consists of solving the Graetz problem by two different methods however, and it was shown that each has their own merit.

The operation of two different digital techniques was used to solve the axial dispersion model, or to be more precise, by the orthogonal collocation method.
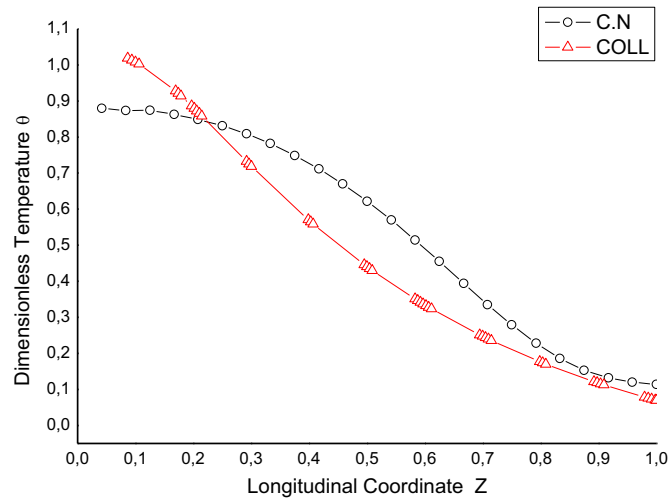
Fig. 7. Comparison of orthogonal collocation ($n=29$) with Crank–Nicholson method ($N=30$).

The first method is the collocation method, which proved to be a stable numerical procedure and could be considered to advance and excellent, as it produces precise results for the low number of mesh numbers, which means low computing time.

The second method is by the finite difference method using the implicit Crank–Nicholson scheme in the second order, associated with the Thomas algorithm for matrix inversion, that is by simultaneously solving the equations. We can consider that the equation is sufficient to solve many physical situations; in fact, this analysis could be used to size a problem, extract a simple description and then guiding the numerical resolution which will be done with a high-performance code.

The model treated in our study uses the convection concepts, and presents a heat transfer inside a circular cylindrical pipe with fully developed laminar flow. The model is composed of a nonlinear partial differential equation which is subject to an initial condition and two boundary conditions. This type of model requires a simulation method adapted to distributed parameter systems. The orthogonal collocation method allowed fast simulations of the system while the finite difference method, is more intensive but is more reliable. The results which come from this study show that the orthogonal collocation method is particularly good, converging quicker than the implicit Crank–Nicholson method, although the latter is better if a high accuracy is required.

From this numerical study, we reached the following conclusions:

-The dimensionless temperature is very high at the beginning of the entrance region of tube and there after decreases exponentially to zero

-For laminar fully developed conditions and for large Peclet numbers, dimensionless temperature goes to a stable value for constant wall temperature

-The orthogonal collocation method giving better results than Crank–Nicholson method.

This study also shows that it would be possible to guide the selection of suitable numerical simulation procedures when dealing with a complex problem of convection. This work also showed the solution to a form of heat transfer in a fairly simple problem of laminar pipe flow. This problem is a fundamental problem, because it is the basics of all possible problems of heat transfer in pipe flow.

The use of exchange coefficients in the method proposed here simplifies the works towards the solution. It was also shown that the collocation method would present a solution with a high degree of accuracy and with very short computer time if low numbers of collocation points were used. On the other hand, if the number of collocation points were used, then the implicit Crank–Nicholson method would give a better accuracy. This means that, it is necessary to juggle the accuracy required and the available computing power.

References

[1] I. Girgin, M. Turker, Axial heat conduction in laminar duct flows, J. Naval. Sci. Eng. 7 (2) (2011) 30–45.
[2] N.R. Braga, L.S. de Barros, L.A. Sphaier, Generalized integral transform solution of extended graetz problems with axial diffusion, ICCM 2014, Cambridge, England (2014) 1–14.
[3] C.J. Hsu, Exact solution to entry-region laminar heat transfer with axial conduction and the boundary condition of the third kind, Chem. Eng. Sci. 23 (5) (1968) 457–468.
[4] M.L. Michelsen, J. Villadsen, The Graetz problem with axial heat conduction, Int. J. Heat Mass Transfer 17 (11) (1974) 1391–1402.
[5] L.S. De Barros, L.A. Sphaier, Enhanced lumped formulations for the graetz problem in infinite domains, In: Proceedings of the 22nd International Congress of Mechanical Engineering (COBEM 2013), November 3-7, Ribeirão Preto, SP, Brazil, 2013, pp. 3299–3306.
[6] A. Acrivos, The extended Graetz problem at low Peclet numbers, Appl. Sci. Res. 36 (1980) 35–40.

[7] M. Ebadian, H. Zhang, An exact solution of extended Graetz problem with axial heat conduction, Int. J. Heat Mass Transfer 32 (9) (1989) 1709–1717.
[8] B. Vick, M.N. Özisik, An exact analysis of low Peclet number heat transfer in laminar flow with axial conduction, Lett. Heat Mass Transfer 8 (1) (1981) 1–10.
[9] J. Villadsen, M. Michelson, Approximate solution of extended graetz problem by orthogonal collocation, J. Eng. Phys. 31 (1) (1976) 809–814.
[10] P.J. Oliveira, P.M. Coelho, F. Tavares de Pinho, The graetz problem for a fene-p fluid in a pipe, In: Proceedings of the 10th Brazilian Congress of Thermal Sciences and Engineering – ENCIT 2004 Brazilian Society of Mechanical Sciences and Engineering – ABCM, Rio de Janeiro, Brazil, Nov. 29 – Dec. 03, 2004.
[11] E. Papoutsakis, D. Ramkrishma, H.C. Lim, The extended Graetz problem with prescribed wall flux, Aiche. J. 26 (5) (1980) 779–786.
[12] Y.P. Shih, J.D. Tsous, Extended Leveque solutions for heat transfer to power fluids in laminart flow in a pipe, Chem. Eng. J. 15 (1978) 55–62.
[13] E. Ower, R.C. Ppankhurst, The measurement of air Flow, Pergamon Press, 1974.
[14] R.B. Bird, W.E. Stewart, E.N. Lightfoot, Transport Phenomena, John Wiley and Sons, New York, 1960.
[15] B.V. Babu, K.K.N. Sastry, Estimation of heat transfer parameters in a trickle-bed reactor using differential evolution and orthogonal collocation, Chem. Eng. Sci. 23 (1998) 327–339.
[16] J. Villadsen, M.L. Michelsen, Solution of Differential Equations Models by Polynomial Approximation, Prentice Hall, Englewood Cliffs, New Jersey, 1978.
[17] W.E. Stewart, K.L. Levien, M. Morari, Simulation of Fractionation by Orthogonal Collocation, Chem. Eng. Sci. 40 (1985) 409.
[18] J. Villadsen, Selected Approximation Methods for Chemical Engineering Problems, Reproset, Copenhagen, 1970.
[19] M.L. Michelsen, Axial dispersion and the collocation method, Chem. Eng. Sci. 1 (1994) 3675.
[20] P. Stanley, in: Bird Allan (Ed.), Electrochemical Chemistry, Vol. 13, Marcel Decker Inc., New York, 1984.
[21] D. Rosenberg, Methods for the Numerical Solution of Partial Differential Equations, American Elsevier Publishing Co., New York, 1969.