A CLASS OF FACET PRODUCING GRAPHS FOR VERTEX PACKING POLYHEDRA

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We examine a family of graphs called webs. For integers \( n \geq 2 \) and \( 1 \leq k \leq \frac{1}{2}n \), the web \( W(n, k) \) has vertices \( V_n = \{1, \ldots, n\} \) and edges \( \{(i, j): j = i + k, \ldots, i + n - k, \text{ for } i \in V_n \} \) (sums mod \( n \)). A characterization is given for the vertex packing polyhedron of \( W(n, k) \) to contain a facet, none of whose projections is a facet for the lower dimensional vertex packing polyhedra of proper induced subgraphs of \( W(n, k) \). Simple necessary and sufficient conditions are given for \( W(n, k) \) to contain \( W(n', k') \) as an induced subgraph; these conditions are used to show that webs satisfy the Strong Perfect Graph Conjecture.

Complements of webs are also studied and it is shown that if both a graph and its complement are webs, then the graph is either an odd hole or its complement.

1 Introduction

We consider a finite, undirected graph \( G = (V, E) \) with vertices \( V \) and edges \( E \) and assume that \( E \) contains neither loops nor multiple edges. A vertex packing (stable set, independent set) is any subset \( P \subseteq V \) for which \( u_i, u_j \in P \) implies \( (u_i, u_j) \notin E \). For brevity we will refer to such a set simply as a packing and we will use \( \mathcal{P}_G \) to denote the family of all packings in \( G \). The problem of determining a member of \( \mathcal{P}_G \) of maximum cardinality has been the subject of several recent investigations in combinatorial optimization (see \([4, 14, 15, 16, 21]\)). This problem is “complete” in the sense of Cook [6] and Karp [1'].

By representing a packing \( P \in \mathcal{P}_G \) by its binary incidence vector \( x^P \) (\( x^P_i = 1 \leftrightarrow u_i \in P \)), we define the “packing polyhedron” whose extreme

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points correspond to members of $\mathcal{P}_G$ as
\[ \mathcal{B}_G = \text{conv. hull} \{ x^P : P \in \mathcal{P}_G \} . \]

The anti-blocking theory of Fulkerson \[7, 8\] shows that for any maximal clique (complete subgraph) with vertices $C \subseteq V$ there is a facet
\[ \sum_{v_j \in C} x_j \leq 1 \]
of the polyhedron $\mathcal{B}_G$. Padberg has shown in \[16\] that certain facets of $\mathcal{B}_G$ can be derived from odd holes (chordless cycles of odd length) in $G$. For the odd hole $H$, such a facet is of the form
\[ \sum_{v_j \in H} x_j + \sum_{v_j \in N(H)} \beta_j x_j \leq \lfloor \frac{1}{2} |H| \rfloor , \]
where $\lfloor \cdot \rfloor$ and $N(H)$ denote, respectively, the greatest integer function, the cardinality function and the neighbors of $H$ in $V \setminus H$. The $\beta_j$ are determined by solving a sequence of $|N(H)|$ weighted vertex packing problems on induced subgraphs of $G$. In \[14\] (see also \[1, 10, 18, 25\]) it is shown that the same technique may be used to define facets of $\mathcal{B}_G$ from any facet
\[ \sum_{v_j \in V'} \alpha_j x_j \leq \alpha_0 \]
of $\mathcal{B}_{G'}$, where $G'$ is the subgraph of $G$ induced by $V' \subseteq V$. In other words, $\mathcal{B}_G$ inherits certain of its facets from its induced subgraphs.

It is generally the case that not all facets of $\mathcal{B}_G$ are due to subgraphs of $G$. The full graph $G$ may be facet producing, in the sense that for any $G' \subset G$ induced by $V' \subset V$, the projection $\sum_{v_j \in V'} \alpha_j x_j \leq \alpha_0$ on the facet $\sum_{v_j \in V} \alpha_j x_j \leq \alpha_0$ for $\mathcal{B}_G$ is not a facet for $\mathcal{B}_{G'}$. Examples of such facet producing graphs are odd holes and odd anti-holes (complements of holes). In this paper we examine a class of facet producing graphs which properly subsume these examples.

For integers $n \geq 2$ and $k$, $1 \leq k \leq \frac{1}{2} n$, let $W(n, k)$ denote the graph with vertices $V_n = \{ 1, \ldots, n \}$ and edges
\[ \{(i, j) : j = i + k, \ldots, i + n - k \text{ for } i \in V_n \} . \]
Such graphs will be called webs. The web $W(8, 3)$ is shown in Fig. 1.

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1 As has become customary in the literature, we use halfspaces rather than hyperplanes to denote facets (faces of dimension $n - 1$). Also, here we are generally interested in non-trivial facets (other than $x_j \geq 0$) of $\mathcal{B}_G$.

2 Sums here are taken mod $n$ to lie in the range $1, \ldots, n$. This will be the case whenever we refer to sums of elements of $V_n$.

3 Webs are a special case of the "star-shaped polygons" studied by Turner in \[24\].
It is easy to see that $W(n, k)$ is regular of degree $n - 2k + 1$ and that $W(n, k)$ has exactly $n$ maximum packings of size $k$ given by 
\{i, i + 1, \ldots, i + k - 1\}, \ i \in V_n$. Observe also that $W(n, 1)$ is an $n$-clique and for integer values of $s \geq 2$, $W(2s + 1, s)$ is an odd hole and $W(2s + 1, 2)$ is an odd anti-hole.

In the following section we show that $W(n, k)$ is facet producing for $\mathcal{G} W(n, k)$ if and only if $k > 1$ and $n$ and $k$ are relatively prime; the resulting facet is $\sum_{j \in V_n} x_j \leq k$. Subgraphs of webs are studied in Section 3, where we show that $W(n, k) \supseteq W(n', k')$ as an induced subgraph if and only if $nk' \geq n'k$ and $n(k' - 1) \leq n'(k - 1)$. In Section 4 we examine complements of webs and indicate several implications of this material for perfect graphs.

### 2. Facet producing webs

For $n \geq 1$ and $1 \leq k \leq n$ let $A(n, k)$ denote the $n \times n$ binary matrix given by

$$a_{ij} = \begin{cases} 1 & \text{for } j = i, i + 1, \ldots, i + k - 1 \text{ (sums mod } n) \text{ and } i = 1, \ldots, n, \\ 0 & \text{else.} \end{cases}$$

Notice that when $n \geq 2$ and $1 \leq k \leq \frac{1}{2} n$, $A(n, k)$ is the incidence matrix of maximum cardinality packings with vertices in $W(n, k)$.

**Lemma 2.1.** $A(n, k)$ is invertible if and only if $n$ and $k$ are relatively prime.

**Proof.** (only if) Suppose $n$ and $k$ are not relatively prime and let $g > 1$ be a common divisor of $n$ and $k$. Thus we have $n = gn'$ and $k = gk'$. Choose distinct $i$ and $i'$, $1 \leq i, i' \leq g$, and define $I = \{i, i + k, \ldots, i + (n' - 1)k\}$ and $I' = \{i', i' + k, \ldots, i' + (n' - 1)k\}$, with arithmetic mod $n$. Now the construction of $A(n, k)$ shows that

$$\sum_{i \in I} a_i = \sum_{i' \in I'} a_i = k' \cdot 1_n,$$

where $a_i$ denotes the $i$th row of $A(n, k)$ and $1_n$ is an $n$-vector of 1's. Furthermore, we must have $I \cap I' = \emptyset$, since $i + rk = i' + r'k$ with

\[\text{Proofs of this result also are given in [18,19].}\]
0 < r, r < n' i would imply either i = i' or k divides i - i'. Hence
A(n, k) has linearly dependent rows.

(ii) Consider \( \lambda \in \mathbb{R}^n \) for which \( A(n, k)\lambda = 0_n \). Using pairs of consecutive rows from \( A(n, k) \) we obtain \( \lambda_1 = \lambda_{k+1}, \lambda_2 = \lambda_{k+2}, \ldots, \lambda_n = \lambda_k \). Since \( n \) and \( k \) are relatively prime, \( k \) generates the group of integers mod \( n \). Thus we have \( \lambda_1 = \lambda_{k+1} = \lambda_{2k+1} = \ldots, \) so that \( \lambda_1 = \ldots = \lambda_n \). Since \( A(n, k)\lambda = 0_n \), we conclude that \( \lambda_1 = \ldots = \lambda_n = 0 \). Consequently \( A(n, k) \) has independent columns.

We can now characterize those webs for which the inequality

\[
\sum_{j \in V_n} x_j \leq k
\]

is a facet of \( \mathcal{B}_{\mathcal{W}(n,k)} \).

**Theorem 2.2.** The inequality (1) is a facet of \( \mathcal{B}_{\mathcal{W}(n,k)} \) if and only if \( n \) and \( k \) are relatively prime.

**Proof.** By definition, (1) is a facet of \( \mathcal{B}_{\mathcal{W}(n,k)} \) if and only if \( \mathcal{W}(n, k) \) contains \( n \) maximum packings of cardinality \( k \) with corresponding affinely independent incidence vectors. Since \( A(n, k) \) is the incidence matrix of maximum packings with vertices in \( \mathcal{W}(n, k) \), the former statement is true if and only if \( A(n, k) \) is invertible. The desired conclusion now follows directly from Lemma 2.1.

An interesting alternative proof of sufficiency in Theorem 2.2 is provided by the work of Chvátal [4]. An edge \( e \) of a graph \( G \) is called critical if its removal increases the size of a maximum packing, i.e., if

\[
\max_{P \in \mathcal{P}(G)} |P| < \max_{P \in \mathcal{P}(G \setminus e)} |P|.
\]

Let \( k = \max_{P \in \mathcal{P}(G)} |P| \) in \( G = (V, E) \), where \( |V| = n \) and let \( E^* \) denote the set of critical edges of \( G \). Chvátal has shown that if \( G^* = (V, E^*) \) is connected, then \( \sum_{j \in V} x_j \leq k \) is a facet of \( \mathcal{B}_G \). Now if \( n \) and \( k \) are relatively prime, the path \( 1, k + 1, 2k + 1, \ldots, (n - 1)k + 1, 1 \) is a hamiltonian cycle for \( \mathcal{W}(n, k) \). Since each edge of this cycle is critical, (1) is a facet of \( \mathcal{B}_{\mathcal{W}(n,k)} \).

Theorem 2.2 does not characterize those webs which are actually facet producing, but to do this we need only exclude cliques.
Theorem 2.3. $W(n, k)$ produces the facet (1) for $\mathcal{B}_{W(n, k)}$ if and only if $k > 1$ and $n$ and $k$ are relatively prime.

Proof. (only if) If $n$ and $k$ are not relatively prime, we apply Theorem 2.2 to conclude that (1) is not a facet of $\mathcal{B}_{W(n, k)}$. On the other hand, if $k = 1$, then $W(n, k)$ is an $n$-clique. Let $V \subseteq V_n$, $V \neq \emptyset$, and consider the projection of (1) onto the subspace $\{x \in \mathbb{R}^n : x_j = 0$ for $j \in V\}$. In this case the subgraph of $W(n, 1)$ induced by $V' = V_n \setminus V$ is the clique $W(|V'|, 1)$, so the projection of (1), $\sum_{j \in V'} x_j < 1$, is a facet of $\mathcal{B}_{W(|V'|, 1)}$. Thus $W(n, k)$ is not facet producing.

(if) Clearly (1) is a facet of $\mathcal{B}_{W(n, k)}$. We need only show that each of its projections does not yield a facet of the packing polyhedron for the appropriate subgraph. Let $V \subseteq V_1$, $V \neq \emptyset$, and consider the projection of (1) onto $\{x \in \mathbb{R}^n : x_j = 0$ for $j \in V\}$. The removal of $|V|$ vertices of $W(n, k)$ corresponds to the removal of $|V|$ columns of $A(n, k)$. The resulting submatrix clearly has at most $n - |V|$ rows with $k$ unit entries. Suppose there are exactly $n - |V|$ such rows and let $R$ denote the index set of these rows. Notice that the submatrix of $A(n, k)$ with rows $R$ and columns which correspond to $V_n \setminus V$ must have all row and column sums equal to $k$. Without loss of generality, let $1 \in R$. Then $k > 1$ implies $a_{12} = 1$, which in turn implies $2 \in R$. Similarly, for $i = 3, ..., n$, $a_{i-1,i} = 1$ implies $i \in R$. Hence $R = V_n$, contradicting $V \neq \emptyset$. Consequently the subgraph $G' \subseteq W(n, k)$ induced by $V' = V_n \setminus V$ contains less than $n - |V|$ packings of size $k$. Thus $\sum_{j \in V'} x_j < k$ is not a facet of $\mathcal{B}_{G'}$.

Let $\mathcal{G}$ be a family of subsets of $I = \{1, ..., n\}$ with the property that $I_1 \subset I_2 \in \mathcal{G}$ implies $I_1 \in \mathcal{G}$. The pair $S = (I, \mathcal{G})$ is called an independence system and the members of $\mathcal{G}$ are called independent sets. Let $\mathcal{B}_S$ denote the polyhedron whose extreme points correspond to members of the independence system $S$. Padberg [18] has obtained a result analogous to Theorem 2.2 for the independence system defined by feasible 0-1 solutions to an inequality constrained knapsack problem. He also mentions that a similar construction can be used to obtain facets for polyhedra defined by arbitrary independence systems (see also [1, 10, 14, 25]).

Theorem 2.2 represents such a construction for the independence system $(V_n, \mathcal{P}_{W(n, k)})$. Indeed, since invertibility of $A(n, k)$ implies affine independence of its rows, Lemma 2.1 shows that if $n$ and $k$ are relatively prime and $A(n, k)$ is the incidence matrix of $n$ maximum independent sets with $I$ in $S = (I, \mathcal{G})$, then $\sum_{j=1}^n x_j \leq k$ is a facet of $\mathcal{B}_S$. 
3. Induced subgraphs

The high degree of symmetry in \( W(n, k) \) leads to surprisingly simple necessary and sufficient conditions for \( W(n', k') \) to be an induced subgraph of \( W(n, k) \) (see Theorem 3.3). We will use the notation \( W(n, k) \supseteq W(n', k') \) to indicate that \( W(n', k') \) is an induced subgraph of \( W(n, k) \). Obvious necessary conditions for such containment are \( n \geq n' \) and \( k \geq k' \), which we assume henceforth. The following theorem states that \( W(n, k) \supseteq W(n', k') \) it is both necessary and sufficient that we be able to select \( n' \) vertices from \( V_n \) for which the set of \( k \) consecutive vertices of \( W(n, k) \) beginning with any chosen vertex contains \textit{exactly} \( k' \) chosen vertices.

**Theorem 3.1.** \( W(n, k) \supseteq W(n', k') \) if and only if there exists a subset \( V' = \{i_1, \ldots, i_{n'}\} \subseteq V_n \) with the property that \( |V' \cap \{i_j, i_j + 1, \ldots, i_j + k - 1\}| = k' \) for each \( 1 \leq j \leq n' \).

**Proof.** (if) Given \( V' \) as described, we must also have \( |V' \cap \{i_j, i_j + 1, \ldots, i_j - k + 1\}| = k' \) for each \( 1 \leq j \leq n' \), or else the former condition would fail for an appropriate choice of \( j, 1 \leq j \leq n' \). Thus the subgraph of \( W(n, k) \) induced by \( V' \) is \( W(n', k') \).

(only if) Suppose \( V' = \{i_1, \ldots, i_{n'}\} \subseteq V_n \) induces \( W(n', k') \) and choose \( i_j \in V' \). Also define \( S = \{i_j, i_j + 1, \ldots, i_j + k - 1\} \) and \( T = \{i_j, i_j - 1, \ldots, i_j - k + 1\} \). Since \( S \in \mathcal{P}_{W(n,k)} \), \( S \cap V' \in \mathcal{P}_{W(n', k')} \), so we must have \( |S \cap V'| \leq k' \). Similarly, \( |T \cap V'| \leq k' \). There are exactly \( 2(k' - 1) \) vertices in \( W(n', k') \) which are not incident to \( i_j \). Since \( |S \cap V'| < k' \) implies \( |T \cap V'| > k' \), which contradicts the size of a maximum packing in \( W(n', k') \), we must have \( |S \cap V'| = k' \).

**Corollary 3.2.** \( W(n, k) \supseteq W(n', k') \) if and only if the following system is feasible:

\[
\begin{align*}
\sum_{i=1}^{n} x_i &= n', \\
\sum_{j=1}^{k} x_{i+j} &= k', \\
\sum_{j=1}^{k-1} x_{i+j} &\leq k' - 1,
\end{align*}
\]
(The index sums \( i + j \) are taken mod \( n \)).

Proof. Eq. (2) insures that \(|V'| = n'\), where \( V' = \{i : x_i = 1\} \). If \( i \in V' \), then from (3),

\[
\sum_{j=1}^{k-1} x_{i+j} \leq k' - 1 ,
\]

and from (4),

\[
\sum_{j=1}^{k-1} x_{i+j} \geq k' - 1 .
\]

Hence \(|V' \cap \{i, i + 1, \ldots, i + k - 1\}| = k'\). Thus the existence of a feasible solution to (2)–(5) implies \( W(n, k) \supseteq W(n', k') \), by Theorem 3.1. On the other hand, if \( W(n, k) \supset W(n', k') \), we choose \( V' \) as described in Theorem 3.1 and define \( x \) by \( x_i = 1 \) for \( i \in V' \) and \( x_i = 0 \) otherwise. It is clear that \( x \) satisfies (2), (3) and (5). If \( x_i = 1 \), then \(|V' \cap \{i, i + 1, \ldots, i + k - 1\}| = k'\), so that \( \sum_{j=1}^{k-1} x_{i+j} = k' - 1 \) and (4) is satisfied. If \( x_i = 0 \) and (4) is not satisfied, let \( i' = i - j \) where \( j \) is the least index for which \( i - j \in V' \). In this case

\[
|V' \cap \{i', i' + 1, \ldots, i' + k - 1\}| < k' ,
\]

which contradicts \( i' \in V' \). Hence (4) also holds for \( x \).

By using the system (2)–(5) we obtain the more useful conditions for \( W(n, k) \supseteq W(n', k') \) given by

**Theorem 3.3.** \( W(n, k) \supseteq W(n', k') \) if and only if

\[
\begin{align*}
nk' & \geq n'k , \\
n(k' - 1) & \leq n'(k - 1) .
\end{align*}
\]

Proof. (only if) The first condition of (6) is obtained by summing the \( n \) relations of (3) over \( i \) and then applying (2). The second condition of (6) is obtained similarly using (4).

(If) We define the set \( V' = \{i_1, \ldots, i_n\} \) by

\[^5\text{[s]} \text{ denotes the smallest integer no less than } s.\]
\[ i_j = \lfloor jn/n' \rfloor, \quad 1 \leq j \leq n'. \]

Notice that \( j = n' \) implies \( i_j = n \), so that this process identifies exactly \( n' \) vertices in \( V_n = \{1, \ldots, n\} \). Since \( n/n' \geq 1 \), these vertices are distinct. We will prove that \( V' \) induces \( W(n', k') \) by verifying the conditions of Theorem 3.1. Let \( i_j \in V' \), \( S = \{i_j, i_j + 1, \ldots, i_j + k - 1\} \) and define the integer \( r \) by \( r/n = \lfloor (n'/n) - j \rfloor \). Since \( jn/n' \leq i_j < (jn/n') + 1 \), we must have \( 0 \leq r/n < n'/n \). The definition of \( i_j \) also shows that

\[
|V' \cap S| = 1 + \left\lfloor \frac{r}{n} + (k - 1) \frac{n'}{n} \right\rfloor \leq 1 + \frac{r}{n} + (k - 1) \frac{n'}{n} < 1 + \frac{kn'}{n} \leq 1 + k',
\]

where the last inequality follows from the first condition of (6). On the other hand, the second condition of (6) implies \((k - 1)n'/n \geq k' - 1\), from which we conclude that

\[ k' - 1 \leq \left\lfloor \frac{r}{n} + (k - 1) \frac{n'}{n} \right\rfloor. \]

Combining this with our previous relation yields

\[ k' \leq 1 + \left\lfloor \frac{r}{n} + (k - 1) \frac{n'}{n} \right\rfloor = |V' \cap S| < 1 + k'. \]

Hence \( |V' \cap S| = k' \).

The proof of sufficiency in Theorem 3.3 actually determines the induced subgraph \( W(n', k') \subseteq W(n, k) \) when the conditions (6) are satisfied. This construction may be interpreted as a division of the interval \([0, n]\) into \( n' \) segments, each of length \( n'/n' \). If the right-hand endpoint of the \( j \)th such segment, \( 1 \leq j \leq n' \), lies in the interval \((i - 1, i] \), \( 1 \leq i \leq n \), then we let \( i \in V' \). This is demonstrated in

**Example 3.4.** The graph of Fig. 1 is \( W(8, 3) \). The conditions (6) indicate that \( W(8, 3) \supset W(5, 2) \). Proceeding as described we obtain

\[
\begin{array}{cccccccc}
& & & & & & & \\
\frac{8}{5} & | & \frac{16}{5} & | & \frac{24}{5} & | & \frac{32}{5} & | & \frac{40}{5} \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

and so \( V' = \{2, 4, 5, 7, 8\} \) induces a 5-hole in \( W(8, 3) \). From symmetry considerations we see that \( W(8, 3) \), in fact, contains the eight distinct 5-holes \( \{i, i + 2, i + 3, i + 5, i + 6\} \), for \( 1 \leq i \leq 8 \).
The conditions (6) also imply that the containment indicated in Theorem 3.3 is proper if and only if \( n > n' \) and \( k > k' \), except for cliques.

**Corollary 3.5.** \( W(n, k) \supseteq W(n, k') \) implies \( k = k' \). For \( k > 1 \), \( W(n, k) \supseteq W(n', k) \) implies \( n = n' \).

Since \( n' = \lfloor n/k \rfloor \) is the largest integer satisfying (6) with \( k' = 1 \), we also obtain

**Corollary 3.6.** The size of a maximum cardinality clique in \( W(n, k) \) is \( \lfloor n/k \rfloor \).

The following theorem uses the technique described in [14, 16] to derive facets for \( \mathcal{B}_{W(n,k)} \) from an induced subgraph \( W(n', k') \subseteq W(n, k) \).

**Theorem 3.7.** Suppose \( W(n, k) \supseteq W(n', k') \) and \( W(n', k') \) is induced by \( V' \subseteq V_n \). If \( W(n', k') \) is facet producing, then a facet of \( \mathcal{B}_{W(n,k)} \) is given by

\[
(7) \quad \sum_{j \in V} x_j \leq k'.
\]

**Proof.** Since the size of a maximum packing in \( W(n', k') \) is \( k' \), it is clear that (7) is a support for \( \mathcal{B}_{W(n,k)} \). Thus we need only demonstrate \( n \) packings in \( W(n, k) \) whose incidence vectors are affinely independent.
and satisfy (7) at equality. The maximum packings in \( W(n', k') \) give \( n' \) of these because \( W(n', k) \) is facet producing. The remaining \( n - n' \) packings are chosen as follows. Let \( i \in V_n \setminus V' \) and define \( j \) as the smallest index for which \( i - j = i' \in V' \). By Theorem 3.1, the set

\[
S_i = V' \cap \{i', i' + 1, \ldots, i' + k - 1\}
\]

must satisfy \( |S_i| = k' \) and because \( k' > 1 \) (see Theorem 2.3), \( i \in \{i', i' + 1, \ldots, i' + k - 1\} \). Since \( S_i \cup \{i\} \) is a subset of the packing \( \{i', i' + 1, \ldots, i' + k - 1\} \) in \( W(n, k) \), it follows that \( S_i \cup \{i\} \) is a packing in \( W(n, k) \). Thus the required \( n - n' \) packings are given by \( S_i \cup \{i\} \), \( i \in V_n \setminus V' \).

Notice that Theorem 3.7 shows that the facet (7) of \( \mathcal{B}_{W(n, k)} \) is obtained by the technique described in [14, 16] regardless of the particular ordering chosen for the vertices in \( V_n \setminus V' \). In addition we point out that the stronger assumption that \( W(n', k') \) be facet producing, rather than simply requiring \( n' \) and \( k' \) to be relatively prime, is needed to avoid obvious difficulties with cliques in \( W(n, k) \) which are not maximal in \( W(n, k) \). Of course, if \( W(n', k') \) is a maximal clique in \( W(n, k) \), it is associated with a facet for \( \mathcal{B}_{W(n, k)} \) of the form (7).

4. Complements of webs

By the complement of the graph \( G = (V, E) \) we mean the graph \( \bar{G} = (V, \bar{E}) \), where \( \bar{E} = \{(i, j) : i, j \in V, i \neq j, (i, j) \not\in E\} \). The complement of the web \( W(n, k) \), denoted \( \bar{W}(n, k) \), it not necessarily a web. For example, \( \bar{W}(8, 3) \) is not a web (see Fig. 1), since the size of a maximum packing in \( \bar{W}(8, 3) \) is \( \lfloor 8/3 \rfloor = 2 \) and \( \bar{W}(8, 3) \) is regular of degree \( 4 \neq n - 2k + 1 = 5 \). Of course, odd holes and odd anti-holes have complements which are webs; we now show that these are the only such webs.

**Theorem 4.1.** \( \bar{W}(n, k) \) is a web if and only if \( W(n, k) \) is an odd hole or odd anti-hole

**Proof.** Sufficiency is obvious. To prove necessity note that \( W(n, k) \) is regular of degree \( n - 2k + 1 \) and, since \( \bar{W}(n, k) \) is a web, it is regular of degree \( n - 2\lfloor n/k \rfloor + 1 \) (see Corollary 3.6). Thus
\[(n - 2k + 1) + (n - 2\lfloor n/k \rfloor) + 1 = n - 1,\]

or

\[k + \lfloor n/k \rfloor = \frac{1}{2}(n + 3).\]

We may rewrite the last relation as

\[k + \frac{n}{k} - \frac{r}{k} = \frac{1}{2}(n + 3),\]

where \(r = n \mod k\). Thus \(n\) and \(k\) must satisfy the polynomial relation

\[2k^2 - (n + 3)k + 2(n - r) = 0.\]

Consequently,

\[k = \frac{1}{4}(n + 3) \pm \sqrt{n^2 - 10n + 9 + 16r}.\]

Because \(k\) must be integer-valued, \(n^2 - 10n + 9 + 16r\) must be a perfect square. This implies \(r = 1\). Thus we have either \(k = \frac{1}{2}(n - 1)\) or \(k = 2\). In the former case \(W(n, k)\) is an odd hole and in the latter, an odd anti-hole (note \(k = 2\) and \(n\) even violates \(k + \lfloor n/k \rfloor = \frac{1}{2}(n + 3)\)).

Complements of \(d\) facets for the packing polyhedra.

**Theorem 4.2.** If \(W(n, k)\) is facet producing, then

\[(8) \quad \sum_{j \in V_n} x_j < \lfloor n/k \rfloor\]

is a facet of \(\mathfrak{W}_{W(n,k)}\).

**Proof.** Consider the maximum clique in \(W(n, k)\) given by the vertex set \(C_1 = \{1, 1 + k, \ldots, 1 + (\lfloor n/k \rfloor - 1)k\}\). Since \(W(n, k)\) is facet producing, we have \(\lfloor n/k \rfloor k < n\) (see Theorem 2.3). Subtracting \(k - 1\) from both sides of this relation yields \(1 + (\lfloor n/k \rfloor - 1)k < n - k + 1\). Consequently, the vertex set \(C_1 \cup \{n\}\{1\}\) is also a clique in \(W(n, k)\). Thus the edge \((n, 1)\) in \(W(n, k)\) is critical. By the same argument we conclude that each of the edges \((i, i + 1)\) for \(i \in V_n\) is critical in \(W(n, k)\). Thus (see the discussion following Theorem 2.2) (8) is a facet of \(\mathfrak{W}_{W(n,k)}\).

The subject of perfect graphs has stimulated much recent research in extremal combinatorics (see, for example, \([2, 3, 5, 7, 8, 9, 12, 13, 17, 20, 22, 23]\)). For the graph \(G\) let \(\pi(G)\) be the size of a minimum partition of
the vertices of $G$ into cliques and let $\omega(G)$ be the size of a maximum vertex packing in $G$. $G$ is called \textit{\pi-perfect} if $\omega(H) = \pi(H)$ for each induced subgraph $H \subseteq G$. \textit{\gamma-perfection} is defined similarly using $\gamma(G) = \pi(\overline{G})$, the size of a minimum partition of the vertices of $G$ into packings, and $\lambda(G) = \omega(\overline{G})$, the size of a maximum clique in $G$. A graph which is both \gamma-perfect and \pi-perfect is called \textit{perfect}. The \textit{perfect graph theorem} asserts that \gamma-perfection is equivalent to perfection for $G$ (see Lovász [12, 13] and Fulkerson [7, 8, 9]).

Vertex minimal imperfect graphs have been called \textit{critical} (see Sachs [20]). The \textit{strong perfect graph conjecture} [2] asserts that the only critical graphs are odd holes and odd anti-holes or, equivalently, that $G$ is perfect if and only if no induced subgraph $H \subseteq G$ is an odd hole or odd anti-hole. This conjecture has been proven true for planar graphs by Tucker [23]. Since the polyhedron

$$\{x \in \mathbb{R}^n : Ax \leq 1_m, x \geq 0_n\},$$

where $A$ is an $m \times n$ binary matrix, each of whose rows is essential, has integer extreme points if and only if $A$ is the incidence matrix of maximal cliques with vertices in a perfect graph (see [9]), the above conjecture, if true, would provide an interesting graphical characterization of certain integer polyhedra.

Facet producing webs share many properties with critical graphs. Let $G$ be a critical graph with $n$ vertices for which $\omega(G) = k$. Then it can be shown that $\lambda(G) = (n - 1)/k = \lfloor n/k \rfloor$. Note that this implies $n$ and $k$ are relatively prime. Since a graph is perfect if and only if the non-trivial facets of its packing polyhedron correspond to its maximal cliques (see Fulkerson [9]), $G$ is facet producing. Padberg [17] has shown that every vertex of $G$ must be in exactly $k$ maximum packings in $G$ and also that $\sum_{j=1}^n x_j \leq k$ is a facet of $\mathcal{B}_G$. Sachs [20] points out that $\deg(v) \leq n - 2k + 1$ for each vertex of a critical graph, where $\deg(v)$ denotes the degree of $v$ in $G$. Since $G$ is critical if and only if $\overline{G}$ is critical, this implies that $\overline{\deg}(v) \leq n - (2(n - 1)/k) + 1$, where $\overline{\deg}(v)$ denotes the degree of $v$ in $\overline{G}$. Now $n - 1 = \deg(v) + \overline{\deg}(v)$, so that these relations imply that $n$ and $k$ must satisfy the following polynomial inequality when $G$ is critical:

$$2k^2 - (n + 3)k + 2(n - 1) \leq 0.$$  

(9)

It is interesting to note that the zeroes of this polynomial correspond to odd holes and odd anti-holes (see the proof of Theorem 4.1). There are, however, other pairs of integers $(n, k)$ which satisfy (9) that
could also correspond to critical graphs. The "smallest" such pair of interest is \((10, 3)\). Does there exist a 10-vertex counterexample to the strong perfect graph conjecture? If so, this graph is not \(W(10, 3)\), since \(W(10, 3) \supset W(5, 2)\), as can be seen by Theorem 3.3. In fact, the only critical webs are odd holes and odd anti-holes.

**Theorem 4.3.** For \(n \geq 8\) and \(2 < k < \lfloor \frac{1}{2} n \rfloor\), \(W(n, k)\) contains an odd hole or odd anti-hole as an induced subgraph.

**Proof.** Theorem 3.3 shows that

\[
W(n, k) \supset W(2s + 1, 2) \iff \begin{cases} 
2n \geq (2s + 1)k, \\
n \leq (2s + 1)(k - 1),
\end{cases}
\]

and

\[
W(n, k) \supset W(2t + 1, t) \iff \begin{cases} 
nt \geq (2t + 1)k, \\
n(t - 1) \leq (2t + 1)(k - 1),
\end{cases}
\]

Letting \(s = 2, 3, \ldots\) in (10), we see that \(W(n, k)\) contains an odd anti-hole if and only if \(k\) is in some interval

\[
\left[ \left\lfloor \frac{n}{5} + 1, \frac{2n}{5} \right\rfloor, \left\lfloor \frac{n}{7} + 1, \frac{2n}{7} \right\rfloor \right], \ldots, \left[ \frac{n}{2s + 1} + 1, \frac{2n}{2s + 1} \right], \ldots.
\]

Similarly, for \(t = 2, 3, \ldots\) in (11), \(W(n, k)\) contains an odd hole if and only if \(k\) is in some interval

\[
\left[ \left\lfloor \frac{n}{5} + 1, \frac{2n}{5} \right\rfloor, \left\lfloor \frac{2n}{7} + 1, \frac{3n}{7} \right\rfloor \right], \ldots, \left[ \frac{(t - 1)n}{2t + 1} + 1, \frac{tn}{2t + 1} \right], \ldots.
\]

To prove the theorem we will show that the intervals (12), (13) cover the integer values of \(k\) in the range \(2 < k < \lfloor \frac{1}{2} n \rfloor\). We first establish upper limits on \(s\) and \(t\) for fixed \(n\). The lower limit of the intervals of (12) is a decreasing function of \(s\). Thus, in order to cover \(k = 3\) we must choose \(s\) large enough so that \(n/(2s + 1) + 1 \leq 3\). This implies \(s \geq \frac{1}{2}(n - 2)\).

Similarly, \(n\) odd and \(k < \lfloor \frac{1}{2} n \rfloor\) implies \(k \leq \frac{1}{2}(n - 3)\) while \(n\) even and \(k < \lfloor \frac{1}{2} n \rfloor\) implies \(k \leq \frac{1}{2}(n - 2)\). Thus (13) shows that we must have \(tn/(2t + 1) \geq \frac{1}{4}(n - 2)\). This implies that \(t \geq \frac{1}{4}(n - 2)\). Thus we need only consider \(2 \leq s, t \leq \frac{1}{4}(n - 2)\).

---

\(^6\) A slightly different proof of this theorem has been given by G. Nemhauser and T. King (private communication).
The overlap of successive intervals in (12) is a decreasing function of $s$. To see this, for $s \geq 2$ let $f(s)$ denote this value for the intervals of (12) corresponding to $s$ and $s+1$; i.e., define

$$f(s) = \frac{2n}{2s+3} - \frac{n}{2s+1} - 1, \quad s = 2, 3, \ldots.$$ 

Since the first derivative of $f(s)$ is nonpositive for $s \geq 2$, $f$ is decreasing for $s \geq 2$. A similar situation holds for the intervals of (13). Here we define

$$g(t) = \frac{tn}{2t+1} - \frac{tn}{2t+3} - 1, \quad t = 2, 3, \ldots,$$

and note that $g$ is decreasing for $t \geq 2$.

Since $f(s)$ and $g(t)$ are decreasing functions and we have $s, t \leq \frac{1}{4}(n - 2), \frac{1}{4}(n + 2)$, it suffices to show that $f\left(\frac{1}{4}(n - 2)\right) \geq 0$ and $g\left(\frac{1}{4}(n - 2)\right) \geq 0$. By direct substitution we obtain after simplification that

$$f\left(\frac{1}{4}(n - 2)\right) \geq 0 \iff n \geq 12$$

and

$$g\left(\frac{1}{4}(n - 2)\right) \geq 0 \iff n \geq 8.$$ 

Thus the theorem holds for $n \geq 12$.

For $8 \leq n \leq 11$, Theorem 3.3 shows that $W(8, 3)$, $W(9, 3)$, $W(10, 3)$, $W(10, 4)$ and $W(11, 4)$ contain $W(5, 2)$ and $W(11, 3) \supset W(7, 2)$.

In view of Theorem 4.3 (or Theorem 4.1), the strong perfect graph conjecture is equivalent to the statement that a critical graph must be a web. A similar equivalence is provided in [51]. Theorem 4.3 also implies

**Corollary 4.4.** The only perfect webs are, for appropriate values of $n$,

- $W(n, 1)$ (cliques),
- $W(2n, 2)$ (even anti-holes),
- $W(2n, n)$ (pairwise disjoint edges).

Even though all webs except those given in Corollary 4.4 contain odd holes or their complements, it is interesting that if we exclude $W(5, 2)$ (which is both a hole and an anti-hole) no web contains both odd holes and odd anti-holes.
Theorem 4.5. If \( W(n, k) \supseteq W(2s + 1, 2) \) and \( W(n, k) \supseteq W(2t + 1, t) \), then either \( s = 2 \) or \( t = 2 \).

Proof. From (10) and (11) we see that
\[
(s + \frac{1}{2})k \leq n \leq (2s + 1)(k - 1)
\]
and
\[
\left(2 + \frac{1}{t}\right)k \leq n \leq \frac{2t + 1}{t - 1}(k - 1).
\]
Note that
\[
s \geq 2 \Rightarrow s + \frac{1}{2} \geq 2\frac{1}{2}, 2s + 1 \geq 5
\]
and
\[
t \geq 2 \Rightarrow 2 + \frac{1}{t} \leq 2\frac{1}{2}, \frac{2t + 1}{t - 1} \leq 5.
\]
Thus the above restrictions are equivalent to
\[
(14) \quad (s + \frac{1}{2})k \leq n \leq \frac{2t + 1}{t - 1}(k - 1).
\]
Now if \( s > 2 \) and \( t > 2 \), \((s + \frac{1}{2})\) is minimized by \( s = 3 \) and \((2t + 1)/(t - 1)\) is maximized by \( t = 3 \). In this case, if (14) is to be satisfied, we must have
\[
\frac{7}{2}k \leq n \leq \frac{7}{2}(k - 1),
\]
which is impossible. Thus we must have either \( s = 2 \) or \( t = 2 \).

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