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Study of a Perturbed Boundary Optimal Control System

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Abstract—This article deals with the study of the existence and behaviour of the optimal control and the state of a perturbed boundary control linear system when the space of admissible controls has finite dimension.

Keywords—Perturbed state, Boundary optimal control.

INTRODUCTION AND PLACEMENT OF THE PROBLEM

In this paper, we are concerned with the boundary optimal control problem when the state satisfies

$$\begin{aligned} -\Delta y_\varepsilon &= 0, && \text{on } \Omega, \\ \frac{\partial}{\partial v} y_\varepsilon + \varepsilon\, y_\varepsilon &= u_\varepsilon, && \text{at } \Gamma = \partial\Omega, \\ \int_\Gamma y_\varepsilon \, d\Gamma &= 0, && y_\varepsilon \in H^1(\Omega), \end{aligned} \tag{$(P_1)(u_\varepsilon)$}$$

where $\frac{\partial}{\partial v} y_\varepsilon$ is the normal derivative of y_ε, Ω is a regular and bounded open set in the Euclidean space $\mathcal{R}^n$, u_ε is the optimal control solution of the problem

$$J_\varepsilon(u_\varepsilon) = \min\left\{J_\varepsilon(v);\quad v \in \mathcal{U}_{\rm ad}\right\}, \tag{P_2}$$

where

$$J_\varepsilon(v) := \int_\Gamma (y_\varepsilon(v) - z_1)^2 \, d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial v} y_\varepsilon(v) - z_2\right)^2 d\Gamma;$$

$y_\varepsilon(v)$ is a solution of $(P_1)(v)$, $v \in \mathcal{U}_{\rm ad}$, $\mathcal{U}_{\rm ad}$ is a closed convex set in $L^2(\Gamma)$, and z_1 and z_2 are fixed functions in the space $L^2(\Gamma)$ (decision functions).

The author in [1] dealt with this system, which is a mathematical modelization for problems of layers phenomena in aerodynamics. He took a cost functional J_ε in the form

$$J_\varepsilon(v) := \int_\Gamma (y_\varepsilon(v) - z_d)^2 \, d\Gamma + N \int_\Gamma v^2 \, d\Gamma,$$

where $N > 0$. In this case, $J_\varepsilon(v)$ is a definite positive and strictly convex function. Here, we loose the positivity definiteness of the cost functional, so the techniques of [1] cannot be used

We dedicate this article to our friend and colleague M. Azzouzi with good wishes.

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here. The space $\mathcal{U}_{\rm ad}$ of admissible controls considered here will be an arbitrary linear subspace of finite dimension in $L^2(\Gamma)$.

This paper is organised as follows. In the first section, we prove the existence of the state in some Sobolev space (Theorem 1.1) and show, in Theorem 1.2, that the problem (P_2) has a solution. In the second section, we establish our main result, Theorem 2.1, where we study the convergence of u_ε and $y_\varepsilon(u_\varepsilon)$ (when $\varepsilon \to 0$). We end the paper with some concluding remarks.

1. EXISTENCE OF THE PERTURBED STATE AND CONTROL FOR THE SYSTEM: $(P_1)(u_\varepsilon)$ AND (P_2)

The space of admissible controls $\mathcal{U}_{\rm ad}$ will be a linear subspace of $\mathcal{U}$ with finite dimension $m \geq 1$ where

$$\mathcal{U} := \left\{ v \in L^2(\Gamma) : \int_\Gamma v\, d\Gamma = 0 \right\}; \tag{1.1}$$

$H^1(\Omega)$ is the usual Sobolev space with its scalar product and associated norm. We look for solutions (i.e., the states) of the system $(P_1)(u_\varepsilon)$ in the space

$$V := \left\{ y \in H^1(\Omega) : \int_\Gamma y\, d\Gamma = 0 \right\}. \tag{1.2}$$

THEOREM 1.1. *For all $v \in \mathcal{U}_{\rm ad}$, there exists a unique solution of the problem $(P_1)(v)$, denoted by $y_\varepsilon(v)$, in the space V.*

PROOF. We use a variational formulation of the problem $(P_1)(v)$. This yields a continuous symmetric bilinear form defined on $V \times V$ by

$$\mathcal{A}_\varepsilon(y,q) := \int_\Omega \nabla y\, \nabla q\, dx + \varepsilon \int_\Gamma y\, q\, d\Gamma. \tag{1.3}$$

We know [2] that the norm defined in $H^1(\Omega)$ by

$$\mathcal{P}(y) := \left[\int_\Omega |\nabla y|^2\, dx + \varepsilon \int_\Gamma y^2\, d\Gamma \right]^{1/2} \tag{1.4}$$

is equivalent to the usual norm $\|y\|_{H^1(\Omega)}$ in $H^1(\Omega)$. Then, there exists a constant $\alpha_\varepsilon > 0$ such that

$$(\mathcal{A}_\varepsilon(y,y))^{1/2} \geq \alpha_\varepsilon \|y\|_{H^1(\Omega)}. \tag{1.5}$$

Thus, $\mathcal{A}_\varepsilon$ is a continuous coercive bilinear form. The theorem of Lax-Milgram [3,4] gives the existence of a unique solution, denoted by $y_\varepsilon(v)$, satisfying for all $q \in V$,

$$\mathcal{A}_\varepsilon\left(y_\varepsilon(v), q\right) := \int_\Gamma v\, q\, d\Gamma. \tag{1.6}$$

The representation theorem of Riesz [5] implies that there exists a unique linear (continuous) isomorphism A_ε from V onto V' (the dual space of V) and a unique element denoted by $\Psi_\varepsilon(v)$ in the space V', such that

$$A_\varepsilon\, y_\varepsilon(v) = \Psi_\varepsilon(v). \tag{1.7}$$

Then, the solution of the problem $(P_1)(v)$ is given by

$$y_\varepsilon(v) = A_\varepsilon^{-1} \Psi_\varepsilon(v), \tag{1.8}$$

where A_ε^{-1} is the inverse operator of A_ε. This completes the proof.

REMARK 1.1. One can verify that the map Ψ_ε is a linear operator from $\mathcal{U}_{\rm ad}$ into V'.

THEOREM 1.2. *There exists a nonvanishing subset X_ε of $\mathcal{U}_{\rm ad}$, such that for all $u_\varepsilon \in X_\varepsilon$, we have*

$$J_\varepsilon(u_\varepsilon) = \min\{J_\varepsilon(v); \quad v \in \mathcal{U}_{\rm ad}\}.$$

PROOF. To prove the existence of X_ε, it suffices to prove that the two following conditions are satisfied [5]:

(i) The map: $v \to J_\varepsilon(v)$ is convex and l.s.c. (i.e., lower semicontinuous) on the space $\mathcal{U}_{\rm ad}$.
(ii) For all sequences (v_n) of elements in $\mathcal{U}_{\rm ad}$, such that $\|v_n\|_{L^2(\Gamma)} \to +\infty$, then $J_\varepsilon(v_n) \to +\infty$, when $n \to +\infty$.

It is easy to verify that the functional cost J_ε is convex. By using formula (1.3)–(1.6) we can see that the map $v \in \mathcal{U}_{\rm ad} \to y_\varepsilon(v) \in H^1(\Omega)$ is continuous. On the other hand, the trace theorem and the continuity of the norm ensure the continuity of the map

$$J_\varepsilon : v \in \mathcal{U}_{\rm ad} \to \|y_\varepsilon(v) - z_1\|^2_{L^2(\Gamma)} + \left\| \frac{\partial}{\partial v} y_\varepsilon(v) - z_2 \right\|^2_{L^2(\Gamma)}.$$

So condition (i) is satisfied. Condition (ii) results from the next Lemma.

LEMMA 1.1. *The map $\mathcal{B}_\varepsilon : \mathcal{U} \to L^2(\Gamma)$, which associates to each $v \in \mathcal{U}$ the element $\mathcal{B}_\varepsilon(v) := y_\varepsilon(v)\big|_\Gamma$ is an injective linear map into $L^2(\Gamma)$. As a consequence, there exists a constant $C_\varepsilon > 0$. In fact, $C_\varepsilon = \left(\|\mathcal{B}_\varepsilon^{-\Lambda}\|\right)^{-1}$, where $\mathcal{B}_\varepsilon^{-\Lambda}$ is the inverse operator of $\mathcal{B}_\varepsilon$ defined on the range $\mathcal{B}_\varepsilon(\mathcal{U}_{\rm ad})$, such that*

$$C_\varepsilon \|v\|_{L^2(\Gamma)} \le \|y_\varepsilon(v)\|_{L^2(\Gamma)}, \qquad \text{for all } v \in \mathcal{U}_{\rm ad}. \tag{1.9}$$

PROOF. It is clear that $\mathcal{B}_\varepsilon$ is a linear map. Let $v \in \mathcal{U}$ such that $\mathcal{B}_\varepsilon(v) := y_\varepsilon(v)\big|_\Gamma = 0$. Then, $y_\varepsilon(v)$ is in the space $H_0^1(\Omega)$ and is a solution of the following problem:

$$\begin{aligned} -\Delta y_\varepsilon(v) &= 0, && \text{on } \Omega, \\ \frac{\partial}{\partial v} y_\varepsilon(v) &= v, && \text{at } \Gamma = \partial\Omega, \\ \int_\Gamma y_\varepsilon(v)\, d\Gamma &= 0, && y_\varepsilon(v) \in H^1(\Omega). \end{aligned}$$

Using Green's formula, we obtain

$$\int_\Omega |\nabla y_\varepsilon(v)|^2 \, dx + \int_\Gamma v\, y_\varepsilon(v)\, d\Gamma = 0.$$

Since $\int_\Gamma v\, y_\varepsilon(v)\, d\Gamma = 0$ and $y_\varepsilon(v) \in H_0^1(\Omega)$, we have $y_\varepsilon(v) = 0$ on Ω. Consequently, $v = 0$. The existence of C_ε results from the fact that $\mathcal{B}_\varepsilon$ is linear injective and $\mathcal{U}_{\rm ad}$ has a finite dimension.

REMARKS 1.2.

(i) Since $\mathcal{U}_{\rm ad}$ is a finite dimensional space, to decide the unicity of the solution of (P_2), one can use the Hessian function associated to J_ε, after fixing (for example) an orthonormal basis of $\mathcal{U}_{\rm ad}$.
(ii) One can verify that there exist two constants $C_1 > 0$ and $C_2 > 0$, such that: $C_1 \le \|\mathcal{B}_\varepsilon\| \le C_2$ for all $0 < \varepsilon < 1$.

2. STUDY OF THE CONVERGENCE OF THE STATE y_ε AND CONTROL u_ε

The main result of this article is the following theorem.

THEOREM 2.1. *We have the following statements:*

(i) *The control u_ε converges strongly in the space $L^2(\Gamma)$, to $u \in \mathcal{U}_{\text{ad}}$, satisfying $J(u) = \min\{J(v);\, v \in \mathcal{U}_{\text{ad}}\}$, where $J(v) := \int_\Gamma (y(v) - z_1)^2\, d\Gamma + \int_\Gamma (v - z_2)^2\, d\Gamma$, and $y(v)$ is the solution of the problem*

$$\begin{aligned} -\Delta y(v) &= 0, && \text{on } \Omega, \\ \frac{\partial}{\partial v} y(v) &= v, && \text{at } \Gamma = \partial\Omega, \\ \int_\Gamma y(v)\, d\Gamma &= 0, && y(v) \in H^1(\Omega). \end{aligned} \qquad (P_3)(v)$$

(ii) *The state y_ε converges strongly in the space $H^1(\Omega)$ to the state $y(u)$, solution of the system $(P_3)(u)$.*

PROOF. As the control 0 is in the space $\mathcal{U}_{\text{ad}}$, we have $J_\varepsilon(u_\varepsilon) \le \|z_1\|^2_{L^2(\Gamma)} + \|z_2\|^2_{L^2(\Gamma)}$; then there exists a constant $C_1 > 0$ (independent of ε), such that $\|y_\varepsilon\|_{L^2(\Gamma)} \le C_1$.

Starting from the fact that $\|\frac{\partial}{\partial v} y_\varepsilon(u_\varepsilon) - z_2\|^2_{L^2(\Gamma)} \le \|z_1\|^2_{L^2(\Gamma)} + \|z_2\|^2_{L^2(\Gamma)}$, one can show, by using $\frac{\partial}{\partial v} y_\varepsilon(u_\varepsilon) + \varepsilon\, y_\varepsilon(u_\varepsilon) = u_\varepsilon$, that there exists a constant $C_2 > 0$, such that, for all $0 < \varepsilon < 1$,

$$\|u_\varepsilon\|_{L^2(\Gamma)} \le C_2. \tag{2.1}$$

Since $\mathcal{U}_{\text{ad}}$ is of finite dimension, u_ε converges strongly to an element u in the space $\mathcal{U}_{\text{ad}}$.

Using the variational formulation of the problem $(P_1)(u_\varepsilon)$, we deduce that there exists a constant $C_3 > 0$ such that $\|y_\varepsilon\|_{H^1(\Omega)} \le C_3$, independently of ε ($y_\varepsilon(u_\varepsilon)$ is denoted by y_ε). Consequently, the state y_ε converges weakly in the space $H^1(\Omega)$ to an element $y(u)$ (denoted by y), which is a solution of the problem $(P_3)(u)$.

In order to prove the strong convergence of the state y_ε to y in $H^1(\Omega)$, it suffices to prove that $\|\nabla y - \nabla y_\varepsilon\|_{L^2(\Omega)}$ converges to 0, when $\varepsilon \to 0$. We have

$$\|\nabla y - \nabla y_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega |\nabla y_\varepsilon|^2\, dx - 2\int_\Omega \nabla y\, \nabla y_\varepsilon\, dx + \int_\Omega |\nabla y|^2\, dx. \tag{2.2}$$

We remark that the application of Green's formula to the problem $(P_1)(u_\varepsilon)$ gives

$$\int_\Omega |\nabla y_\varepsilon|^2\, dx = -\varepsilon \int_\Gamma y_\varepsilon^2\, d\Gamma + \int_\Gamma y_\varepsilon\, u_\varepsilon\, d\Gamma.$$

Then, the trace theorem [2] and the continuity of the restriction of the linear map $\mathcal{B}_\varepsilon$ to $\mathcal{U}_{\text{ad}}$ (cf. Lemma 1.1) allow us to assert that $\int_\Omega |\nabla y_\varepsilon|^2\, dx$ converges to $\int_\Gamma y\, u\, d\Gamma$, when $\varepsilon \to 0$. Consequently, $\|\nabla y - \nabla y_\varepsilon\|_{L^2(\Omega)}$ converges, when $\varepsilon \to 0$, to $\int_\Gamma u\, y\, d\Gamma - \int_\Omega |\nabla y|^2\, dx$; this quantity vanishes because y is a solution of $(P_3)(u)$.

Again, by the trace theorem and the continuity of the norm in $L^2(\Gamma)$, we obtain that for all $v \in \mathcal{U}_{\text{ad}}$

$$J(u) = \lim_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) \le \lim_{\varepsilon \to 0} J_\varepsilon(v) = J(v). \tag{2.3}$$

3. CONCLUSIONS

(1) We have established the existence of the state and the control for the perturbed boundary optimal control systems $(P_1)(u_\varepsilon)$ and (P_2), for a functional cost J_ε which is not strictly convex and is defined on the boundary. We have considered $\mathcal{U}_{\mathrm{ad}}$, the space of admissible controls, as an arbitrary linear subspace with finite dimension of the space $\mathcal{U} := \left\{ v \in L^2(\Gamma) : \int_\Gamma v \, d\Gamma = 0 \right\}$. Our results remain valid if $\mathcal{U}_{\mathrm{ad}}$ is replaced by the affine space $w + \mathcal{U}_{\mathrm{ad}}$, where w is an element of $\mathcal{U}$.

It is interesting to look at the questions treated here for other closed convex sets in $L^2(\Gamma)$, for instance, we can replace, again, $\mathcal{U}_{\mathrm{ad}}$ by $\mathcal{W} + \mathcal{U}_{\mathrm{ad}}$, where $\mathcal{W}$ is a closed and bounded convex set in $\mathcal{U}$, and our results are still valid.

(2) Consider the (linear) perturbed boundary optimal control system on a bounded and regular open set in $\mathbb{R}^n$ $(Q_1)(u_\varepsilon)$ and (Q_2):

$$\begin{aligned} -\Delta y_\varepsilon + \beta(x)\, y_\varepsilon = 0, \qquad & \text{on } \Omega, \\ \frac{\partial}{\partial v} y_\varepsilon + \varepsilon\, y_\varepsilon = u_\varepsilon, \qquad & \text{at } \Gamma = \partial\Omega, \end{aligned} \tag{$(Q_1)(u_\varepsilon)$}$$

$$\begin{aligned} & J_\varepsilon(u_\varepsilon) = \min \{ J_\varepsilon(v); \quad v \in \mathcal{U}_{\mathrm{ad}} \}; \\ & J_\varepsilon(v) := \int_\Gamma (y_\varepsilon(v) - z_1)^2 \, d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial v} y_\varepsilon(v) - z_2 \right)^2 d\Gamma; \end{aligned} \tag{(Q_2)}$$

$\mathcal{U}_{\mathrm{ad}} := \mathcal{W} + \mathcal{W}_1$, with $\mathcal{W}_1$ a closed and boundary convex set, $\mathcal{W}$ a linear subspace of finite dimension $(m \geq 1)$ in $L^2(\Gamma)$, and $\beta(x)$ is an essentially bounded and positive function defined on Ω, for which we suppose the existence of $\beta_0 > 0$, such that $\beta(x) \geq \beta_0$ for all $x \in \Omega$. Then, our techniques may be adapted to treat this example, and yield similar results concerning the existence of the control u_ε, the state y_ε, and the study of their behaviour.

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