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An existence theorem for stationary discs in almost complex manifolds

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Abstract

An existence theorem for stationary discs of strongly pseudo-convex domains in almost complex manifolds is proved. More precisely, it is shown that, for all points of a suitable neighborhood of the boundary and for any vector belonging to certain open subsets of the tangent spaces, there exists a unique stationary disc passing through that point and tangent to the given vector. This result gives a generalization of a theorem of B. Coupet, H. Gaussier and the second author, originally proved only for almost complex structures which are small deformations of an integrable one.

Keywords: Almost complex manifolds; Stationary discs

0. Introduction

Analysis on almost complex manifolds recently became a rapidly growing research area in modern geometric analysis, due to the impulse given by the fundamental paper of Gromov [4], where deep connections between almost complex and symplectic structures have been discovered. Strongly pseudoconvex domains in almost complex manifolds play a substantial role in Gromov's approach. However, many basic questions on this class of domains are still open. The aim of our paper is to study a class of invariants of these domains, which generalize the so-

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called stationary discs, introduced by L. Lempert for analyzing the Kobayashi metric of bounded domains in $\mathbb{C}^n$.

The family of stationary discs consists of a special class of holomorphic discs that are attached to the boundary of a bounded domain. They were considered for the first time in the celebrated paper [10]. In that paper, Lempert showed that the extremal discs for the Kobayashi metric of a strongly convex domain $D \subset \mathbb{C}^n$, coincide with the stationary discs, i.e., with the holomorphic discs admitting a meromorphic lift to the cotangent bundle $T^*\mathbb{C}^n$, with the boundary attached to the conormal bundle of ∂D and with exactly one pole of order one at the origin. Lempert's idea of stationary discs turned out to be a very important and fruitful tool in a variety of topics of research, like e.g. Kobayashi metric, boundary regularity problems, moduli spaces of bounded domains, complex Monge–Ampère equations.

Many authors developed the theory of stationary discs. We limit ourselves to mention the recent Tumanov's paper [13], where a very simple definition of stationary discs (equivalent to Lempert's definition) is given and is immediately generalizable to the almost complex setting. The wide range of applicability of the stationary discs, even for questions non-related to the boundary of pseudoconvex domains, is made evident f.i. in [13] and it represents one of the main motivations for studying them also in the almost complex case.

The concept of stationary discs of a domain D in an almost complex manifold (M, J) (i.e., of J-holomorphic maps $f : \bar{\Delta} \subset \mathbb{C} \to D$, with $f(\partial\Delta) \subset \partial D$ and admitting a lifted map $\hat{f} : \bar{\Delta} \to T^*M$ with the same properties of Lempert's stationary discs) was used for the first time in [1]. In that paper, the authors proved that if D admits a diffeomorphism $\phi : D \to B^n \subset \mathbb{C}^n$ so that $J' = \phi_*(J)$ is a sufficiently small deformation of the standard complex structure J_{st}, then for any $p_o \in D$ and any $v \in T_{p_o}D$, there exists a unique stationary disc $f : \Delta \to D$, with $f(0) = p_o$ and $f_*(\frac{\partial}{\partial x}) \in \mathbb{R}v$. Moreover, using the stationary discs passing through a given point p_o, for any domain D with the above properties, they also proved the existence of a natural diffeomorphism $\Phi : D \to B^n \subset \mathbb{C}^n$, with $\phi(p_o) = 0$, in perfect analogy with what occurs for the convex domains in $\mathbb{C}^n$. Such diffeomorphism $\Phi : D \to B^n$ is called *(generalized) Riemann map* and it is a biholomorphic invariant of (D, p_o). In fact, if $f : D \to D'$ is a (J, J')-biholomorphism, $\mathcal{C}^1$-smooth up to the boundary, between D and another pseudoconvex domain $D' \subset M'$, then the Riemann map $\phi' : D' \to B^n$ of $(D', f(p_o))$ is exactly the map $\phi' = f \circ \phi$.

In this paper, we are interested in the existence and uniqueness problem for stationary discs of a strongly pseudoconvex domain D in an almost complex manifold (M, J) with no further assumptions on J or D. Our main result is the following.

Theorem 0.1. *Let D be a smoothly bounded strongly pseudoconvex domain in an almost complex manifold (M, J). Then there exists a neighborhood $\mathcal{U}$ of the boundary $\Gamma = \partial D$ such that for any $q \in \mathcal{U} \cap D$ and any vector v of a suitable open neighborhood of a codimension one complex subspace $V_q \subset T_qM$, there exists a unique stationary J-holomorphic disc $f : \bar{\Delta} \to \mathcal{U} \cap \bar{D}$, with $f(\partial\Delta) \subset \Gamma$ and with a lift $\hat{f}$ on T^*M, Hölder continuous up to $\partial\Delta$, so that $f(0) = q$ and $df(\frac{\partial}{\partial x}|_0) \in \mathbb{R}v$.*

We want to point out that, by definition, the lift $\hat{f} : \bar{\Delta} \to T^*M$ of a stationary disc $f : \bar{\Delta} \to D$ (see Definition 1.3 below) maps $\partial\Delta$ into the so-called "conormal bundle" of ∂D, which is a totally real submanifold of T^*M [12]. By the reflection principle for pseudoholomorphic curves [7], one immediately obtains that the stationary discs of this theorem are actually smooth up to the boundary.

In addition, the proof gives precise information on the complex subspaces $V_q \subset T_q M$, $q \in \mathcal{U} \cap D$. In fact, let us fix a point $p \in \Gamma = \partial D$ and let $[q_0, p]$ be a segment with endpoints p and $q_0 \in D$ which is orthogonal to ∂D with respect to some Riemannian metric on M. Then there exist a neighborhood $\mathcal{U}$ of ∂D, which satisfies the claim of theorem, and a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of p on which:

(a) there is a system of coordinates centered at p;
(b) Γ and J have coordinate expressions in the so-called "standard form" (for the definition, see Section 2).
(c) $[q_0, p] \cap \mathcal{U}'$ is a part of the coordinate axis $z^1 = \cdots = z^{n-1} = \operatorname{Im} z^n = 0$.

Under the identification of $\mathcal{U}'$ with an open subset of $\mathbb{C}^n$ determined by such coordinates, *for any* $q \in [q_0, p] \cap \mathcal{U}'$ *the subspaces* V_q *are given by the complex* $(n-1)$*-dimensional spaces parallel* (*in* $\mathbb{C}^n$) *to the holomorphic tangent space of* ∂D *at* $p = 0$. Of course, a different choice of the considered Riemannian metric corresponds to different possibilities for subspaces V_q.

The main ingredient of our proof consists in a scaling argument, which allows to consider the Riemann–Hilbert problem defining the small stationary discs, as a continuous deformation of the Riemann–Hilbert problem which determines the stationary discs of a suitable "osculating" almost complex structure. After proving the existence of stationary discs for the osculating structure, the result follows as a direct application of the Implicit Function Theorem. We should mention that the "osculating" almost complex structure which occurs in our proof is an object with a relevance of its own and we refer the reader to [3] for a deeper analysis of its properties.

We remark here that, in the standard case, the results of Lempert [10] and M.-Y. Pang [11] give the existence of stationary discs attached to "convexifiable" regions of ∂D. This gives immediately the existence of stationary discs attached to sufficiently small open sets of the boundary of a strongly pseudoconvex domain. Our result can be considered as a natural extension of this fact in the almost complex setting.

Secondly, we should warn the reader that it is not possible to obtain our main theorem as a corollary of the results in [1]. In fact, in order to use the existence theorem of that paper, one should show that for any point p_o, sufficiently close to ∂D, it is possible to find a system of coordinates in which the expressions of J and of ∂D are well approximated by those of J_{st} and ∂B^n at the same time. But one can find such systems of coordinates only if some additional hypotheses on J are assumed (e.g. the vanishing of the Nijenhuis tensor at some boundary point).

An immediate application of our Theorem 0.1 is the non-trivial fact, recently proved in [6], that for any point p_o of a strongly pseudoconvex hypersurface Γ there exists a one-sided neighborhood that can be filled by J-holomorphic discs with boundaries attached to Γ. Furthermore, the following analogue of Fefferman's theorem can be obtained as direct corollary of Theorem 0.1.

Corollary 0.2. *Let* $D \subset M$ *and* $D' \subset M'$ *be two* C^∞ *smoothly bounded, strongly pseudoconvex domains in two almost complex manifolds* (M, J) *and* (M, J'). *Then any biholomorphic map* $f : D \to D'$ *of class* $\mathcal{C}^1(\bar{D})$ *is* $\mathcal{C}^\infty$ *smooth up to the boundary.*

This result was first proved in [1] under the additional assumption that J is small deformation of an integrable structure (this last hypothesis is not explicitly written in the statement of [1, Theorem 3], but it is constantly assumed as true in all that paper). We also remark that Corollary 0.2

holds also if the hypothesis on the boundary $\mathcal{C}^1$-smoothness of f is removed. This stronger result has been proved in [2,3] with quite different methods.

Beside Corollary 0.2, we expect that our result can be used to extend to the almost complex setting many other classical results on the regularity of CR maps. In our opinion, this problem certainly deserve further investigations. It would be also quite interesting and useful a characterization of the stationary discs in terms of some extremality property, analogous with what occurs in the classical case [10,13]. Such a result would imply the invariance of the stationary discs of pseudoconvex domains under biholomorphisms with boundary regularity weaker than $\mathcal{C}^1$.

The structure of the paper is as follows: after a preliminary section, in Section 2 we give the concept of osculating structures and prove the existence of a continuous deformation between the Riemann–Hilbert problem for stationary discs of the given almost complex structure J and attached to a given boundary ∂D and the corresponding problem for the osculating structure. In Section 3 we give the explicit expressions for the Riemann–Hilbert problem determined by the osculating structure and in Section 4 we prove the main theorem by showing that the deformation described in Section 2 satisfies the hypothesis of the general Implicit Function Theorem.

Everywhere in this paper, we will denote by J_{st} the standard complex structure of $\mathbb{C}^n$, given by the multiplication by $i = \sqrt{-1}$, and by Δ the unit disc in $\mathbb{C}$.

1. Preliminaries

1.1. Almost complex structures

Let (M, J) be an almost complex manifold, i.e., a manifold M endowed with an almost complex structure J. We remind that an almost complex structure J is a smooth tensor field of type $(1, 1)$, such that $J_x^2 = -\mathrm{Id}$ at any point $x \in M$.

We recall that if M is a complex manifold (i.e., a manifold endowed with an atlas of complex coordinate charts $\xi : \mathcal{U} \subset M \to \mathbb{C}^n$ that overlap biholomorphically), then it is naturally endowed with an almost complex structure J. In fact, it suffices to consider the tensor field J which, in any holomorphic system of coordinates $\xi = (z^1, \dots, z^n)$, is defined by

$$J_x\left(\frac{\partial}{\partial z^i}\right) = \sqrt{-1}\,\frac{\partial}{\partial z^i}. \tag{1.1}$$

An almost complex structure J is called an (*integrable*) *complex structure* if it is the almost complex structure defined by means of (1.1) by an atlas on M of complex coordinate charts that overlap biholomorphically.

The standard complex structure of $\mathbb{C}^n$ (i.e., the almost complex structure determined by the standard coordinates of $\mathbb{C}^n$) will be always denoted by J_{st}.

Given two almost complex manifolds (M, J) and (M', J'), a map $f : M \to M'$ is called (J, J')-*holomorphic* if $df \circ J_x = J' \circ df_x$ at any $x \in M$. It is well known that, in case M and M' are complex manifolds, then f is holomorphic if and only if the expression of f in two systems of holomorphic coordinates for M and M' satisfies the classical Cauchy–Riemann equations.

Any (J_{st}, J)-holomorphic map $f : \Delta \to M$ from the unit disc to an almost complex manifold (M, J) is usually called a J-*holomorphic disc*.

It is well known that for any complex manifold M, the cotangent bundle T^*M has a natural structure of complex manifold and the complex structures J and $\mathbb{J}$ of M and T^*M, respectively, are such that the natural projection $\pi : T^*M \to M$ is holomorphic. This property has been generalized by Ishihara and Yano in [5] for any almost complex manifold (M, J) and its cotangent bundle T^*M. We summarize the key points of Ishihara and Yano's result in the next proposition. In the following, given a system of coordinates $\xi = (x^1, \dots, x^{2n}) : \mathcal{U} \subset M \to \mathbb{R}^{2n}$ on M, we call "associated system of coordinates on $\pi^{-1}(\mathcal{U}) \subset T^*M$" the set of coordinates $\hat{\xi} = (x^1, \dots, x^{2n}, p_1, \dots, p_{2n})$, which associate to any $\alpha = p_i\, dx^i \in T^*_x M \subset \pi^{-1}(\mathcal{U})$ the coordinates (x^i) of the point x and the components (p_i) of α with respect to the basis dx^i. Moreover, if $J^i_j : \mathcal{U} \to \mathbb{R}$ are the real functions which give the components of a $(1,1)$ tensor $J = J^i_j \frac{\partial}{\partial x^i} \otimes dx^j$, we denote by $J^i_{j,k}$ the partial derivatives $J^i_{j,k} \overset{\text{def}}{=} \frac{\partial J^i_j}{\partial x^k}$.

Proposition 1.1. [5] *For any almost complex manifold (M, J), there exists a unique almost complex structure $\mathbb{J}$ on T^*M with has the following properties*:

(i) *the projection $\pi : T^*M \to M$ is $(\mathbb{J}, J)$-holomorphic*;
(ii) *for any (J, J')-biholomorphism $f : M \to N$ between two almost complex manifolds (M, J) and (N, J'), the natural lifted map $\hat{f} : T^*N \to T^*M$ is $(\mathbb{J}', \mathbb{J})$-holomorphic*;
(iii) *if J is a complex structure, then $\mathbb{J}$ coincides with the natural complex structure of T^*M*;
(iv) *for any system of coordinates on the cotangent bundle, which is associated with some coordinates $\xi = (x^1, \dots, x^{2n}) : \mathcal{U} \subset M \to \mathbb{R}^{2n}$ on M, if J^i_j denote the components of J in the coordinate basis, for any $\alpha \in T^x M^*$, $x \in \mathcal{U}$,*

$$\mathbb{J}|_\alpha = J^a_i(x)\, dx^i \otimes \frac{\partial}{\partial x^a} + J^a_i(x)\, dp_a \otimes \frac{\partial}{\partial p_i} + \frac{1}{2} p_a \bigl(-J^a_{i,j} + J^a_{j,i} + J^a_\ell \bigl(J^\ell_{i,m} J^m_j - J^\ell_{j,m} J^m_i\bigr)\bigr)\, dx^i \otimes \frac{\partial}{\partial p_j}. \tag{1.2}$$

The almost complex structure $\mathbb{J}$ will be called the *canonical lift of J on T^*M*.

1.2. Hypersurfaces and holomorphic distributions

Let $S \subset M$ be a submanifold of an almost complex manifold (M, J). The *J-invariant* (or *J-holomorphic*) *distribution of S* is the collection of subspaces $\mathcal{D}_x \subset T_x S$, $x \in S$, defined by

$$\mathcal{D}_x = \bigl\{ v \in T_x S:\ J(v) \in T_x S \bigr\}. \tag{1.3}$$

Notice that the real subspaces $\mathcal{D}_x$ are given by the real parts of the vectors $V \in T^{\mathbb{C}}_x S \subset T^{\mathbb{C}}_x M$ such that $JV = iV$. If we consider the subbundles $T^{(1,0)}M$ and $T^{(0,1)}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$, given by the $(+i)$- and $(-i)$-eigenspaces of J in the complexified tangent spaces $T^{\mathbb{C}}_x M$, $x \in M$, we may also say that $\mathcal{D}_x$ is given by the real parts of the vectors in the subspaces

$$\mathcal{D}^{\mathbb{C}}_x = T^{\mathbb{C}}_x S \cap T^{(1,0)}_x M.$$

Assume now that Γ is a hypersurface in M. Notice that, in case Γ is (locally) defined as the zero set of a smooth real-valued function ρ (i.e., $\Gamma = \{x \in M:\ \rho(x) = 0\}$), then the 1-form on M defined by

$$\vartheta_x = (d\rho \circ J)|_{T_p \Gamma},$$

is so that

$$\ker \vartheta|_x = \mathcal{D}_x$$

for any $x \in M$. We will call any such form a *defining* 1*-form for* $\mathcal{D}$ and we will call *Levi form at* $x \in \Gamma$ *associated with* ϑ the quadratic form

$$\mathcal{L}_x(v) \stackrel{\text{def}}{=} -d\vartheta_x(v, Jv), \quad \text{for any } v \in \mathcal{D}_x \subset T_x\Gamma.$$

Notice that, for any vector field X with values in $\mathcal{D}$ such that $X_x = v$, we have that

$$\begin{aligned}\mathcal{L}_x(v) &= -d\vartheta_x(X, JX) = -X\big(\vartheta(JX)\big)\big|_x + JX\big(\vartheta(X)\big)\big|_x + \vartheta_x\big([X, JX]\big)\\ &= \vartheta_x\big([X, JX]\big),\end{aligned} \tag{1.4}$$

because $\vartheta(X) \equiv \vartheta(JX) \equiv 0$.

Notice also that, up to multiplication by a non-zero real number, $\mathcal{L}_x$ does not depend on ϑ. Moreover, by polarization formula, we may say that $\mathcal{L}_x$ is the quadratic form associated with the symmetric bilinear form $(\mathbb{L})^s_x$, where $\mathbb{L}_x$ is the bilinear form defined by

$$\mathbb{L}_x : \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R}, \qquad \mathbb{L}_x(v, w) = -d\vartheta_x(v, Jw)$$

and $(\mathbb{L})^s_x$ is the symmetric part of $\mathcal{L}_x$, i.e.,

$$(\mathbb{L}_x)^s(v, w) = -\frac{1}{2}\big(d\vartheta_x(v, Jw) + d\vartheta_x(w, Jv)\big).$$

If J is an integrable complex structure, the bilinear form $\mathbb{L}_x$ is symmetric and it coincides with $(\mathbb{L}_x)^s$.

Definition 1.2. We say that an oriented hypersurface $\Gamma \subset M$ is *strongly pseudoconvex* if the Levi form $\mathcal{L}_x$ is positive definite at every point $x \in \Gamma$, for some $\mathcal{L}_x$ associated to a (local) defining function ρ of Γ such that $d\rho_x(n) < 0$ for any vector $n \in T_xM$, which is considered as pointing outwards according to the orientation of Γ.

In all the following, when Γ is oriented, we will consider only local defining functions satisfying $d\rho|_\Gamma(n) < 0$ for any outwards pointing vector field n.

We conclude this section by recalling the definition of *conormal bundle* $\mathcal{N}^*(\Gamma)$ *of a hypersurface* $\Gamma \subset M$. Consider the submanifold $\pi^{-1}(\Gamma) \subset T^*M$. The conormal bundle $\mathcal{N}^*(\Gamma)$ is defined as the subbundle of $\pi^{-1}(\Gamma)$ given by

$$\mathcal{N}^*(\Gamma) \stackrel{\text{def}}{=} \big\{\alpha \in T^*_xM,\ x \in \Gamma\colon \alpha|_{T_x\Gamma} \equiv 0\big\} \subset \pi^{-1}(\Gamma) \subset T^*M.$$

1.3. Stationary discs of real hypersurfaces in an almost complex manifold

Let (M, J) be an almost complex manifold. Notice that, on any tangent space T_xM and any cotangent space T^*_xM, we may consider the following natural action of $\mathbb{C}$. For any $a + ib \in \mathbb{C}$, $v \in T_xM$ and $\alpha \in T^*_xM$ we set

$$(a + ib) \cdot v \stackrel{\text{def}}{=} av + bJ(v), \qquad (a + ib) \cdot \alpha \stackrel{\text{def}}{=} a\alpha + bJ^*(\alpha), \tag{1.5}$$

where J^* is the complex structure of T^*_xM defined by $J^*(\alpha)(\cdot) \stackrel{\text{def}}{=} \alpha(J(\cdot))$. Notice also that such an action of $\mathbb{C}$ commutes with the push-forwards and pull-backs of holomorphic maps. More precisely, if $f : M \to N$ is a holomorphic map between two almost complex manifolds (M, J) and (N, J'), then

$$f_*(\zeta \cdot v) = \zeta \cdot f_*(v), \qquad f^*(\zeta \cdot \alpha) = \zeta \cdot f^*(\alpha),$$

for any $v \in TM$ and any $\alpha \in T^*N$.

We may now introduce the crucial concept of “stationary disc” in almost complex manifolds.

Definition 1.3. [1,13] A continuous map $f:\bar{\Delta}\to M$ is called *disc attached to a hypersurface* $\Gamma\subset M$ if $f|_{\Delta}$ is J-holomorphic and $f(\partial\Delta)\subset\Gamma$.

An attached disc f is called a *stationary disc of* Γ (or, shortly, Γ-stationary) if there exists a continuous map $\hat{f}:\bar{\Delta}\to T^*M$ such that:

(a) it projects onto f, i.e., $\pi\circ\hat{f}=f$;
(b) $\hat{f}|_{\Delta}$ is $\mathbb{J}$-holomorphic;
(c) $\zeta^{-1}\cdot\hat{f}(\zeta)\in\mathcal{N}^*(\Gamma)\setminus\{\text{zero section}\}$ for any $\zeta\in\partial\Delta$, where “$\cdot$” denotes the action of $\mathbb{C}$ on T^*M defined in (1.5).

If f is stationary, any map $\hat{f}$ satisfying (a)–(c), is called a ($\mathbb{J}$-*holomorphic*) *lift of* f.

Remark. We have to mention that, in the literature on stationary discs, what is usually called “lift” is the meromorphic map $f^*(\zeta)=\zeta^{-1}\cdot\hat{f}(\zeta)$ and not the map $\hat{f}$ (see e.g. [10,13]). It goes without saying that there is a natural 1-1 correspondence between the two kinds of lifts, but, in the following arguments, our new terminology turns out to be slightly more efficient.

We conclude this preliminary section, recalling that the condition for a map $f:\Delta\to\mathcal{U}$ to be J-holomorphic is equivalent to the condition

$$J\circ df\left(\frac{\partial}{\partial x}\right)=df\left(J_{\mathrm{st}}\left(\frac{\partial}{\partial x}\right)\right). \tag{1.6}$$

In fact, the necessity is obvious, while the sufficiency follows from the following argument (see e.g. [6]): if (1.6) holds, by considering the linear maps df_ζ and $J_{f(\zeta)}$ as extended to $\mathbb{C}$-linear maps between the complexified tangent spaces and observing that any vector $v\in T_\zeta\mathbb{C}$ can be expressed as $v=(a+ib)\partial_x$ for some $a+ib\in\mathbb{C}$, we have that for any v

$$J\circ df(v)=(a+ib)\cdot\left(J\circ df\left(\frac{\partial}{\partial x}\right)\right)\overset{(1.6)}{=}(a+ib)\cdot df\left(J_{\mathrm{st}}\left(\frac{\partial}{\partial x}\right)\right)=df\circ J_{\mathrm{st}}(v),$$

which is the J-holomorphicity condition. Replacing $\frac{\partial}{\partial x}=\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}$, condition (1.6) becomes

$$J\circ df\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\bar{\zeta}}\right)-i\,df\left(\frac{\partial}{\partial\zeta}\right)+i\,df\left(\frac{\partial}{\partial\bar{\zeta}}\right)=0.$$

In case M is identified with an open subset of $\mathbb{R}^{2n}\simeq\mathbb{C}^n$ by means of a system of coordinates (see also next Section 2) and recalling that the multiplication by i corresponds to the standard complex structure of $\mathbb{C}^n$, we obtain the following very useful expression for the condition of J-holomorphicity:

$$(J+J_{\mathrm{st}})_{f(\zeta)}\left(df\left(\frac{\partial}{\partial\bar{\zeta}}\right)\right)+(J-J_{\mathrm{st}})_{f(\zeta)}\left(df\left(\frac{\partial}{\partial\zeta}\right)\right)=0. \tag{1.7}$$

2. Standard forms and osculating pairs for almost complex structures and strongly pseudoconvex hypersurfaces

Let (M,J) be an almost complex manifold of real dimension $2n$ and let $\mathcal{U}\subset M$ be an open subset, which admits a system of coordinates. Notice that any coordinate map $\xi=$

$(x^1, y^1, \dots, x^n, y^n): \mathcal{U} \subset M \to \mathbb{R}^{2n} = \mathbb{C}^n$ gives an identification of $\mathcal{U}$ with the open subset $\xi(\mathcal{U}) \subset \mathbb{C}^n = \{(z^1, \dots, z^n),\ z^i \stackrel{\text{def}}{=} x^i + iy^i\}$. In particular, this identification allows to consider on $\mathcal{U}$ the standard complex structure

$$\underset{(\xi)}{J_{\mathrm{st}}} = \frac{\partial}{\partial y^i} \otimes dx^i - \frac{\partial}{\partial x^i} \otimes dy^i = i\frac{\partial}{\partial z^i} \otimes dz^i - i\frac{\partial}{\partial \bar{z}^i} \otimes d\bar{z}^i, \tag{2.1}$$

that we will call *associated with the system of coordinates* ξ. Of course, one should never forget that the identification with an open subset of $\mathbb{C}^n$ and the consequently defined complex structure (2.1) on $\mathcal{U}$ *does* depend on the considered coordinates $(x^1, y^1, \dots, x^n, y^n)$. However, when no confusion may occur, we will often omit the indication of the system of coordinates ξ, denoting the complex structure (2.1) simply by J_{st} and by identifying the open subset $\mathcal{U} \subset M$ with a subset of $\mathbb{C}^n$ with respect to any regard.

Notation. In all the sequel, we will also adopt the following convention on indices: *latin letters* i, j, *etc. will be used to denote indices running between* 1 *and n, while greek letters* α, β, *etc. will be always used to indicate indices running between* 1 *and* $n-1$.

Let $\Gamma \subset M$ be an oriented strongly pseudoconvex hypersurface and $\mathcal{U}$ an open neighborhood of $p \in \Gamma$, identified with an open subset of $\mathbb{C}^n$ by means of a system of coordinates $\xi = (z^1, \dots, z^n): \mathcal{U} \to \mathbb{R}^{2n} = \mathbb{C}^n$ so that $\xi(p) = 0$. It is immediately recognizable that it is always possible to choose the coordinates so that, at $p = 0$, we have that $J|_0 = J_{\mathrm{st}}|_0$, the tangent space $T_0\Gamma$ coincides with the hyperplane $x^n = 0$ and that the J-invariant subspace of $\mathcal{D}_0 \subset T_0\Gamma$ coincides with the span of the vectors parallel to $z^n = 0$. In other words, we may assume that J and Γ are of the form

$$\begin{aligned}
&\Gamma: \quad \rho(z, \bar{z}) = 0, \quad \text{with}\\
&\rho(z, \bar{z}) = 2\,\mathrm{Re}\big(z^n\big) - \mathrm{Re}\big(K_{\alpha\beta} z^\alpha z^\beta\big) - H_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta + O\big(|z|^3\big), \quad K_{\alpha\beta} = K_{\beta\alpha},\ \overline{H_{\alpha\bar{\beta}}} = H_{\beta\bar{\alpha}},
\end{aligned} \tag{2.2}$$

and

$$J|_z = J_{\mathrm{st}}|_z + L_z + O\big(|z|^2\big), \qquad J_{\mathrm{st}} = i\left(\frac{\partial}{\partial z^i} \otimes dz^i - \frac{\partial}{\partial \bar{z}^i} \otimes d\bar{z}^i\right), \tag{2.3}$$

where L_z denotes a real tensor field of type $(1,1)$ which depends linearly on the real coordinates $x^i = \mathrm{Re}(z^i)$ and $y^i = \mathrm{Im}(z^i)$. From the fact that $J^2 \equiv -I$, it follows immediately that $L_z \cdot J_{\mathrm{st}} = -J_{\mathrm{st}} \cdot L_z$ at any $z \in \mathcal{U}$ and hence that the linear map $L_z: T_z\mathbb{C}^n \to T_z\mathbb{C}^n$ (or, more precisely, its $\mathbb{C}$-linear extension $L_z: T_z^{\mathbb{C}}\mathbb{C}^n \to T_z^{\mathbb{C}}\mathbb{C}^n$) is of the form

$$L_z = \big(L^i_{\bar{j}k} z^k + L^i_{\bar{j}\bar{k}} \bar{z}^k\big)\frac{\partial}{\partial z^i} \otimes d\bar{z}^j + \text{c.c.} \tag{2.4}$$

(here and in all the following we will always write "c.c." to indicate the complex conjugate terms of the previous expression).

The following lemma shows that the difference between the Levi forms of Γ at $p = 0$, computed using J_{st} and J is determined exactly by the coefficients $L^n_{\bar{j}k}$ and $L^n_{\bar{j}\bar{k}}$.

Lemma 2.1. *For any real vector $v = v^\alpha \frac{\partial}{\partial z^\alpha}|_0 + \overline{v^\alpha} \frac{\partial}{\partial \bar{z}^\alpha}|_0 \in \mathcal{D}_0 \subset T_0\mathbb{C}^n$, let us denote by $\mathcal{L}_{\mathrm{st}}(v)$ and $\mathcal{L}(v)$ the value at 0 of the Levi form with respect to J_{st} and J, respectively, associated with a defining form ϑ so that $\vartheta|_0 = d\rho \circ J|_0 = d\rho \circ J_{\mathrm{st}}|_0$. Then*

$$\mathcal{L}(v) = \mathcal{L}_{\mathrm{st}}(v) + 2i\overline{v^\alpha} v^\beta \big(L^n_{\bar{\alpha}\beta} - \overline{L^n_{\bar{\beta}\alpha}}\big). \tag{2.5}$$

Proof. We want to compute $\mathcal{L}(v)$ using (1.4). We consider a smooth real vector field $X_z = X^i(z)\frac{\partial}{\partial z^i} + \overline{X^i(z)}\frac{\partial}{\partial \bar{z}^i}$, defined on a neighborhood of the origin and such that $X|_0 = v$ and so that, at any point $z \in \Gamma$, it takes values in the J-holomorphic distribution $\mathcal{D}$. Then, using (1.4),

$$\mathcal{L}(v) = -d\vartheta_0(X, JX) = \vartheta_0\big([X, JX]\big) = (d\rho \circ J_{\mathrm{st}})_0\big([X, JX]\big).$$

Notice that the higher order terms in (2.3) give no contribution to the vector $[X, JX]_0 \in T_0\mathbb{C}^n$ and hence that $[X, JX]_0 = [X, J_{\mathrm{st}}X]_0 + [X, LX]_0$. In particular, we may write

$$\begin{aligned}\mathcal{L}(v) &= (d\rho \circ J_{\mathrm{st}})_0\big([X, J_{\mathrm{st}}X]\big) + (d\rho \circ J_{\mathrm{st}})_0\big([X, LX]\big)\\ &= (d\rho \circ J_{\mathrm{st}})_0\big([X, J_{\mathrm{st}}X]\big) + i\big(dz^n - d\bar{z}^n\big)[X, LX]_0.\end{aligned} \tag{2.6}$$

Now, let us denote by ϑ_{st} the 1-form defined on a neighborhood of 0 as $\vartheta_{\mathrm{st}} \overset{\mathrm{def}}{=} d\rho \circ J_{\mathrm{st}}$. Observe that the restriction of ϑ_{st} to the tangent spaces of Γ gives a defining 1-form for the J_{st}-holomorphic distribution $\mathcal{D}_{\mathrm{st}}$ of Γ and that $-d\vartheta_{\mathrm{st}}|_0(v, J_{\mathrm{st}}v) = \mathcal{L}_{\mathrm{st}}(v)$. So, we may write also that

$$\begin{aligned}(d\rho \circ J_{\mathrm{st}})_0\big([X, J_{\mathrm{st}}X]\big) &= \vartheta_{\mathrm{st}}|_0\big([X, J_{\mathrm{st}}X]\big)\\ &= -d\vartheta_{\mathrm{st}}|_0(X, J_{\mathrm{st}}X) + X\big(\vartheta(J_{\mathrm{st}}X)\big)\big|_0 - J_{\mathrm{st}}X\big(\vartheta(X)\big)\big|_0\\ &= \mathcal{L}_{\mathrm{st}}(v) - X\big(X(\rho)\big)\big|_0 - J_{\mathrm{st}}X\big(J_{\mathrm{st}}X(\rho)\big)\big|_0.\end{aligned} \tag{2.7}$$

By construction, at all points of $\Gamma = \{\rho = 0\}$, the vector field X belongs to $\mathcal{D} \subset T\Gamma$. From this we get that $X(\rho)|_z = 0$ and $JX(\rho)_z = J_{\mathrm{st}}X(\rho)_z + LX(\rho)|_z + O(|z|^2) = 0$ at any $z \in \Gamma$. Now, recalling that $J_{\mathrm{st}}X|_0 \in T_0\Gamma$, we have the following values for the directional derivatives $X(X(\rho))|_0$ and $J_{\mathrm{st}}X(J_{\mathrm{st}}X(\rho))|_0$:

$$X\big(X(\rho)\big)\big|_0 = 0, \qquad J_{\mathrm{st}}X\big(J_{\mathrm{st}}X(\rho)\big)\big|_0 = -J_{\mathrm{st}}X\big(LX(\rho)\big)\big|_0. \tag{2.8}$$

By means of (2.8) and (2.7) we get that (2.6) is equal to

$$\mathcal{L}(v) = \mathcal{L}_{\mathrm{st}}(v) + J_{\mathrm{st}}X\big(LX(\rho)\big)\big|_0 + i\big(dz^n - d\bar{z}^n\big)[X, LX]_0. \tag{2.9}$$

On the other hand, recalling that the $\mathbb{C}$-valued functions $X^i(z)$ satisfy $X^n(0) = 0$ and $X^\alpha(0) = v^\alpha$, we may easily compute what we need, i.e.,

$$\begin{aligned}[X, LX]_0 &= X_0(LX) - (L_0X_0)(X) = v\bigg(L_z\bigg(X^i(z)\frac{\partial}{\partial z^i} + \overline{X^i(z)}\frac{\partial}{\partial \bar{z}^i}\bigg)\bigg)\bigg|_0\\ &= \big(v^\alpha \overline{v^\beta} L^i_{\bar{\beta}\alpha} + \overline{v^\alpha}\,\overline{v^\beta} L^i_{\bar{\beta}\bar{\alpha}}\big)\frac{\partial}{\partial z^i}\bigg|_0 + \big(\overline{v^\alpha} v^\beta \overline{L^i_{\bar{\beta}\alpha}} + v^\alpha v^\beta \overline{L^i_{\bar{\beta}\bar{\alpha}}}\big)\frac{\partial}{\partial \bar{z}^i}\bigg|_0,\end{aligned} \tag{2.10}$$

$$J_{\mathrm{st}}X\big(LX(\rho)\big)\big|_0 = iv^\alpha \overline{v^\beta} L^n_{\bar{\beta}\alpha} - i\overline{v^\alpha}\,\overline{v^\beta} L^n_{\bar{\beta}\bar{\alpha}} - i\overline{v^\alpha} v^\beta \overline{L^n_{\bar{\beta}\alpha}} + iv^\alpha v^\beta \overline{L^n_{\bar{\beta}\bar{\alpha}}}. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9), formula (2.5) follows. □

From the previous lemma, it follows that if the coefficients $L^n_{\bar{\alpha}\beta}$ are vanishing, then the Levi form at 0 of Γ with respect to J coincides with the Levi form of Γ with respect to J_{st}. This fact suggests the following definition and motivates the next proposition.

Definition 2.2. We say that J and Γ are *in standard form in the system of coordinates* $(z^1, \dots, z^n)$ if:

(a) J is of the form

$$J = J_{\mathrm{st}} + \left(L^i_{\bar{j}k} z^k + L^i_{\bar{j}\bar{k}} \bar{z}^k\right) \frac{\partial}{\partial z^i} \otimes d\bar{z}^j + \overline{\left(L^i_{\bar{j}k} z^k + L^i_{\bar{j}\bar{k}} \bar{z}^k\right)} \frac{\partial}{\partial \bar{z}^i} \otimes dz^j + O\left(|z|^2\right),$$

with coefficients $L^i_{j\bar{k}}$ so that

$$L^n_{\bar{\alpha}\beta} = 0 \quad \text{and} \quad L^n_{\bar{\alpha}\bar{\beta}} = -L^n_{\bar{\beta}\bar{\alpha}} \quad \text{for any } 1 \leqslant \alpha, \beta \leqslant n-1; \tag{2.12}$$

(b) Γ admits a defining function on a neighborhood of the origin of the form

$$\rho(z, \bar{z}) = 2\,\mathrm{Re}\left(z^n\right) - \sum_{\alpha=1}^{n-1} \left|z^\alpha\right|^2 + O\left(|z|^3\right). \tag{2.13}$$

Proposition 2.3. *For any $p \in \Gamma$, there exists a neighborhood $\mathcal{U}$ of p and a system of complex coordinates $\xi = (z^1, \dots, z^n)$ with $z(p) = 0$, in which J and Γ are in standard form. In particular, up to a positive scalar multiple, the Levi form of Γ with respect to J coincides with the Levi form with respect to J_{st} at $p = 0$ (as usual, J_{st} denotes the complex structure associated with the coordinates $(z^1, \dots, z^n)$).*

Proof. There is no loss of generality if we assume that Γ and J are of the form (2.2) and (2.3). Now, consider the change of coordinates

$$z^\alpha = w^\alpha, \qquad z^n = w^n + \frac{i}{2} L^n_{\bar{\alpha}\beta} \bar{w}^\alpha w^\beta + \frac{i}{4} L^n_{\bar{\alpha}\bar{\beta}} \bar{w}^\alpha \bar{w}^\beta.$$

The defining function ρ remains of the form (2.2) even when it is written in terms of w and $\bar{w}$, while J becomes of the form

$$\begin{aligned}
J|_w = {} & i\left(\frac{\partial}{\partial w^i} \otimes dw^i - \frac{\partial}{\partial \bar{w}^i} \otimes d\bar{w}^i\right) + L_w + O\left(|w|^2\right) \\
& - \left(L^n_{\bar{\alpha}\beta} w^\beta + \frac{1}{2}\left(L^n_{\bar{\alpha}\bar{\beta}} + L^n_{\bar{\beta}\bar{\alpha}}\right) \bar{w}^\alpha\right) \frac{\partial}{\partial w^n} \otimes d\bar{w}^\alpha \\
& - \left(\overline{L^n_{\bar{\alpha}\beta}} \bar{w}^\beta + \frac{1}{2}\left(\overline{L^n_{\bar{\alpha}\bar{\beta}}} + \overline{L^n_{\bar{\beta}\bar{\alpha}}}\right) w^\alpha\right) \frac{\partial}{\partial \bar{w}^n} \otimes dw^\alpha
\end{aligned}$$

and it satisfies condition (a) of Definition 2.2. So, assuming that (a) holds, from (2.5) we have that $\mathcal{L}_{\mathrm{st}}|_0 = \mathcal{L}|_0$ and hence that the matrix $H_{\alpha\bar{\beta}}$ is positive definite. Notice also that any change of coordinates of the form $z^\alpha = U^\alpha_\beta w^\beta$, $z^n = w^n$, gives an expression for J, which still satisfies (a), while it changes the coefficients $H_{\alpha\bar{\beta}}$ into the coefficients $H'_{\alpha\bar{\beta}} = U^\gamma_\alpha H_{\gamma\bar{\delta}} \overline{U^\delta_\beta}$. Hence, by means a suitable change of coordinates, we may always assume that $H_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$. Finally, by performing the change of coordinates $z^\alpha = w^\alpha$ and $z^n = w^n - \frac{1}{2} K_{\alpha\beta} w^\alpha w^\beta$, which also leaves the property (a) unchanged, we obtain a system of coordinates in which also (b) of Definition 2.2 is satisfied. □

We conclude this section introducing the crucial concept of osculating pairs and the proof that the existence problem for small stationary disc can be reduced to the analysis of deformations of the osculating structures.

Definition 2.4. Let J and Γ be in standard form in a system of complex coordinates $(z^1, \dots, z^n)$. We call *osculating pair for* (J, Γ) *at* $p = 0$ the pair (J^0, Γ^0) given by the hypersurface (locally J_{st}-biholomorphic to the unit sphere)

$$\Gamma^0\colon\ \rho^0(z, \bar{z}) = 2\,\mathrm{Re}\big(z^n\big) - \sum_{\alpha=1}^{n-1} \big|z^\alpha\big|^2, \tag{2.14}$$

and the almost complex structure

$$J^0 \stackrel{\mathrm{def}}{=} J_{\mathrm{st}} + A_{\bar{\alpha}\bar{\beta}} \bar{z}^\beta \frac{\partial}{\partial z^n} \otimes d\bar{z}^\alpha + \overline{A_{\bar{\alpha}\bar{\beta}}} z^\beta \frac{\partial}{\partial \bar{z}^n} \otimes dz^\alpha, \quad \text{with } A_{\bar{\alpha}\bar{\beta}} = L^n_{\bar{\alpha}\bar{\beta}} \tag{2.15}$$

where we denoted by $L^i_{\bar{j}\bar{k}}$ and $L^i_{\bar{j}k}$ the coefficients of the linear part of J as in (2.4).

Remark 2.5. In case M has real dimension 4, the indices α and β may assume only the value 1 and, by (2.12), $A_{\bar{1}\bar{1}} = 0$. In other words, when $\dim_{\mathbb{R}} M = 4$, the only possible osculating pair is $(J_{\mathrm{st}}, \Gamma^0)$. If this is the case, all claims of next section and our main result can be considered as consequences of the well-known properties of the stationary discs of the unit ball with respect to the standard complex structure J_{st}.

Consider now the anisotropic dilations

$$\phi_t : \mathbb{C}^n \to \mathbb{C}^n, \quad \phi_t(z) = \left(\frac{1}{t} z^1, \dots, \frac{1}{\delta} z^{n-1}, \frac{1}{t^2} z^n\right), \quad t \in \mathbb{R},$$

and the pairs (J^t, Γ^t) of almost complex structures $J^t \stackrel{\mathrm{def}}{=} \phi_{t*}(J)$ and hypersurfaces $\Gamma^t \stackrel{\mathrm{def}}{=} \phi_t(\Gamma)$.

A very simple computation shows that, for any real value of t, the pair (J^t, Γ^t) is in standard form and with (J^0, Γ^0) as osculating pair. Furthermore, the functions which give the components of J^t and the defining functions $\rho^t \stackrel{\mathrm{def}}{=} \frac{1}{t}(\rho \circ \phi_t^{-1})$ of the oriented hypersurfaces Γ^t, tend uniformly on compacta to the components of J^0 and the defining function ρ^0. Finally, it can be checked that the partial differential equations and the boundary conditions which define the lifts of the stationary discs of the hypersurfaces Γ^t with respect to the structures J^t depends continuously on t and they can be considered as continuous deformations of the corresponding equations and boundary conditions for the lifts of stationary discs for (J^0, Γ^0). We formalize this fact in the following proposition, which will turn out to be the key point for the proof of our main result.

Proposition 2.6. *For any $p \in \Gamma$ and any $\epsilon > 0$, there exist a neighborhood $\mathcal{U}$ of p, a system of complex coordinates $\xi = (z^1, \dots, z^n)$ with $z(p) = 0$ and a smooth family of pairs (J^t, Γ^t) of almost complex structure J_t and real hypersurfaces Γ^t, $t \in [-1, 1] \subset \mathbb{R}$, such that*

- (i) *(J^t, Γ^t) are in standard form for any t and, for all of them, the osculating pair at $p = 0$ coincides with (J^0, Γ^0);*
- (ii) *$(J^1, \Gamma^1) = (J, \Gamma)$ and for any $t \neq 0$, there exists a (J, J^t)-biholomorphism $\phi_t : \mathcal{U} \to \mathcal{U}$ such that $\phi_t(\Gamma) = \Gamma^t$.*

Moreover, the partial differential boundary problem which defines the lifts of stationary discs of (J, Γ) in a sufficiently small neighborhood of the origin is equivalent to an arbitrarily small, continuous deformation of the corresponding problem for the osculating pair (J^0, Γ^0)

3. The equations for the stationary discs of an osculating pair

In this section, (J^0, Γ^0) will denote a fixed osculating pair, i.e., a given almost complex structure J^0 on $\mathbb{C}^n$ of the form (2.15) together with the boundary of the Siegel domain Γ^0, defined in (2.14). We denote by $(x^1, y^1, \dots, x^n, y^n, u_1, v_1, \dots, u_n, v_n)$ the system of coordinates on $T^*\mathbb{C}^n$, associated with real coordinates $x^i = \mathrm{Re}(z^i)$ and $y^i = \mathrm{Im}(z^i)$ (see the definition in Section 1.1). This means that any 1-form $\alpha \in T^*_z\mathbb{C}^n$ is written in terms of such coordinates as $\alpha = u_i\, dx^i|_z + v_i\, dy^i|_z$. On the other hand, it is quite useful to consider the u^i and v^i as real and imaginary parts of some complex coordinates of $T^*\mathbb{C}^n = \mathbb{C}^{2n}$. More precisely, we will consider on $T^*\mathbb{C}^n$ the complex coordinates $(z^1, \dots, z^n, P_1, \dots, P_n)$, with $P_i = \frac{1}{2}(u_i - v_i)$, so that any 1-form $\alpha \in T^*_z\mathbb{C}^n$ will be written as

$$\alpha = P_i\, dz^i + \bar{P}_i\, d\bar{z}^i.$$

Using (1.2) we may compute the components of the lift $\mathbb{J}^0$ in the real coordinates (x^i, y^i, u_j, v_j) and then re-express $\mathbb{J}^0$ using the complex basis associated with the complex coordinates (z^i, P_j). Some tedious but straightforward computations show that $\mathbb{J}^0$ is

$$\mathbb{J}^0 = i\left(\frac{\partial}{\partial z^k} \otimes dz^k - \frac{\partial}{\partial \bar{z}^k} \otimes d\bar{z}^k\right) + A_{\bar{\alpha}\bar{\beta}}\bar{z}^\beta \frac{\partial}{\partial z^n} \otimes d\bar{z}^\alpha + \overline{A_{\bar{\alpha}\bar{\beta}}} z^\beta \frac{\partial}{\partial \bar{z}^n} \otimes dz^\alpha$$
$$+ i\left(\frac{\partial}{\partial P_k} \otimes dP_k - \frac{\partial}{\partial \bar{P}_k} \otimes d\bar{P}_k\right) + \overline{A_{\bar{\alpha}\bar{\beta}}} z^\beta \frac{\partial}{\partial \bar{P}_\alpha} \otimes dP_n + A_{\bar{\alpha}\bar{\beta}}\overline{z^\beta} \frac{\partial}{\partial P_\alpha} \otimes d\bar{P}_n. \quad (3.1)$$

For the interested reader, we give here some more detailed indications on how (3.1) is obtained. As we mentioned, we have to apply formula (1.2) by considering as real coordinates x^i and p_a the coordinates $x^i = \mathrm{Re}(z^i)$, $x^{n+i} = y^i = \mathrm{Im}(z^i)$ and $p_a = u_a = \frac{1}{2}\mathrm{Re}(P_a)$ and $p_{a+n} = v_a = -\frac{1}{2}\mathrm{Im}(P_a)$. In order to simplify the notation in the next formulae, in place of the indices "j" and "$j+n$" we are going to use the indices j_r and j_i, respectively, to indicate in a more expressive way when we refer to a quantity related to a real part ("r") or to an imaginary part ("i") of a complex quantity. As usual, all latin indices j, k etc. will be considered running between 1 and n, while greek indices α, β, etc. will run just between 1 and $n-1$. Capital letters A, B, etc. will be used for indices that can be both of the form j_r and of the form j_i.

Using these conventions and from the definition of the osculating structure J^0, we have that the components J^A_B of J^0 are

$$\begin{aligned}
&J^{\alpha_r}_{\beta_r} = 0, \qquad J^{\alpha_i}_{\beta_r} = \delta^\alpha_\beta, \qquad J^{n_r}_{\beta_r} = \mathrm{Re}(A_{\bar{\beta}\bar{\gamma}})x^\gamma + \mathrm{Im}(A_{\bar{\beta}\bar{\gamma}})y^\gamma,\\
&J^{n_i}_{\beta_r} = -\mathrm{Re}(A_{\bar{\beta}\bar{\gamma}})y^\gamma + \mathrm{Im}(A_{\bar{\beta}\bar{\gamma}})x^\gamma,\\
&J^{\alpha_r}_{\beta_i} = -\delta^\alpha_\beta, \qquad J^{\alpha_i}_{\beta_i} = 0, \qquad J^{n_r}_{\beta_i} = -\mathrm{Re}(A_{\bar{\beta}\bar{\gamma}})y^\gamma + \mathrm{Im}(A_{\bar{\beta}\bar{\gamma}})x^\gamma,\\
&J^{n_i}_{\beta_i} = -\mathrm{Re}(A_{\bar{\beta}\bar{\gamma}})x^\gamma - \mathrm{Im}(A_{\bar{\beta}\bar{\gamma}})y^\gamma,\\
&J^{j_r}_{n_r} = 0, \qquad J^{j_i}_{n_r} = \delta^j_n, \qquad J^{j_r}_{n_i} = -\delta^j_n, \qquad J^{j_i}_{n_i} = 0.
\end{aligned} \quad (3.2)$$

In particular, the only non-trivial values of the partial derivatives $J^A_{B,C}$ are

$$\begin{aligned}
&J^{n_r}_{\alpha_r,\beta_r} = \mathrm{Re}(A_{\bar{\alpha}\bar{\beta}}), \qquad J^{n_r}_{\alpha_r,\beta_i} = \mathrm{Im}(A_{\bar{\alpha}\bar{\beta}}), \qquad J^{n_i}_{\alpha_r,\beta_r} = \mathrm{Im}(A_{\bar{\alpha}\bar{\beta}}),\\
&J^{n_i}_{\alpha_r,\beta_i} = -\mathrm{Re}(A_{\bar{\alpha}\bar{\beta}}),\\
&J^{n_r}_{\alpha_i,\beta_r} = \mathrm{Im}(A_{\bar{\alpha}\bar{\beta}}), \qquad J^{n_r}_{\alpha_i,\beta_i} = -\mathrm{Re}(A_{\bar{\alpha}\bar{\beta}}), \qquad J^{n_i}_{\alpha_i,\beta_r} = -\mathrm{Re}(A_{\bar{\alpha}\bar{\beta}}),\\
&J^{n_i}_{\alpha_i,\beta_i} = -\mathrm{Im}(A_{\bar{\alpha}\bar{\beta}}).
\end{aligned}$$

Hence, we may compute the coefficients of the second line terms in (1.3) and we get that

$$u_n\big(-J^{n_r}_{\alpha_r,\beta_r}+J^{n_r}_{\beta_r,\alpha_r}-J^{n_i}_{\alpha_r,\beta_i}+J^{n_i}_{\beta_r,\alpha_i}\big)+v_n\big(-J^{n_i}_{\alpha_r,\beta_r}+J^{n_i}_{\beta_r,\alpha_r}+J^{n_r}_{\alpha_r,\beta_i}-J^{n_r}_{\beta_r,\alpha_i}\big)=0.$$

Similarly, we have that all other coefficients of that line are vanishing. On the other hand, using (3.2) and expressing the terms of the first line of (1.2) using the complex coordinates z^i and P_j, one immediately gets (3.1).

Coming back to (3.1), it is clear that the matrix $(\mathbb{J}^i_j)$, representing $\mathbb{J}^0$ in the basis $(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \bar z^i},\frac{\partial}{\partial P_j},\frac{\partial}{\partial \bar P_i})$, is

$$\big(\mathbb{J}^i_j\big)=\begin{pmatrix} iI_n & \mathcal{A}(\bar z) & 0 & 0\\ \bar{\mathcal{A}}(z) & -iI_n & 0 & 0\\ 0 & 0 & iI_n & \bar{\mathcal{A}}^t(z)\\ 0 & 0 & \mathcal{A}^t(\bar z) & -iI_n\end{pmatrix} \tag{3.3}$$

with

$$\mathcal{A}^\alpha_j(\bar z)=\mathcal{A}^n_n(\bar z)=0,\qquad \mathcal{A}^n_\alpha(\bar z)=A_{\bar\alpha\bar\beta}\bar z^\beta.$$

Recall also that, by (2.12), the coefficients $A_{\bar\alpha\bar\beta}$ satisfy

$$A_{\bar\alpha\bar\beta}=-A_{\bar\beta\bar\alpha}\quad\Rightarrow\quad A_{\bar\alpha\bar\alpha}=0\quad \text{for any } 1\leqslant\alpha\leqslant n-1.$$

From (3.3), a direct computation gives the matrix representing the linear map $(\mathbb{J}^0+J_{\mathrm{st}})^{-1}\cdot(\mathbb{J}^0-J_{\mathrm{st}})$ in the previous complex basis. This matrix is

$$\begin{pmatrix} 0 & -\frac{i}{2}\mathcal{A}(\bar z) & 0 & 0\\ \frac{i}{2}\bar{\mathcal{A}}(z) & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{i}{2}\bar{\mathcal{A}}^t(z)\\ 0 & 0 & -\frac{i}{2}\mathcal{A}^t(\bar z) & 0\end{pmatrix}$$

and hence, from (1.7), we obtain that a map $\hat f=(f^1,\dots,f^n,g_1,\dots,g_n):\Delta\to T^*\mathbb{C}^n$ is $\mathbb{J}^0$-holomorphic if and only if it satisfies the following p.d.e. system:

$$\begin{aligned} &\frac{\partial f^\alpha}{\partial\bar\zeta}=0,\qquad \frac{\partial f^n}{\partial\bar\zeta}-\frac{i}{2}A_{\bar\alpha\bar\beta}\overline{f^\beta}\overline{\left(\frac{\partial f^\alpha}{\partial\zeta}\right)}=0,\\ &\frac{\partial g_\alpha}{\partial\bar\zeta}+\frac{i}{2}\overline{A_{\bar\alpha\bar\beta}}f^\beta\overline{\left(\frac{\partial g^n}{\partial\zeta}\right)}=0,\qquad \frac{\partial g_n}{\partial\bar\zeta}=0. \end{aligned}\tag{3.4}$$

We now want to write down the boundary condition for a $\mathbb{J}^0$-holomorphic disc $\hat f$ in order to be the lift of a stationary disc. Observe that, by (1.5), the action of a complex number ζ on a real form $\alpha=P_j\,dz^j+\bar P_j\,d\bar z^j\in T^*\mathbb{C}^n$ is

$$\begin{aligned}\zeta\cdot\big(P_j\,dz^j+\bar P_j\,d\bar z^j\big)&\stackrel{\text{def}}{=}P_j\big(\mathrm{Re}(\zeta)\,dz^j+\mathrm{Im}(\zeta)\big(dz^j\circ J^0\big)\big)\\ &\quad+\overline{P_j}\big(\mathrm{Re}(\zeta)\,d\bar z^j+\mathrm{Im}(\zeta)\big(d\bar z^j\circ J^0\big)\big)\\ &=\zeta P_j\,dz^j+\overline{(\zeta P_j)}\,d\bar z^j+\big(\mathrm{Im}(\zeta)A_{\bar\alpha\bar\beta}\overline{z^\beta}P_n\big)\,d\overline{z^\alpha}\\ &\quad+\big(\mathrm{Im}(\zeta)\overline{A_{\bar\alpha\bar\beta}}z^\beta\overline{P_n}\big)\,dz^\alpha.\end{aligned}$$

Using the fact that the fibers of $\mathcal{N}(\Gamma^0)$ are generated over $\mathbb{R}$ by the 1-forms

$$d\rho^0\big|_z=dz^n-\sum_{\alpha=1}^{n-1}\bar z^\alpha\,dz^\alpha+d\bar z^n-\sum_{\alpha=1}^{n-1}z^\alpha\,d\bar z^\alpha,\quad z\in\Gamma^0,$$

it is quite immediate to realize that a $\mathbb{J}^0$-holomorphic disc $\hat{f} = (f^1, \dots, f^n, g_1, \dots, g_n)$ satisfies condition (c) of Definition 1.3 if and only if for any $\zeta \in \partial\Delta$ there is a real number $\lambda_\zeta \neq 0$ such that

$$2\,\mathrm{Re}\big(f^n(\zeta)\big) - \sum_{\alpha=1}^{n-1} \big|f^\alpha(\zeta)\big|^2 = 0, \tag{3.5}$$

$$\begin{aligned} g_i(\zeta)\,dz^i + \overline{g_i(\zeta)}\,d\bar{z}^i &= \lambda_\zeta\big(\zeta \cdot d\rho^0\big|_{f(\zeta)}\big) \\ &= \lambda_\zeta\bigg(-\zeta\,\overline{f^\alpha(\zeta)}\,dz^\alpha + \zeta\,dz^n - \frac{i}{2}(\zeta - \bar{\zeta})\overline{A_{\bar{\alpha}\bar{\beta}}}\,f^\beta(\zeta)\,dz^\alpha \\ &\qquad - \bar{\zeta}\,f^\alpha(\zeta)\,d\bar{z}^\alpha + \bar{\zeta}\,d\bar{z}^n - \frac{i}{2}(\zeta - \bar{\zeta})A_{\bar{\alpha}\bar{\beta}}\,\overline{f^\beta(\zeta)}\,d\bar{z}^\alpha\bigg). \end{aligned} \tag{3.6}$$

This immediately implies that $\lambda_\zeta = \bar{\zeta}\,g_n(\zeta) = \zeta\,\overline{g_n(\zeta)}$ and hence that (3.6) is equivalent to

$$\begin{aligned} &g_\alpha(\zeta) + \bigg(\overline{f^\alpha(\zeta)} + \frac{i}{2}\overline{A_{\bar{\alpha}\bar{\beta}}}\,f^\beta(\zeta)\bigg)g^n(\zeta) - \frac{i}{2}\overline{A_{\bar{\alpha}\bar{\beta}}}\,f^\beta(\zeta)\overline{g^n(\zeta)} = 0, \\ &\bar{\zeta}\,g_n(\zeta) - \zeta\,\overline{g_n(\zeta)} = 0, \end{aligned} \tag{3.7}$$

for any $\zeta \in \partial\Delta$.

In brief, we have proved the following lemma.

Lemma 3.1. *A map $\hat{f} = (f^1, \dots, f^n, g_1, \dots, g_n) : \bar{\Delta} \to \mathbb{C}^{2n}$ represents the lift of a stationary disc of Γ^0 if and only if it satisfies the generalized Riemann–Hilbert problem given by the p.d.e. system* (3.4) *and the boundary conditions* (3.5) *and* (3.7).

Now, recalling that $A_{\bar{1}\bar{1}} = 0$, by a direct inspection it is possible to check that, for any $a \in \mathbb{C}^*$ and $\lambda \in R^*$, the map $\hat{f}_{a,\lambda} = (f_a(\zeta); g_{a,\lambda}) : \bar{\Delta} \to \mathbb{C}^{2n}$, defined by

$$f_a(\zeta) = \bigg(a\zeta, 0, \dots, 0, \frac{|a|^2}{2}\bigg), \tag{3.8}$$

$$g_{a,\lambda}(\zeta) = \bigg(-\lambda\bar{a}, \frac{i\lambda}{2}\overline{A_{\bar{2}\bar{1}}}\,a\big(-\zeta^2 - |\zeta|^2 + 2\big), \dots, \frac{i\lambda}{2}\overline{A_{\overline{n-1}\bar{1}}}\,a\big(-\zeta^2 - |\zeta|^2 + 2\big), \lambda\zeta\bigg) \tag{3.9}$$

is a lift of the stationary discs $f_a : \bar{\Delta} \to \mathbb{C}^n$ of Γ^0.

Furthermore, notice that any map of the form $z^n = \tilde{z}^n$, $z^\alpha = U^\alpha_\beta \tilde{z}^\beta$, with $U^\alpha_\beta \in U_n$, leaves Γ^0 invariant and sends J^0 into an almost complex structure J'^0, which is still of the form (3.1). It follows immediately that the discs (3.8) and (3.9), computed with the coefficients $A'_{\bar{\alpha},\bar{\beta}}$ of J', are images under the previous transformation of lifts of stationary discs for (J^0, Γ^0). In particular, it follows that for any point $z_o = (0, \dots, 0, z^n_o)$, with $\rho^0(z_o) > 0$, and any vector $v_o \in \mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial z^\alpha}\}_{1 \leqslant \alpha \leqslant n-1}$, there exists a stationary disc f for (J^0, Γ^0) so that $f(0) = z_o$ and $f_*(\partial_{\mathrm{Re}\,\zeta}|_0) = \lambda v_o$ for some $\lambda \in \mathbb{R}^*$.

4. Proof of the Main Theorem

The key point of the proof consists in showing that the 1-parameter family of non-linear operators, which defines the lifts of stationary discs for the pairs (J^t, Γ^t) of Proposition 2.6

satisfies the hypothesis of the general Implicit Function Theorem (see e.g. [9, p. 353]) at the disc $\hat{f}_{a,\lambda}$, defined in (3.8) and (3.9), and with the differential problem determined by the osculating pair (J^0, Γ^0). After that, our main result will follow immediately.

So, for some fixed $0 < \delta < 1$, let us consider the Fréchet derivative $\mathcal{F}'|_{\hat{f}_{a,\lambda}} = (\mathcal{F}_1|_{\hat{f}_{a,\lambda}}; \mathcal{F}_2|_{\hat{f}_{a,\lambda}})$ of the non-linear operator

$$\mathcal{F} = (\mathcal{F}_1; \mathcal{F}_2) : \mathcal{C}^{1,\delta}\big(\bar{\Delta}, \mathbb{C}^{2n}\big) \to \mathcal{C}^{\delta}\big(\bar{\Delta}, \mathbb{C}^{2n}\big) \times \mathcal{C}^{\delta}\big(\partial\Delta, \mathbb{R} \times \mathbb{C}^{n-1} \times i\mathbb{R}\big)$$

defined by the l.h.s. of the p.d.e. system (3.4) and of the boundary conditions (3.5) and (3.7). A straightforward computation shows that for any $\hat{h} = (h; k) : \bar{\Delta} \to \mathbb{C}^{2n}$, one has $\mathcal{F}_1'|_{\hat{f}_{a,\lambda}}(\hat{h}) = (\mathcal{H}(h; k); \mathcal{K}(h; k))$ where

$$\mathcal{H}^{\alpha}(h; k) = \frac{\partial h^{\alpha}}{\partial \bar{\zeta}}, \qquad \mathcal{H}^{n}(h; k) = \frac{\partial h^{n}}{\partial \bar{\zeta}} - \frac{i\bar{a}}{2} A_{\bar{1}\bar{\alpha}} \overline{h^{\alpha}} + \frac{i\bar{a}}{2} A_{\bar{1}\bar{\alpha}} \overline{\frac{\partial h^{\alpha}}{\partial \zeta}} \bar{\zeta},$$

$$\mathcal{K}^{\alpha}(h; k) = \frac{\partial k_{\alpha}}{\partial \bar{\zeta}} + \frac{ia}{2} \overline{A_{\bar{\alpha}\bar{1}}} \zeta \overline{\frac{\partial k_n}{\partial \zeta}} + \frac{i\lambda}{2} \overline{A_{\bar{\alpha}\bar{\beta}}} h^{\beta}, \qquad \mathcal{K}^{n}(h; k) \stackrel{\text{def}}{=} \frac{\partial k^{n}}{\partial \bar{\zeta}}.$$

With similar computations one gets also the components of the second part of the Fréchet derivative $\mathcal{F}_2'|_{\hat{f}_{a,\lambda}}(\hat{h}) = (\mathcal{M}^0(h; k), \mathcal{M}^1(h; k), \dots, \mathcal{M}^{n-1}(h; k), \mathcal{M}^n(h; k))$, i.e., the maps

$$\mathcal{M}^0(h; k) : \partial\Delta \to \mathbb{R}, \qquad \mathcal{M}^{\alpha}(h; k) : \partial\Delta \to \mathbb{C}, \qquad \mathcal{M}^{n}(h; k) : \partial\Delta \to i\mathbb{R},$$

with

$$\begin{aligned}
\mathcal{M}^0(h; k)(\zeta) &= h^n(\zeta) + \overline{h^n(\zeta)} - \bar{a} h^1(\zeta)\bar{\zeta} - a\overline{h^1(\zeta)}\zeta, \\
\mathcal{M}^{\alpha}(h; k)(\zeta) &= k_{\alpha}(\zeta) + \delta^1_{\alpha} \bar{a}\bar{\zeta} k_n(\zeta) + \frac{ia}{2} \overline{A_{\bar{\alpha}\bar{1}}} \zeta \big(k_n(\zeta) - \bar{k}_n(\zeta)\big) \\
&\quad + \lambda\left(\overline{h^{\alpha}(\zeta)} + \frac{i}{2} \overline{A_{\bar{\alpha}\beta}} h^{\beta}(\zeta)\right)\zeta - \frac{i\lambda}{2} \overline{A_{\bar{\alpha}\bar{\beta}}} h^{\beta}(\zeta)\bar{\zeta}, \\
\mathcal{M}^n(h; k)(\zeta) &= \bar{\zeta} k_n(\zeta) - \zeta \overline{k_n(\zeta)}.
\end{aligned}$$

The following lemma is the key point of our proof.

Lemma 4.1. *The linear operator $\mathcal{F}'|_{\hat{f}_{a,\lambda}}$ is surjective and the map*

$$(h, k) \in \ker \mathcal{F}'\big|_{\hat{f}_{a,\lambda}} \to \left(h^i(0) \text{ with } 1 \leqslant i \leqslant n,\ \frac{\partial h^i}{\partial \zeta}(0) \text{ with } i \geqslant 2,\right.$$
$$\left.\operatorname{Im}\left(\bar{a}\frac{\partial h^1}{\partial \zeta}(0)\right), \operatorname{Re}\left(\frac{\partial k_n}{\partial \zeta}(0)\right)\right) \in \mathbb{C}^{2n-1} \times \mathbb{R}^2$$

is a linear isomorphism between $\ker \mathcal{F}'|_{\hat{f}_{a,\lambda}}$ *and* $\mathbb{C}^{2n-1} \times \mathbb{R}^2$.

Proof. Consider the classical Cauchy–Green transform, i.e., the operator $T_{CG} : \mathcal{C}^{\delta}(\Delta, \mathbb{C}) \to \mathcal{C}^{1,\delta}(\Delta, \mathbb{C})$, defined by

$$T_{CG}(\varphi)(\zeta) = \frac{1}{2\pi i} \iint\limits_{\Delta} \frac{\varphi(\eta)}{\eta - \zeta}\, d\eta \wedge d\bar{\eta}.$$

It is well known that it is inverse to the $\bar{\partial}$-operator, i.e., satisfies $\frac{\partial T_{CG}(\varphi)}{\partial \bar{\zeta}} = \varphi$ (see e.g. [14, Section I.8]). With the help of T_{CG}, the operators $\mathcal{H}$ and $\mathcal{K}$ can be written as

$$\mathcal{H}^\alpha(h;k) = \frac{\partial h^\alpha}{\partial \bar\zeta}, \qquad \mathcal{H}^n(h;k) = \frac{\partial}{\partial \bar\zeta}\left(h^n + \frac{i\bar a}{2} A_{\bar 1\bar\alpha}\overline{h^\alpha}\bar\zeta - i\bar a A_{\bar 1\bar\alpha} T_{CG}\left(\overline{h^\alpha}\right)\right), \tag{4.1}$$

$$\mathcal{K}^\alpha(h;k) = \frac{\partial}{\partial \bar\zeta}\left(k_\alpha + \frac{ia}{2}\overline{A_{\bar\alpha\bar 1}}\zeta\overline{k_n} + \frac{i\lambda}{2}\overline{A_{\bar\alpha\bar\beta}} T_{CG}\left(h^\beta\right)\right), \qquad \mathcal{K}^n(h;k) = \frac{\partial k^n}{\partial \bar\zeta}. \tag{4.2}$$

From this expression and using the Cauchy–Green transform, one can directly show that $\mathcal{F}_1'|_{\hat f_{a,\lambda}}$ is surjective. By linearity of the operator, this implies that the surjectivity of $\mathcal{F}'|_{\hat f_{a,\lambda}}$ is proved if we can show that, for any $\Phi \in \mathcal{C}^\delta(\partial\Delta, \mathbb{R}\times\mathbb{C}^{2n-1}\times i\mathbb{R})$, there exists a solution $\hat h = (h,k)$ to the Riemann–Hilbert problem

$$\mathcal{F}_1'\big|_{\hat f_{a,\lambda}}(\hat h) = 0, \qquad \mathcal{F}_2'\big|_{\hat f_{a,\lambda}}(\hat h) = \Phi. \tag{4.3}$$

It follows immediately from (4.1) and (4.2) that $\hat h = (h,k)$ satisfies $\mathcal{F}_1'|_{\hat f_{a,\lambda}}(\hat h) = 0$ if and only if the components h^α and k^n are holomorphic, while the components h^n and k^α are of the form

$$h^n = -\frac{i\bar a}{2} A_{\bar 1\bar\alpha}\overline{h^\alpha}\bar\zeta + i\bar a A_{\bar 1\bar\alpha}\overline{\mathcal{I}\left(h^\alpha\right)} + \tilde h^n, \tag{4.4}$$

$$k^\alpha = -\frac{ia}{2}\overline{A_{\bar\alpha\bar 1}}\zeta\overline{k_n} - \frac{i\lambda}{2}\overline{A_{\bar\alpha\bar\beta}} h^\beta\bar\zeta + \tilde k^\alpha \tag{4.5}$$

for some holomorphic functions $\tilde h^n$ and $\tilde k^\alpha$. In (4.4) we used the symbol "$\mathcal{I}$" to denote the operator which associates to any holomorphic function ϕ on the unit disc, the unique holomorphic function $\mathcal{I}(\phi)$ such that $\frac{\partial \mathcal{I}(\phi)}{\partial\zeta} = \phi$ and $\mathcal{I}(\phi)(0) = 0$.

From (4.4) and (4.5), it follows that (4.3) admits a solution for an arbitrary Φ if and only if there exists a holomorphic disc $(h^\alpha, \tilde h^n; \tilde k^\alpha, k^n) : \bar\Delta \to \mathbb{C}^{2n}$ which satisfies the following conditions for any $\zeta \in \partial\Delta$:

$$\begin{aligned}
&\tilde h^n(\zeta) + \overline{\tilde h^n(\zeta)} - \frac{i\bar a}{2} A_{\bar 1\bar\alpha}\overline{h^\alpha(\zeta)}\bar\zeta + \frac{ia}{2}\overline{A_{\bar 1\bar\alpha}} h^\alpha(\zeta)\zeta + i\bar a A_{\bar 1\bar\alpha}\overline{\mathcal{I}\left(h^\alpha\right)(\zeta)} - ia\overline{A_{\bar 1\bar\alpha}}\mathcal{I}\left(h^\alpha\right)(\zeta) \\
&\quad - \bar a h^1(\zeta)\bar\zeta - a\overline{h^1(\zeta)}\zeta = \Phi^0(\zeta),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\tilde k_\alpha(\zeta) - ia\overline{A_{\bar\alpha\bar 1}}\,\overline{k_n(\zeta)}\zeta + \delta^1_\alpha \bar a k_n(\zeta)\bar\zeta - i\lambda\overline{A_{\bar\alpha\bar\beta}} h^\beta(\zeta)\bar\zeta + \frac{ia}{2}\overline{A_{\bar\alpha\bar 1}}\zeta k_n(\zeta) \\
&\quad + \lambda\left(\overline{h^\alpha(\zeta)} + \frac{i}{2}\overline{A_{\bar\alpha\bar\beta}} h^\beta(\zeta)\right)\zeta = \Phi^\alpha(\zeta),
\end{aligned} \tag{4.7}$$

$$\bar\zeta k_n(\zeta) - \zeta\overline{k_n(\zeta)} = i\Phi^n(\zeta). \tag{4.8}$$

We recall that, by the regularity assumed on Φ, each map Φ^i can be written as sum of a unique Fourier power series $\Phi^i(\zeta) = \sum_{m\in\mathbb{Z}} \Phi^i_m\zeta^m$ (see e.g. [8, Remark VII.7.5]). So, if we consider the expressions of the holomorphic functions as sum of power series

$$h^\alpha(\zeta) = \sum_{m\geqslant 0} h^\alpha_m\zeta^m, \qquad \tilde h^n = \sum_{m\geqslant 0} h^n_m\zeta^m, \qquad \tilde k_\alpha = \sum_{m\geqslant 0} k_{\alpha,m}\zeta^m,$$

$$k^n(\zeta) = \sum_{m\geqslant 0} k_{n,m}\zeta^m,$$

we obtain the following conditions on the coefficients h^i_m and $k_{i,m}$, with $m \geqslant 0$ (for reader's convenience, we point out that Eqs. (4.9) are obtained from (4.8), Eqs. (4.10)–(4.13) come from (4.7) and (4.14) and (4.15) are consequences of (4.6)):

$$k_{n,m} = i\Phi^n_{m-1} \quad \text{for all } m \geqslant 3, \qquad k_{n,2} - \overline{k_{n,0}} = i\Phi^n_1, \qquad k_{n,1} - \overline{k_{n,1}} = i\Phi^n_0, \tag{4.9}$$

$$\overline{h^\alpha_m} = \frac{1}{\lambda}\Phi^\alpha_{1-m} + \frac{ia}{\lambda}\overline{A_{\bar\alpha\bar 1}}\,\overline{k_{n,m}} \quad \text{for all } m \geqslant 3,$$

$$\overline{h^\alpha_2} - i\overline{A_{\bar\alpha\bar\beta}}h^\beta_0 = \frac{ia}{\lambda}\overline{A_{\bar\alpha,\bar 1}}\,\overline{k_{n,2}} - \frac{\delta^1_\alpha}{\lambda}\bar a k_{n,0} + \frac{1}{\lambda}\Phi^\alpha_{-1}, \tag{4.10}$$

$$k_{\alpha,0} = ia\overline{A_{\bar\alpha\bar 1}}\,\overline{k_{n,1}} - \delta^1_\alpha \bar a k_{n,1} + i\lambda\overline{A_{\bar\alpha\bar\beta}}h^\beta_1 - \lambda\overline{h^\alpha_1} + \Phi^\alpha_0, \tag{4.11}$$

$$k_{\alpha,1} = ia\overline{A_{\bar\alpha\bar 1}}\,\overline{k_{n,0}} - \delta^1_\alpha \bar a k_{n,2} - \frac{ia}{2}\overline{A_{\bar\alpha\bar 1}}k_{n,0} + i\lambda\overline{A_{\bar\alpha\bar\beta}}\left(h^\beta_2 - \frac{h^\beta_0}{2}\right) - \lambda\overline{h^\alpha_0} + \Phi^\alpha_1, \tag{4.12}$$

$$k_{\alpha,m} = -\delta^1_\alpha \bar a k_{n,m+1} - \frac{ia}{2}\overline{A_{\bar\alpha\bar 1}}k_{n,m-1} + i\lambda\overline{A_{\bar\alpha\bar\beta}}\left(h^\beta_{m+1} - \frac{h^\beta_{m-1}}{2}\right) + \Phi^\alpha_m$$
$$\text{for all } m \geqslant 2, \tag{4.13}$$

$$h^n_m = ia\overline{A_{\bar 1\bar\alpha}}h^\alpha_{m-1}\left(\frac{1}{m} - \frac{1}{2}\right) + \bar a h^1_{m+1} + \Phi^0_m \quad \text{for all } m \geqslant 2, \tag{4.14}$$

$$h^n_1 = \frac{ia}{2}\overline{A_{\bar 1\bar\alpha}}h^\alpha_0 + \bar a h^1_2 + a\overline{h^1_0} + \Phi^0_1, \qquad h^n_0 + \overline{h^n_0} = \bar a h^1_1 + a\overline{h^1_1} + \Phi^0_0. \tag{4.15}$$

This system is immediately seen to be solvable by substitutions for any choice of the Fourier coefficients Φ^i_m and this concludes the proof of the surjectivity.

Let us now consider $\ker \mathcal{F}'|_{\hat f_{a,\lambda}}$. It is clear that it is linearly isomorphic with the space of holomorphic discs $(h^\alpha, \tilde h^n, \tilde k^\alpha, k^n) : \bar\Delta \to \mathbb{C}^{2n}$, whose power series coefficients satisfy the above equations with all terms Φ^i_m's set equal to 0. In particular, it can be checked that any solution is uniquely determined by the values of h^i_0, for $1 \leqslant i \leqslant n$, by the real numbers $\mathrm{Re}(k_{n,1})$ and $\mathrm{Im}(\bar a h^1_1)$ and by the complex numbers h^i_1 for $2 \leqslant i \leqslant n$ (to check this claim, recall that $A_{\bar 1\bar 1} = 0$ and then solve all the equations by substitutions in the following order: (4.15), $(4.10)_2$ for $\alpha = 1$, (4.9), (4.10) and then all the others). From this, the second claim follows immediately. □

From Lemma 4.1, we see that the continuous family of operators

$$\left(\mathcal{F}^{(t)}, \mathcal{V}^{(t)}\right) : \mathcal{C}^{1,\delta}(\bar\Delta) \to \mathcal{C}^\delta\left(\bar\Delta, \mathbb{C}^{2n}\right) \times \mathcal{C}^\delta\left(\partial\Delta, \mathbb{C}^{n-1} \times \mathbb{R}\right) \times \mathbb{C}^n \times \mathbb{R}^{2n-1} \times \mathbb{R}$$

given by the differential operators $\mathcal{F}^{(t)}$, defining the lifts of stationary discs for the pairs (J^t, Γ^t), and by the evaluation maps

$$\hat f = (f, g) \xrightarrow{\mathcal{V}^{(t)}} \left(f^i(0),\ \frac{a^{-1} \cdot \frac{\partial f^i}{\partial\zeta}(0)}{\mathrm{Re}(a^{-1} \cdot \frac{\partial f^1}{\partial\zeta}(0))},\ \mathrm{Re}\left(\frac{\partial g_n}{\partial\zeta}(0)\right)\right)$$

has invertible Fréchet derivative at any disc $\hat f_{a,\lambda}$.

By the general Implicit Function Theorem, there exists a stationary disc passing through any point z_o on the inward real normal to Γ^0 in a sufficiently small neighborhood of the origin and tangent to any vector of an open neighborhood of $\mathrm{span}_{\mathbb{C}}\{\frac{\partial}{\partial z^\alpha}|_z\}_{1\leqslant\alpha\leqslant n-1}$ for any pair (J^t, Γ^t), $0 \leqslant t < \epsilon$, for ϵ sufficiently small (moreover, this obviously remains true for any point z_o in an open convex cone with vertex at the origin containing the real inward normal of Γ if this cone contains no real lines). By Proposition 2.6, our main theorem follows.

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