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One-step recursive method for solving systems of differential equations

Luis Lara[a, b, *]

[a]Departamento de Física, FCEIA, UNR Av. Pellegini 250, 2000 Rosario, Argentina
[b]Departamento de Sistemas, FRR, UTN, Zevallos 1120, 2000 Rosario, Argentina

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Abstract

In this work an iteration one-step method to integrate systems of nonlinear ordinary differential equations with initial values is presented

Keywords: Ordinary differential equations; Taylor polynomial; Numerical methods

1. Introduction

Differential equations with strong or weak nonlinearities have been of great interest because they play an important role in dynamics, physics, etc. In this work we consider the following nonlinear equation:

$$y'(x) = Ay(x) + F(y(x)), \tag{1}$$

where $' = \mathrm{d}/\mathrm{d}x$, $y \in R^m$, and A is a constant real matrix of dimension $m \times m$, $F \in R^m$ is a polynomial function and $x \in (-\infty, \infty)$.

A large bibliography exists on numerical methods to solve nonlinear systems of differential equations [1–3,7,9,11,12,14,15], in particular, there are interesting works with nonlinear oscillations [6,8,10]. In this paper a one-step iteration method is presented for initial value problems, based on the solution of the non-homogeneous linear systems [4]. In the work (see [10]) F is approximated by a Taylor's polynomial. In our case, we first approximate y by Taylor's polynomial. We demonstrate that this strategy is more efficient than the continuous analytic continuation (CAC) [5]. Moreover the uniform convergence to the exact solution is demonstrated and the local error is determined.

The paper is organized in the following sections: In Section 2 the recursive method is developed, based on the solution of the linear autonomous systems. In Section 3 the convergence of the method is studied. In Section 4 the local error is determined. In Section 5 the implementation of the recusive equations is presented for a simple case. In Section 6 numerical results are presented. In Section 7 the conclusions are drawn.

* Corresponding author at: Departamento de Física, FCEIA, UNR Av. Pellegini 250, 2000 Rosario, Argentina. Tel.: +54 341 4214586.
E-mail address: lplara@arnet.com.ar.

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2. Recursive solution

In this section, a one-step method of integration is developed to solve Eq. (1), with initial conditions, based on the theory of the non-homogeneous linear systems. Let us divide the integration interval $[x_0, x_0 + H]$ in n equal parts by the points $x_0, x_1, \ldots, x_n$, where the longitude of each element is $x_{i+1} - x_i = h_n = H/n, i = 0, \ldots, n-1$.

Let us consider the initial condition $y(x_0) = y_0$ for the integration in the interval $x_0 \leqslant x \leqslant x_1$. The integral curve is defined for the solution of the linear problem:

$$
\begin{aligned}
&y_n'(x) = Ay_n(x) + F(T_q(y_n(x), x_0, x)),\\
&y(x_0) = y_0, \quad x_0 \leqslant x \leqslant x_1,
\end{aligned}
\tag{2}
$$

where the number of intervals n parameterizes the solution and $T_q(y(x), x_0, x)$ is the Taylor's polynomial of degree q around x_0. By the explicit integration of the Eq. (2) we find $y_n(x)$ in the intervals $x_0 \leqslant x \leqslant x_1$. Starting from the initial value $y_n(x_1)$ we obtain the solution in the second interval, and so forth. Then, for each interval we have the approximate explicit solution of Eq. (1).

In this way we determine $y_n(x)$ in $x_0 \leqslant x \leqslant x_0 + H$. In Section 3, we will demonstrate that the sequence $y_n(x)$ uniformly converges to the solution of Eq. (1), when $n \to \infty$.

3. Convergence of the method

Let us consider the equation

$$
\begin{aligned}
&y'(x) = Ay(x) + F(y(x)),\\
&y(x_0) = y_0,
\end{aligned}
\tag{3}
$$

where $y \varepsilon R^m$, A is a constant real matrix of dimension $m \times m$, $F \varepsilon R^m$ is polynomial and $x_0 \leqslant x \leqslant x_0 + H$. We will demonstrate the sequence $y_n(x)$ uniformly converges to the solution $y(x)$ satisfying $y(x_0) = y_0$.

The existence and uniqueness of the solution of Eq. (3) is given by the existence and uniqueness theorem of Ref. [4]. The theorem also determines the interval $[x_0, x_0 + H]$, $H \leqslant \infty$ where $y(x)$ is analytic.

We will demonstrate that the succession of the function $y_n(x)$ that begins in the point (x_0, y_0) with step $h_n = H/n$ uniformly converges in the segment $x_0 \leqslant x \leqslant x_0 + H$.

Based on Section 2 the solution $y_n(x)$ satisfies

$$
y_n'(x) = Ay_n(x) + F(T_q(y_n(x), x_k, x)), \quad x_k \leqslant x \leqslant x_{k+1}, \quad k = 0, \ldots, n-1. \tag{4}
$$

that we can rewrite as

$$
y_n'(x) = Ay_n(x) + F(y_n(x)) + g_n(x), \tag{5}
$$

where

$$
g_n(x) = F(T_q(y_n(x), x_k, x)) - F(y_n(x)),
$$

since F is polynomical and y_n is analytical in $[x_0, x_0 + H]$. We can bound g_n by

$$
\|g_n(x)\| < \varepsilon_n, \quad k = 0, 1, \ldots, \infty, \tag{6}
$$

if $n > N_{\varepsilon_n}$, where $\varepsilon_n \to 0$ when $n \to \infty$, $g_n(x) = \nabla F \cdot (T_q - y_n) + \mathrm{O}(T_q - y_n)^2$, $T_q - y_n = \partial y(x_k)/\partial x(x - x_k) + \mathrm{O}(x - x_k)^2$, since $\|x - x_k\| \leqslant h_n$ and $h_n \to 0$ when $n \to \infty$.

Integrating the Eq. (5) between x_0 and x with $\|x - x_0\| \leqslant H$, and using the initial condition $y(x_0) = y_0$, we formally obtain (see details in Section 5)

$$
y_n(x) = y_0 + \int_{x_0}^{x} Ay_n(x')\,\mathrm{d}x' + \int_{x_0}^{x} F(y_n(x'))\,\mathrm{d}x' + \int_{x_0}^{x} g_n(x')\,\mathrm{d}x', \tag{7}
$$

where n is positive integer. Now, considering an integer $m \geqslant 0$, we have

$$
y_{n+m}(x) = y_0 + \int_{x_0}^{x} Ay_{n+m}(x')\,\mathrm{d}x' + \int_{x_0}^{x} F(y_{n+m}(x'))\,\mathrm{d}x' + \int_{x_0}^{x} g_{n+m}(x')\,\mathrm{d}x'. \tag{8}
$$

Subtracting Eq. (8) from Eq. (7) and taking the norm, we obtain

$$\|y_{n+m}(x)-y_n(x)\| \leqslant \int_{x_0}^{x} \|A\|\|y_{n+m}(x')-y_n(x')\|\,\mathrm{d}x' + \int_{x_0}^{x} \|F(y_{n+m}(x'))-F(y_n(x'))\|\mathrm{d}x' + \int_{x_0}^{x} \|g_{n+m}(x')\|\,\mathrm{d}x' + \int_{x_0}^{x} \|g_n(x')\|\,\mathrm{d}x',$$

in $x_0 \leqslant x \leqslant x_0+H$, since F is polynomial $\|F(y_{n+m}(x))-F(y_n(x))\| \leqslant \|\nabla F(y_n)\|\|y_{n+m}(x)-y_n(x)\| \leqslant K\|y_{n+m}(x)-y_n(x)\|$ then, the maximum norm is bounded by

$$\max\|y_{n+m}(x)-y_n(x)\| \leqslant \|A\|H \max\|y_{n+m}(x)-y_n(x)\| + HK\max\|y_{n+m}(x)-y_n(x)\| + \max\int_{x_0}^{x}\|g_{n+m}(x')\|\,\mathrm{d}x' + \max\int_{x_0}^{x}\|g_n(x')\|\,\mathrm{d}x',$$

using the Ineq. (6)

$$\max\|y_{n+m}(x)-y_n(x)\| \leqslant H\|A\|\max\|y_{n+m}(x)-y_n(x)\| + HK\max\|y_{n+m}(x)-y_n(x)\| + (\varepsilon_n+\varepsilon_{n+m})H,$$

because $\|x-x_0\| \leqslant H$, then

$$\max\|y_{n+m}(x)-y_n(x)\| \leqslant (\varepsilon_n+\varepsilon_{n+m})\frac{H}{1-H\|A\|-KH},$$

using the Ineq. (6), we have that $\varepsilon_n \to 0$, $\varepsilon_{n+m} \to 0$ when $n \to \infty$. In this way, we have demonstrated that the succession of functions $y_n(x)$ uniformly converges in $x_0 \leqslant x \leqslant x_0+H$.

Now, we will demonstrate that the limit of the succession is the solution $y(x)$ of the Eq. (1). Let

$$z(x) = \lim_{n\to\infty} y_n(x),$$

using successive constructions, we obtain

$$z(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} Ay_n(x')\,\mathrm{d}x' + \lim_{n\to\infty}\int_{x_0}^{x} F(y_n(x'))\,\mathrm{d}x' + \lim_{n\to\infty}\int_{x_0}^{x} g_n(x')\,\mathrm{d}x'.$$

Since $y_n(x)$ is uniformly continuous

$$\lim_{n\to\infty}\int_{x_0}^{x} Ay_n(x')\,\mathrm{d}x' = \int_{x_0}^{x} A\lim_{n\to\infty} y_n(x')\,\mathrm{d}x' = \int_{x_0}^{x} Az(x')\,\mathrm{d}x',$$

since F is polynomial

$$\lim_{n\to\infty}\int_{x_0}^{x} F(y_n(x'))\,\mathrm{d}x' = \int_{x_0}^{x} F\left(\lim_{n\to\infty} y_n(x')\right)\mathrm{d}x' = \int_{x_0}^{x} F(z(x'))\,\mathrm{d}x',$$

and by construction of g_n and using Eq. (6) $\lim_{n\to\infty}\int_{x_0}^{x} g_n(x')\,\mathrm{d}x' = 0$, then

$$z(x) = y_0 + \int_{x_0}^{x} Az(x')\,\mathrm{d}x' + \int_{x_0}^{x} F(z(x'))\,\mathrm{d}x'.$$

Therefore $z(x)$ is the solution of Eq. (1), then $y(x) = z(x)$. The uniqueness of the limit is given by the existence and uniqueness theorem (see [4]).

4. Local error

In this section we study the splitting between the exact and numeric solutions in each step. The cause of this difference is the finite number of segments in which the interval $x_0 \leqslant x \leqslant x_0 + H$ is divided. For simplicity we consider the system

$$\begin{aligned} y'(x) &= z(x) - F(y(x), z(x)), \\ z'(x) &= -y(x) - G(y(x), z(x)), \end{aligned} \tag{9}$$

where F and G are polynomial. Given the initial values y_k, z_k at x_k, integrating Eq. (9) we obtain for $y(x)$:

$$\begin{aligned} y(x) = {} & y_k + (z_k - F(y_k, z_k))(x - x_k) - \frac{1}{2!}\left(y_k + G(y_k, z_k) + \frac{\mathrm{d}}{\mathrm{d}x}F(y(x_k), z(x_k))\right)(x - x_k)^2 \\ & + \frac{1}{3!}\left(-z_k + F(y_k, z_k) - \frac{\mathrm{d}}{\mathrm{d}x}G(y(x_k), z(x_k)) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}F(y(x_k), z(x_k))\right)(x - x_k)^3 + \cdots. \end{aligned} \tag{10}$$

Since we are interested in determining the local error of the method developed in Section 2, we analyze $y(x)$. Using Eqs. (2), (9) can be rewritten as

$$\begin{aligned} y'(x) &= z(x) - F(T_q(y, x_k, x), T_p(z, x_k, x)), \\ z'(x) &= -y(x) - G(T_q(y, x_k, x), T_p(z, x_k, x)), \end{aligned} \tag{11}$$

where $x_k < x < x_k + h_n$, $T_q(y, x_k, x)$ and $T_p(z, x_k, x)$ are the Taylor's polynomials of order q and p of y and z around x_k. To simplify the notation, we suppress the subindex n in the variable y and z. Integrating Eq. (11) with the initial values y_k, z_k in x_k, we obtain

$$\begin{aligned} y(x) = {} & y_k + (z_k - F(T_q(y, x_k, x_k), T_p(z, x_k, x_k)))(x - x_k) \\ & - \frac{1}{2!}(y_k + G(T_q(y, x_k, x_k), T_p(z, x_k, x_k))) + \frac{\mathrm{d}}{\mathrm{d}x}F(T_q(y, x_k, x_k), T_p(z, x_k, x_k))(x - x_k)^2 \\ & + \frac{1}{3!}(-z_k + F(T_q(y, x_k, x_k), T_p(z, x_k, x_k))) - \frac{\mathrm{d}}{\mathrm{d}x}G(T_q(y, x_k, x_k), T_p(z, x_k, x_k)) \\ & - \frac{\mathrm{d}^2}{\mathrm{d}x^2}F(T_q(y, x_k, x_k), T_p(z, x_k, x_k))(x - x_k)^3 + \cdots. \end{aligned} \tag{12}$$

Since T_q, T_p are Taylor's polynomial of order q and p, we obtain

$$\begin{aligned} \frac{\mathrm{d}^m}{\mathrm{d}x^m}T_q(y, x_k, x = x_k) &= \frac{\mathrm{d}^m}{\mathrm{d}x^m}y(x_k), \quad m = 0, 1, \ldots, q, \\ \frac{\mathrm{d}^m}{\mathrm{d}x^m}T_p(z, x_k, x = x_k) &= \frac{\mathrm{d}^m}{\mathrm{d}x^m}z(x_k), \quad m = 0, 1, \ldots, p. \end{aligned}$$

Using these equalities

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}F(T_q(y, x_k, x_k), T_p(z, x_k, x_k)) &= \frac{\partial}{\partial y}F(y(x_k), z(x_k))\frac{\mathrm{d}}{\mathrm{d}x}T_q(y, x_k, x_k) \\ &\quad + \frac{\partial}{\partial z}F(y(x_k), z(x_k))\frac{\mathrm{d}}{\mathrm{d}x}T_p(z, x_k, x_k) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}F(y(x_k), z(x_k)), \end{aligned}$$

in the same way, $\mathrm{d}^m F(y(x_k), z(x_k))/\mathrm{d}x^m = \mathrm{d}^m F(T_q(y, x_k, x_k), T_p(z, x_k, x_k))/\mathrm{d}x^m$ and $\mathrm{d}^m\, G(y(x_k), z(x_k))/\mathrm{d}x^m = \mathrm{d}^m G(T_q(y, x_k, x_k), T_p(z, x_k, x_k))/\mathrm{d}x^m$ where $m = 0, 1, \ldots, \min(p, q)$.

Comparing Eqs. (10) and (12) if we approximate y up to the order q and z up to order $q - 1$, then the solution of Eq. (11) coincides, up to the order $q + 1$, with the solution of Eq. (9).

Summarizing, the solutions $y_n(x)$, $z_n(x)$ are estimates of $y(x)$, $z(x)$ since we approximate $F(y(x), z(x))$ by $F(T_q(y_n(x), x_k, x), T_p(z_n(x), x_k, x))$ and $G(y(x), z(x))$ by $G(T_q(y_n(x), x_k, x), T_p(z_n(x), x_k, x))$ in $x_k \leqslant x \leqslant x_k + h_n$.

Therefore, in each iteration, the local error is $e_k = \|y(x_k) - y_n(x_k)\| = \mathrm{O}(h_n^{q+2})$, $k = 0, \ldots, n-1$. Therefore, at same order, our method is more efficient than CAC.

This result is easily generalizable to Eq. (1) and it is also another way to prove the convergence since the continuous analytic continuation method is uniformly convergent.

5. Applications

In order to present the detailed implementation of the method, we have chosen a particular equation

$$\begin{aligned} y'(x) &= z(x) - P(y(x), z(x)), \\ z'(x) &= -y(x), \end{aligned} \tag{13}$$

where P is a polynomial function. Then using Eqs. (13) the general expression of the Taylor polynomial is obtained: $y(x) \simeq T_r(y(x), x_k, x)$ and $z(x) \simeq T_q(z(x), x_k, x)$. According to Section 2, in the interval $x_k \leqslant x \leqslant x_{k+1}$, $P(y, z)$ is approximated by

$$P(T_r(y(x), x_k, x), T_q(z(x), x_k, x)) = \sum_{j=0}^{m} c_j^{(k)} (x - x_k)^j, \tag{14}$$

using this equation, we determine the coefficients $c_j^{(k)} = c_j^{(k)}(y_n(x_k), z_n(x_k))$. Naturally, $c_j^{(k)}$ and m depend on each election of P. Substituting the Eq. (14) in the Eq. (13) and integrating it, we obtain

$$\begin{aligned} y_n(x) &= \cos(x - x_k) f_k(x - x_k) + \sin(x - x_k) g_k(x - x_k), \\ z_n(x) &= -\sin(x - x_k) f_k(x - x_k) + \cos(x - x_k) g_k(x - x_k), \end{aligned} \tag{15}$$

for $x_k \leqslant x \leqslant x_{k+1}$, where

$$\begin{aligned} f_k(x - x_k) &= y_n(x_k) + \frac{1}{2} \sum_{j=0}^{n} i^{j+1} c_j^{(k)} (\gamma(1 + j, -i(x - x_k)) - (-1)^j \gamma(1 + j, i(x - x_k))) \\ &\quad + \sum_{j=0}^{n} j! \sin\left(\frac{j\pi}{2}\right) c_j^{(k)}, \\ g_k(x - x_k) &= z_n(x_k) + \frac{1}{2} \sum_{j=0}^{n} i^j c_j^{(k)} (\gamma(1 + j, -i(x - x_k)) + (-1)^j \gamma(1 + j, i(x - x_k))) \\ &\quad - \sum_{j=0}^{n} j! \cos\left(\frac{j\pi}{2}\right) c_j^{(k)}, \end{aligned} \tag{16}$$

where $i^2 = -1$ and the incomplete gamma function satisfies $\gamma(\alpha, z) = \int_z^\infty t^{\alpha-1} \exp -t \,\mathrm{d}t$.

Using Eqs. (15) and (16) we evaluate $y_n(x)$, $z_n(x)$, in x_{k+1}, and we obtain

$$\begin{aligned} y_n(x_{k+1}) &= \cos(h_n) f_k(h_n) + \sin(h_n) g_k(h_n), \\ z_n(x_{k+1}) &= -\sin(h_n) f_k(h_n) + \cos(h_n) g_k(h_n), \end{aligned} \tag{17}$$

where the terms of the r.h.s. depend on x_k, $y_n(x_k)$ and $z_n(x_k)$. This allows to determine our iteration procedure that formally is written as

$$\begin{aligned} y_n(x_{k+1}) &= F(x_k, y_n(x_k), z_n(x_k)), \\ z_n(x_{k+1}) &= G(x_k, y_n(x_k), z_n(x_k)), \end{aligned}$$

where $y_0(x_0) = y(x_0)$, $z_0(x_0) = z(x_0)$. Naturally, in each segment, the coefficients $c_j^{(k)}$, f_k, g_k they are re-evaluated using Eqs. (14) and (16). In the following section, these results are implemented to two well-known cases of nonlinear oscillations.

6. Numerical examples

We implement the results of the previous section to two well-known cases. We compare our results with the methods continuous analytic continuation (CAC) [5] and the command NDSolve of the software Mathematica [13] (the default method, Automatic, automatically switches between backward differentiation formulas and Adams multistep methods, depending on stiffness). The considered cases correspond to systems that possess strong nonlinear oscillations [6,8,10].

The first example corresponds to

$$\begin{aligned} y'(x) &= z(x) - (1 - y(x)^2 - z(x)^2), \\ z'(x) &= -y(x), \end{aligned} \tag{18}$$

with the initial values $y(0) = 1$ and $z(0) = 0$, which has the periodic explicit solution $y(x) = \cos x$ and $z(x) = \sin(x)$. This curved solution has the invariant $c(x) = y^2 + z^2 = 1$. We integrate Eq. (18) by using of the three methods: the proposed method developing y and z until the third order, Taylor polynomials (CAC) developing y and z until the third order and by the command NDSolve. We determine the discreet distance d between the invariant $c(x) = 1$ and the numeric result c_k,

$$d(c(x_k), c_k) = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(1 - c_k)^2},$$

where n is the number of iterations. The value of d gives us a measure of the precision of the numeric result. Using an integration step $h = 0.01$ and different periods of the solution, we obtained the following results for the distance d

Periods	1	10	100
Proposed	5×10^{-10}	4×10^{-10}	4×10^{-10}
Taylor	3×10^{-7}	2×10^{-5}	2×10^{-5}
NDSolve	6×10^{-8}	6×10^{-8}	6×10^{-8}

Based on the numeric results we see that: (1) the methods maintain invariant c_k with good precision, (2) the precision obtained with the proposed method is substantially better than in the two other cases.

As a second example we consider the van der Pol equation that has one limit cycle

$$\begin{aligned} y'(x) &= z(x) - \mu(\tfrac{1}{3}y(x)^3 - y(x)), \\ z'(x) &= -y(x). \end{aligned} \tag{19}$$

Since we do not have an explicit solution, we consider as reference the obtained one by NDSolve with initial integration step $h = 0.001$. Developing y until the third order and using the recursive Eqs. (17), we compare the obtained results with the CAC method of third order. In the same way as in the previous example, we use the distance d to evaluate the precision of the solutions. Now, we use the distance

$$d(y(x_k), y_k) = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(y(x_k) - y_k)^2},$$

where $y(x_k)$ is the reference solution obtained with NDSolve, y_x is the discreet solution obtained with the proposed method or CAC and n is the number of iterations. To obtain the limit cycle, we choose $\mu = 1$, $x_0 = 1.85$ and $y_0 = 0$. The integration step $h = 0.1$ for the left table and $h = 0.01$ for the right table. Integrating in different periods of the solution, we compare d for the proposed method and CAC.

Periods	1	10	100
Proposed	10^{-5}	10^{-4}	10^{-3}
Taylor	3×10^{-4}	10^{-3}	10^{-2}

Periods	1	10	100
Proposed	10^{-9}	10^{-8}	10^{-7}
Taylor	3×10^{-7}	10^{-6}	10^{-5}

Although the distance d is a measure of the global error, we obtain better results with our method. With regard to the local and global error, it is interesting to remark that the result is consistent with those of Section 3.

7. Conclusions

In this paper we have developed a fixed-one-step iteration integration method, for non linear systems of ordinary differential equations with given initial value. It is based on the analytic solution of the non-homogeneous linear equation. We demonstrate the uniform convergence and we determine the local error and at the same order, our method is more efficient than CAC. Even better results are obtained with respect to other classic methods. The numeric implementation is relatively simple and shows that the method is stable even for a great number of steps.

References

[1] E.N. Aksan, A. Ozdes, T. Ozis, Appl. Math. Comput. 176 (2006) 270–279.
[2] E. Babolian, A. Vahidi, G. Asadi Cordshooli, Appl. Math. Comput. 167 (2005) 1150–1155.
[3] S. Chapra, R. Canale, Numerical Methods for Engineers with Programming and Software Applications, McGraw-Hill, London, 1999.
[4] C. Chicone, Ordinary Differential Equations with Applications, Springer, New York, 1999.
[5] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, London, 1962.
[6] J.-H. He, Chaos Solitons and Fractals 26 (2005) 827–833;
J.-H. He, Appl. Math. Comput. 135 (2003) 73–79.
[7] A.C. Hindmarsh, L.R. Petzold, Practical Numerical Algorithms Part I, Comput. Phys. 9 (1) (1995) 34–41;
A.C. Hindmarsh, L.R. Petzold, Practical Numerical Algorithms Part II, Comput. Phys. 9 (2) (1995) 148–155
[8] H.-M. Liu, Chaos, Solitons and Fractals 23 (2005) 577–579.
[9] W. Press, B. Flannery, et al., Numerical Recipes, Cambridge University Press, New York, 1986;
E. Hairer, S.P. Norsett, G. Wanner, Solving Ordinary Differential Equations I, second ed., Springer, New York, 2000.
[10] J.I. Ramos, Chaos, Solitons and Fractals 27 (2006) 1229–1238.
[11] S. Strogatz, Nonlinear Dynamics and Chaos, Addison-Wesley, New York, 1994;
C.H. Edwards Jr., D.E. Pennney, Elementary Differential Equations with Boundary Value Problems, third ed., Prentice-Hall, New York, 1993
[12] W. Wang, Z. Li, Appl. Math. Comput. 48 (2004) 237–252;
W. Wang, Z. Li, Appl. Math. Comput. 172 (2006) 568–583.
[13] S. Woffram, Mathematica, A System for Doing Mathematics by Computer, second ed., Addison-Wesley, New York, 1991.
[14] J. Zheng, T. Sederberg, R. Johnson, Appl. Math. Comput. 48 (2004) 237–252.
[15] J. Zheng, T. Sedeberg, R. Johnson, Appl. Numer. Math. 48 (2004) 237–252.