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Projections onto Translation—Invariant Subspaces of $L_1(G)$

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Let G be a locally compact abelian group. A translation-invariant subspace in $L_1(G)$ may or may not be complemented depending on the structure of its hull in $\hat{G}$. Techniques for deciding this complementation problem in a variety of situations are developed and illustrated with examples. A complete characterization is obtained for those ideals with a discrete hull. © 1984 Academic Press, Inc.

0. INTRODUCTION

In [13], Rosenthal gave necessary conditions on a subset A of Γ, the dual group of a locally compact abelian group G, for an ideal $I \subset L_1(G)$ with $\text{hull}(I) = A$ to be complemented in $L_1(G)$. For the special case $G = R$, the first two authors [1] were able to complete the characterization of the complemented ideals. In this paper, we expand the investigation to other locally compact abelian groups. We are not able to determine in general whether a given ideal is complemented; however, we have been able to prove some useful theorems and uncover some interesting phenomena which do not occur in $L_1(R)$.

Rosenthal proved in [13] that if an ideal $I \subset L_1(G)$ is complemented, then $h(I) = \{\gamma \in \Gamma = \hat{G} : \hat{f}(\gamma) = 0 \text{ for all } f \in I\}$ is an element of the coset ring of Γ with the discrete topology. Obviously $h(I)$ is closed in Γ in the usual topology; and thus, a necessary condition for an ideal I to be complemented in $L_1(G)$ is that $h(I)$ be a closed subset of Γ which is an element of the coset

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ring of Γ_d, i.e., Γ with the discrete topology. In case of $G = R$, Rosenthal [13] proved that such sets were of the form $\bigcup_{i=1}^{n} (\alpha_i Z + \beta_i) \backslash F$, where $\alpha_i, \beta_i, i = 1,..., n$, are real numbers and F is a finite set. Subsequently, other authors [2, 3, 12, 16, 17] showed that in general a closed subset of Γ which belongs to the coset ring of Γ_d is of the form $\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash \bigcup_{j=1}^{n_i} \gamma_{ij} + \Gamma_{ij})$, where $\gamma_i, \gamma_{ij}, j = 1,..., n_i, i = 1,..., n$, are elements of $\Gamma, \Gamma_i, i = 1,..., n$, are closed subgroups of Γ, and $\Gamma_{ij}, j = 1,..., n_i$, are clopen subgroups of Γ_i. Moreover, they showed that these sets are strong Ditkin sets and, thus, in particular they are sets of spectral synthesis.

In the case of $L_1(R)$, the characterization of the complemented ideals is that I is complemented if and only if $h(I) = \bigcup_{i=1}^{n} (\alpha_i Z + \beta_i) \backslash F$ as above with $\{\alpha_i\}$ pairwise rationally dependent, see [1]. Unfortunately, any characterization for general $L_1(G)$ cannot be so simple. Several things contribute to the simplicity of the result in $L_1(R)$: the elements of the coset ring to be dealt with are discrete, independent translations of the cosets do not alter the characterizing conditions, and these characterizing conditions only involve pairs of cosets. We will show that in various situations the failure of these properties causes difficulties. Because we do not have a complete characterization of the complemented ideals, we will emphasize examples in this exposition to illustrate the basic difficulties of the complementation question.

The first section of this paper is devoted to notation, definitions, and some simple lemmas. In the second section, we generalize the characterization of complemented ideals in $L_1(R)$ to case of ideals in $L_1(G)$ with discrete hulls. The main result here is Theorem 2.3 which loosely speaking says that the proper generalization of the algebraic condition of rational dependence used in $L_1(R)$ is the topological condition of uniform seperation of the cosets. The third section is devoted to the creation of an inductive procedure for proving implementation in a large number of cases. This procedure is complicated, but it seems to be needed even to prove this result: an ideal I in $L_1(R^n)$ is complemented if the hull A is a finite union of affine subspaces. In the fourth and final section some possible directions for further work are described and an extension of one of the results of Section 2 is proved. This theorem says that if Γ_1 and Γ_2 are closed subgroups of G, then $I(\Gamma_1 \cup \Gamma_2)$ is complemented if and only if $\Gamma_1 + \Gamma_2$ is closed modulo $\Gamma_1 \cap \Gamma_2$.

Let us remark that while most proofs are given for general locally compact abelian groups, the reader will find that in most cases little of interest is lost in assuming that the group is R^n or one of its closed subgroups. In fact, the following examples from R^2 and R^3 will be used to illustrate most of the results that we prove.

Let $I(A)$ denote $\{f \in L_1(G): \hat{f}(a) = 0 \text{ for all } a \in A\}$. To avoid the use of large numbers of parentheses, we will follow the convention that algebraic operations and Cartesian products precede unions, intersections, and set differences, unless otherwise indicated.

0.1. EXAMPLES. (i) $I(\bigcup_{i=1}^{n} {}_{\theta_i}R) \subset L_1(R^2)$, where ${}_{\theta_i}R$ denotes the line $x \tan(\theta_i) = y$. This is complemented. See Section 3.

(ii) $I(Z \times R \cup R \times Z) \subset L_1(R^2)$. This is complemented. See Section 1.

(iii) $I(Z \times R \cup R \times Z \cup {}_{\theta}R) \subset L_1(R^2)$. This is complemented if and only if $\tan(\theta)$ is rational. See Sections 1 and 3.

(iv) $I(R \times Z \times \{0\} \cup \{0\} \times \sqrt{2}\, Z \times R) \subset L_1(R^3)$. This is not complemented. Sections 2 and 4.

(v) $I(R \times Z \times \{0\} \cup \{0\} \times \sqrt{2}\, Z \times R \cup \{0\} \times R \times \{0\}) \subset L_1(R^3)$. This is complemented. See Section 4.

Even from these few examples, one can see that there are grave difficulties in formulating a conjecture for a characterization of the complemented ideals in terms of their hulls. The main unresolved question is whether there is a geometrical, topological, or algebraic condition on the hull which is necessary and sufficient for the ideal to be complemented in $L_1(R)$. Recently, the first named author has been able to use the techniques of this paper to give a complete characterization of the complemented ideals in $L_1(R^2)$. The characterization is not easily stated and this work will be published elsewhere.

1. DEFINITIONS, NOTATION, AND PRELIMINARY RESULTS

Throughout this paper, G (possibly with subscripts) will be a Hausdorff locally compact abelian (LCA) group and H will be a closed subgroup of G. The dual of G, $\hat{G}$, will usually be denoted by Γ. Because our basic examples are in $L_1(R^n)$, we will use additive notation in both G and Γ unless otherwise noted. See Rudin [15] for standard notation and facts.

If H is a closed subgroup of G, then $H^\perp = \{\gamma \in \Gamma : \gamma(h) = 1 \text{ for all } h \in H\}$ and we will identify $H^\perp$ with $\widehat{G/H}$ in the canonical manner. The map π_H (or if the subgroup H is fixed, then just π) will denote both the quotient map of groups $\pi_H : G \to G/H$ and the induced map $\pi_H : L_1(G) \to L_1(G/H)$ given by $\pi_H f(x) = \int_H f(x+y)\, dm_H(y)$ for a.e. $[m_{G/H}]$, $x \in G/H$, where m_H is a Haar measure on H. If H is compact, then $m_H(H) = 1$; if H is not compact, then we will always assume suitable normalizations so that the formula

$$\int f\, dm_G = \int_{G/H} \int_H f(x+y)\, dm_H(y)\, dm_{G/H}(x)$$

is correct for all $f \in L_1(G)$.

Let us remark that the quotient map $\pi_H : L_1(G) \to L_1(G/H)$ has many right inverses. In particular, if ϕ is a locally bounded, locally measurable function on G, i.e., $\phi \in L_\infty^{\mathrm{loc}}(G)$, satisfying

$$\int_H \phi(x+y)\, dm_H(y) = 1 \qquad \text{a.e. } [m_{G/H}],$$

then $Sf(g) = f(g+H)\phi(g)$, for $f \in L_1(G/H)$, defines an isometry of $L_1(G/H)$ into $L_1(G)$ such that $\pi_H S = I_{L_1(G/H)}$, the identity operator. Reiter [11, Chap. 8, Sect. 1.8], proves that such ϕ, actually satisfying stronger properties, always exist and, following Reiter [11], we call such a function a *Bruhat function.* In Theorem 4.4, we will need $\phi \in L_\infty(G)$ and uniformly continuous. Indeed, it is not hard to show there exists a m_G-measurable set M such that $m_H(M+x) = m_H(M + x \cap H) = 1$ for all $x \in G$. Then let $f \in C_c(G)$, $\int_G f(x)\, dm_G(x) = 1$, and let $\phi = f * 1_M$. Then ϕ is uniformly continuous Bruhat function with $|\phi| \leqslant \|f\|_\infty$.

One simple and well-known consequence of the existence of S is that $\pi_H(L_1(G)) = L_1(G/H)$. Moreover, we have $\hat{f}(\gamma) = \widehat{\pi_H f}(\gamma)$ for $\gamma \in H^\perp = \widehat{G/H}$ and $f \in L_1(G)$. We wish to carry this one step further. Suppose that A is a closed subset of Γ. We would like to know that $\pi_H(I(A)) = I(A \cap H^\perp)$, as a subspace of $L_1(G/H)$. A sufficient condition for this equality to hold is that A be a strong Ditkin set (see Definition 1.1) because this implies that $A \cap H^\perp$ is a set of spectral synthesis. Actually $A \cap H^\perp$ is also a strong Ditkin set.

1.1. Definition. A subset A of Γ is said to be a *strong Ditkin set* if there is a net $\{\mu_\alpha\}$ of measures in $M(G)$ such that

(i) $\|\mu_\alpha\| \leqslant M < \infty$ for all α,

(ii) $\lim_\alpha \|\mu_\alpha * f\|_1 = 0$ for $f \in I(A)$,

(iii) $\hat{\mu}_\alpha = 1$ on a neighborhood of A, for all α (the neighborhood depends on α).

The class of strong Ditkin sets is closed under finite unions and intersections. Moreover, if A and B are strong Ditkin subsets of Γ, then

$$I(A \cap B) = I(A) + I(B).$$

This follows from a general argument using approximate identities. See Gilbert [3], Rosenthal [12], and Rudin [14]. Also, because $A \cup B$ is a set of spectral synthesis, $I(A \cup B) = I(A) \cap I(B)$.

As was mentioned in the Introduction, we will be concerned only with certain special closed subsets of Γ. Let us denote the coset ring of Γ by $\Omega(\Gamma)$ and let $\Omega_c(\Gamma)$ be all closed sets in Γ which are in the ring $\Omega(\Gamma_d)$, where Γ_d

denotes Γ with the discrete topology. We may state Rosenthal's results as follows.

1.1. PROPOSITION. *If I is a complemented translation invariant subspace of $L_1(G)$, then $h(I) \in \Omega_c(\Gamma)$.*

Also, summarizing the results of [2, 13, and 17], we have

1.2. PROPOSITION. *If $A \in \Omega_c(\Gamma)$, then A is a strong Ditkin set and A is of the form $\bigcup_{i=1}^n \gamma_i + (\Gamma_i \setminus \bigcup_{j=1}^{n_i} \gamma_{ij} + \Gamma_{ij})$, where $\{\gamma_i : i = 1,..., n\} \subset \Gamma$, $\{\gamma_{ij} : j = 1,..., n_i\} \subset \Gamma_i$, Γ_i is a closed subgroup of Γ and Γ_{ij} is a clopen subgroup of Γ_i for all $i = 1,..., n; j = 1,..., n_i$.*

Because the elements of $\Omega_c(\Gamma)$ are sets of spectral synthesis, we can use the notation $I(A)$ to denote the ideal with $h(I(A)) = A$.

Let us now recall that the translation-invariant projections are given by convolution against idempotent measures [15] and that by Cohen's theorem, the Fourier transform of such a measure is the characteristic function of a set in $\Omega(\Gamma)$. Rosenthal observed that there are ideals which are complemented, but not by a translation invariant projection. In particular, $I(Z)$ is complemented in $L_1(R)$ by

$$Pf(x) = f(x) - \sum_{n \in Z} f(x + 2\pi n)\, 1_{[0,2\pi]}(x)$$

for all $f \in L_1(R)$. The abstract version of this was proved in [10]. We include a proof of this result here because it provides intuition for later arguments. For $\mu \in M(G)$, define $\check{\mu}(A) = \mu(-A)$ for all Borel sets A. Then define $C_\mu : L_1(G) \to L_1(G)$ by $C_\mu(f) = \mu * f$. Then $\widehat{C_\mu(f)} = \hat{\mu} \cdot \hat{f}$ for all $f \in L_1(G)$.

1.3. PROPOSITION. *Let Γ_1 be a closed subgroup of Γ and let $A \in \Omega(\Gamma_1)$. Then $I(A)$ is complemented in $L_1(G)$.*

Proof. Let $H = \Gamma_1^\perp = \{g \in G : \gamma(g) = 1 \text{ for all } \gamma \in \Gamma_1\}$ and let ϕ be a Bruhat function for H. Because $A \in \Omega(\Gamma_1)$, by Cohen's theorem there exists $\mu \in M(G/H)$ such that $\hat{\mu} = 1_A$. Define $Qf(g) = \phi(g)\, C_\mu \pi_H f(g + H)$ and $Pf = f - Qf$ for all $f \in L_1(G), g \in G$. Then Q and P are continuous projections. Notice that for $\alpha \in \Gamma_1$, $\widehat{\pi_H f}(\alpha) = \hat{f}(\alpha)$ and $\widehat{Qf}(\alpha) = \widehat{C_\mu(\pi_H f)}(\alpha) = \hat{\mu}(\alpha)\, \widehat{\pi_H f}(\alpha)$. Hence if $f \in I(A)$, then $C_\mu(\pi_H f) \in L_1(G/H)$ and $\widehat{C_\mu(\pi_H f)} = 0$ on $\widehat{G/H} = \Gamma_1$. So $f \in I(A)$ implies $Qf = 0$. If $\widehat{Qf}(\alpha) = 0$ for all $\alpha \in \Gamma_1$, then $\hat{f}(\alpha) = 0$ for all $\alpha \in A$. Hence, $\ker(Q) = I(A)$ and P is a projection onto $I(A)$. ∎

Let us note that if $A \in \Omega_c(\Gamma)$ and $I(A)$ is complemented, and $\gamma \in \Gamma$, then $I(\gamma + A)$ is complemented. Also, it was observed in [10] that under suitable conditions, if $I(A)$ and $I(B)$ are complemented, then $I(A \cup B)$ is also complemented:

1.4. LEMMA. *If A and B are closed subsets of Γ such that $I(A)$ and $I(B)$ are complemented in $L_1(G)$, and there is a measure $\mu \in M(G)$ such that $\hat{\mu}|_A = 1$ and $\hat{\mu}|_B = 0$, then $I(A \cup B)$ is complemented in $L_1(G)$.*

Proof. Let P_A and P_B be the projections on $I(A)$ and $I(B)$, respectively, and let Q_A and Q_B be their complementary projections. Define Q on $L_1(G)$ by $Qf = (I - C_{\hat{\mu}})\, Q_B f + (C_{\hat{\mu}} Q_A f)$ for $f \in L_1(G)$. Then $Q^*F = Q_B^*(F - C_\mu(F)) + Q_A^*(C_\mu(F))$ for $F \in L_\infty(G)$. If $c \in A \cup B$, then $Q^*c = c$. Thus, Q^* is the identity on the w^*-closed span of $A \cup B$. Also, range $(Q^*) \subset \overline{\text{span}}^{w^*}(A) + \overline{\text{span}}^{w^*}(B) \subset \overline{\text{span}}^{w^*}(A \cup B)$. Because $A \cup B$ is a set of spectral synthesis, $P = I - Q$ is a projection onto $I(A \cup B)$. ∎

We now consider the question of the existence of a measure μ as in Lemma 1.4.

1.5. LEMMA. *Let A and B be closed sets in Γ. The following are equivalent*:

(i) *there exists $\mu \in M(G)$ such that $\hat{\mu}|_A = 1$ and $\hat{\mu}|_B = 0$;*

(ii) *there exists a compact neighborhood W of 0 in Γ such that $A + W \cap B + W = \varnothing$;*

(iii) *there does not exist a pair of nets $\{a_\alpha\} \subset A$, $\{b_\alpha\} \subset B$ such that $\lim_\alpha a_\alpha - b_\alpha = 0$.*

Proof. Suppose (i). Because $\hat{\mu}$ is uniformly continuous, there exists a compact neighborhood W of 0 in Γ such that for all γ_1 with $\hat{\mu}(\gamma_1) = 0$, $|\hat{\mu}(\gamma)| < \frac{1}{2}$ if $\gamma \in \gamma_1 + W$, and for all γ_2 with $\hat{\mu}(\gamma_2) = 1$, $|\hat{\mu}(\gamma)| > \frac{1}{2}$ if $\gamma \in \gamma_2 + W$. So $A + W \cap B + W = \varnothing$. The equivalence of (ii) and (iii) is easy to see. Assume (iii). Then let βA and βB denote the closures of A and B in the Bohr compactification $\beta\Gamma$ of Γ. Condition (iii) means $\beta A \cap \beta B = \varnothing$. Because $L_1(\widehat{\beta\Gamma}) = M(\widehat{\beta\Gamma})$ is a normal algebra on $\widehat{\beta\Gamma}$, there exists $f \in L_1(\widehat{\beta\Gamma})$ such that $\hat{f}|_{\beta A} = 1$ and $\hat{f}|_{\beta A} = 0$. Since $\widehat{\beta\Gamma} = G_d$, f has the form $\sum_{n=1}^\infty c_n 1_{\{g_n\}}$, where $\{g_n\} \subset G$ and $\sum_{n=1}^\infty |c_n| < \infty$. It follows that the measure $\mu = \sum_{n=1}^\infty c_n \delta_{g_n}$ in $M(G)$ has $\hat{\mu}|_A = 1$ and $\hat{\mu}|_B = 0$. ∎

The next lemma will be used in Section 3. It gives a slightly stronger result than the previous lemma for elements in $\Omega_c(\Gamma)$.

1.6. LEMMA. *If $A, B \in \Omega_c(\Gamma)$, and there is a neighborhood W of 0 in Γ such that $A + W \cap B + W = \varnothing$, then there exists a measure $\mu \in M(G)$ with compact support such that $\hat{\mu}|_A = 1$ and $\hat{\mu}|_B = 0$.*

Proof. Let $\beta\Gamma$ denote the Bohr compactification of Γ; and for $A \subset \Gamma$, let βA denote the closure of A in $\beta\Gamma$. Since $A + W \cap B + W = \varnothing$, $\beta A \cap \beta B = 0$. But also $\beta A, \beta B \in \Omega_c(\beta\Gamma)$. Indeed, by Proposition 1.2, $A = \bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i)$, where $\gamma_i \in \Gamma$, Γ_i are closed subgroups of Γ, and B_i is a finite union of clopen cosets of $\Gamma_i, i = 1, ..., n$. If C is a clopen coset of Γ_i, then $C + C \cap (\Gamma_i \backslash C) + C = \varnothing$ and so $\beta C \cap \beta(\Gamma_i \backslash C) = 0$. Also, $\beta\Gamma_i = \beta C \cup \beta(\Gamma_i \backslash C)$. So βC is a clopen coset of $\beta\Gamma_i$. It follows that βB_i is a finite union of clopen cosets of $\beta\Gamma_i$ and $\beta(\Gamma \backslash B_i) = \beta\Gamma \backslash \beta B_i$. Hence, $\beta A = \bigcup_{i=1}^n \gamma_i + (\beta\Gamma_i \backslash \beta B_i)$ and βA is a closed set in $\Omega((\beta\Gamma)_d)$. Also, below we construct $\mu \in M(G_d)$, $G_d = \widehat{\beta\Gamma}$, such that $\hat{\mu} = 1$ on βA, $\hat{\mu} = 0$ on βB, and also μ has compact support. But then μ is a finite linear combination of Dirac masses in G such that $\hat{\mu} = 1$ on A and $\hat{\mu} = 0$ on B.

By the above, we may assume Γ is compact, $A, B \in \Omega_c(\Gamma)$, and $A \cap B = \varnothing$. By Proposition 1.2, and by taking sums of convolutions of the measures constructed below, we may assume that there are closed subgroups Γ_1, Γ_2 in $\Gamma, \alpha_1, \beta_1 \in \Gamma$ such that $A = \alpha_1 + A_1, B = \beta_1 + B_1$ for some $A_1 \in \Omega(\Gamma_1)$, $B_1 \in \Omega(\Gamma_2)$. Hence, it is enough to assume that we have closed subgroups Γ_1, Γ_2 in Γ, $A \in \Omega(\Gamma_1)$, $B \in \Omega(\Gamma_2)$, $\gamma_0 \in \Gamma$ such that $A \cap B + \gamma_0 = \varnothing$, and then construct $\mu \in M(G)$ with compact support such that $\hat{\mu} = 1$ on $A, \hat{\mu} = 0$ on $B + \gamma_0$.

First, assume $\gamma_0 \notin \Gamma_1 + \Gamma_2$. Then there is $g \in G$ such that $\gamma_0(g) = \alpha \neq 1$, and $\gamma(g) = 1$ for $\gamma \in \Gamma_1 + \Gamma_2$. Let $\mu = (1/(1 - \alpha))(\delta_{-g} - \alpha\delta_0)$. Then $\hat{\mu}|_A = 1$ and $\hat{\mu}|_{B+\gamma_0} = 0$. Otherwise $\gamma_0 \in \Gamma_1 + \Gamma_2$, and then there is no harm in assuming $\Gamma = \Gamma_1 + \Gamma_2$. If A is an open subgroup of Γ_1, then $A \cap A + B + \gamma_0 = \varnothing$ and $A + B + \gamma_0 \in \Omega(\Gamma)$ because $\Gamma = \Gamma_1 + \Gamma_2$. Hence, there exists an idempotent measure $\mu \in M(G)$ with $\hat{\mu} = 1_{A+B+\gamma_0}$. The measure μ necessarily has compact support. But then μ satisfies $\hat{\mu} = 0$ on A, $\hat{\mu} = 1$ on $B + \gamma_0$, and $\hat{\mu}$ has compact support. By translating $\hat{\mu}$, this argument also handles the case where A is a coset of an open subgroup of Γ_1. Finally, if $A \in \Omega(\Gamma_1)$, then $A = \bigcup_{i=1}^n A_i$, where each A_i is a coset of an open subgroup of Γ_1. Choose μ_i as above with $\hat{\mu}_i = 0$ on A_i, $\hat{\mu} = 1$ on $B + \gamma_0$, and μ_i has compact support. Let $\mu = \delta_0 - \mu_1 * \cdots * \mu_n$. Then μ has compact support, $\hat{\mu} = 1$ on A, and $\hat{\mu} = 0$ on $B + \gamma_0$. ∎

Remark. We see from the proof that the measure μ constructed above is a finite linear combination of Dirac masses in G.

Let us now consider some examples in $L_1(R^2)$. Let $A = Z \times Z \backslash \{0\} \times Z$ and let $B = \{0\} \times \sqrt{2}\, Z$. By Proposition 1.3, both $I(A)$ and $I(B)$ are complemented. Moreover, A and B are separated as in Lemma 1.5. So by

Lemma 1.4, $I(A \cup B)$ is complemented. We will see in this section that $I(Z \times Z \cup B)$ is not complemented. This shows that unlike the case of $L_1(R)$, the relatively clopen cosets that are removed from a subgroup which is part of the hull of an ideal can have a significant part in determining complementation.

A second instance in which the general case and $L_1(R)$ differ is in the importance of translations. Indeed, $I(\bigcup_{i=1}^n \alpha_i Z + \beta_i)$ is complemented in $L_1(R)$ if and only if $I(\bigcup_{i=1}^n \alpha_i Z)$ is complemented. However, $I(\{0\} \times Z \cup \{1\} \times \sqrt{2}\, Z)$ is complemented in $L_1(R^2)$, but as we will see in this section, $I(\{0\} \times Z \cup \{0\} \times \sqrt{2}\, Z)$ is not complemented.

Now let us consider some nondiscrete examples. First the ideal $I(\{0\} \times R \cup R \times \{0\})$ is complemented in $L_1(R^2)$. Indeed, define

$$Q_1 f(x, y) = \int_{-\infty}^{\infty} f(x, t)\, dt \cdot 1_{[0,1]}(y),$$

$$Q_2 f(x, y) = \int_{-\infty}^{\infty} f(s, y)\, ds \cdot 1_{[0,1]}(x),$$

for all $f \in L_1(R^2)$. Then Q_1 and Q_2 are projections on $L_1(R^2)$ having the same form as Q in the proof of Proposition 1.3. An easy calculation shows that $P = (I - Q_1)(I - Q_2) = (I - Q_2)(I - Q_1)$ is the required projection. Second, the ideal $I(Z \times R \cup R \times Z)$ is complemented in $L_1(R^2)$; this is Example 0.1 (ii). Indeed, define

$$Q_1 f(x, y) = \sum_{n \in Z} f(x, 2\pi n + y)\, 1_{[0,2\pi]}(y),$$

$$Q_2 f(x, y) = \sum_{n \in Z} f(2\pi n + x, y)\, 1_{[0,2\pi]}(x),$$

for $f \in L_1(R^2)$. As above, $P = (I - Q_1)(I - Q_2) = (I - Q_2)(I - Q_1)$ is the required projection.

Both of these examples illustrate a phenomenon not present in $L_1(R)$; namely, the hull in each case has the form $\Gamma_1 \cup \Gamma_2$, where the closed subgroups Γ_1 and Γ_2 do have (many) points close together, i.e., given $\varepsilon > 0$, there are infinitely many points $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$ with $d(\gamma_1, \gamma_2) < \varepsilon$ for a metric d on R^2 in the usual topology. But these examples are also special in that the projections Q_1 and Q_2 which correspond to the subgroups Γ_1 and Γ_2 are commuting projections. In general it seems unlikely that we can find commuting projections like this.

The next proposition shows what is needed to build projections inductively.

1.7. Proposition. *Let $A \in \Omega_c(\Gamma)$ and let Γ_1 be a closed subgroup of Γ. Let $H = \Gamma_1^{\perp}$. Suppose that $B \in \Omega(\Gamma_1)$ and that $\mu \in M(G/H)$ has $\hat{\mu} = 1_{\Gamma_1 \setminus B}$.*

Then $I(A \cup (\Gamma_1 \backslash B))$ is complemented in $I(A)$ if and only if there is a subspace X of $I(A)$ such that $C_\mu \pi|_X$ is an isomorphism of X onto $I((\Gamma_1 \cap A) \cup B) \subset L_1(G/H)$.

Proof. Suppose that X is complementary to $I(A \cup (\Gamma_1 \backslash B))$ in $I(A)$ and that P is a projection of $I(A)$ onto X. Note $\pi(I(A)) = I(A \cap \Gamma_1)$ and thus $C_\mu \pi(I(A)) = I((A \cap \Gamma_1) \cup B)$. Now $\ker(C_\mu \pi) = I(A \cup (\Gamma_1 \backslash B))$ and so $C_\mu \pi|_X$ is one-to-one and onto $I((A \cap \Gamma_1) \cup B)$ as required.

Conversely, define $P_1 f = (C_\mu \pi|_X)^{-1} C_\mu \pi(f)$ for all $f \in I(A)$. Then $P = I - P_1$ is a projection from $I(A)$ onto $I(A \cup (\Gamma_1 \backslash B))$. ∎

In the sequel, we will refer to X as a *lift* of $I((A \cap \Gamma_1) \cup B)$. With the same notation as in Proposition 1.7, we have the following:

1.8. COROLLARY. *Suppose that $I(A \cup (\Gamma_1 \backslash B))$ is complemented in $L_1(G)$. Then $I(A)$ is complemented if and only if $I((A \cap \Gamma_1) \cup B)$ is complemented in $L_1(G/H)$.*

Proof. If $I(A)$ is complemented, let X be the complement of $I(A \cup (\Gamma_1 \backslash B))$ in $I(A)$. Then X is complemented in $L_1(G)$ and by Proposition 1.7 it is isomorphic to $J = I((A \cap \Gamma_1) \cup B)$ in $L_1(G/H)$. Hence, J is isomorphic to a complemented subspace of $L_1(G)$ and thus J is complemented in its second dual J^{**}. But also, by Gilbert [2], $J^\perp$ is complemented in $L_1(G/H)^*$. Hence, J itself is complemented in $L_1(G/H)$. See [7].

Conversely, suppose that $I((A \cap \Gamma_1) \cup B)$ is complemented in $L_1(G/H)$ by a projection P_1. Let X be the complement of $I(A \cup (\Gamma_1 \backslash B))$ in $I(A)$, let $\Phi = (C_\mu \pi|_X)^{-1}$ with $\text{domain}(\Phi) = I((A \cap \Gamma_1) \cup B)$, and let P_2 be a projection from $L_1(G)$ onto $I(A \cup (\Gamma_1 \backslash B))$. Then $P = P_2 + \Phi P_1 C_\mu \pi$ gives a projection of $L_1(G)$ onto $I(A \cup (\Gamma_1 \backslash B)) + X = I(A)$. ∎

Remark. In the first part of this proof, we used a general lemma for Banach spaces. Let J be a closed subspace of Z with Z complemented in Z^{**} and $J^\perp$ complemented in Z^*. Then J is complemented in Z if and only if J is complemented in J^{**}. See [1, 6, 7, 8] for details.

This last corollary gives us a technique for showing that some ideals are not complemented. Indeed, consider $I(Z \times R \cup R \times Z \cup {}_\theta R)$, where $\tan(\theta)$ is irrational; this is Example 0.1 (iii). If this ideal were complemented in $L_1(R^2)$, then the fact that $I(Z \times R \cup R \times Z)$ is also complemented would imply that $I((Z \times R \cup R \times Z) \cap {}_\theta R)$ is complemented in $L_1(R^2/{}_\theta R) \sim L_1(R)$. Because $\tan(\theta)$ is irrational, the two sets $\alpha_1 Z = (Z \times R) \cap {}_\theta R$ and $\alpha_2 Z = (R \times Z) \cap {}_\theta R$ have rationally independent periods α_1 and α_2. Consequently, by Alspach and Matheson [1], $I(\alpha_1 Z \cup \alpha_2 Z) = I((Z \times R \cup$

$R \times Z) \cap {}_\theta R)$ is not complemented in $L_1(R^2/{}_\theta R)$. This contradiction shows that Example 0.1 (iii) is not complemented if $\tan(\theta)$ is irrational.

Our next result refines Corollary 1.8.

1.9. PROPOSITION. *Suppose* $A, B \in \Omega_c(\Gamma)$ *and* $I(A \cup B)$ *is complemented in* $I(A)$ *with complementary subspace* X. *Then*

(i) $I(B) \oplus X = I(A \cap B)$,

(ii) *if in addition* $I(A)$ *and* $I(B)$ *are complemented in* $L_1(G)$, *then* $I(A \cap B)$ *is complemented in* $L_1(G)$.

Proof. It follows that $I(B) + I(A) = I(A \cap B)$ and $I(A) \cap I(B) = I(A \cup B)$ as we observed earlier. Thus $I(B) + I(A) = I(B) + I(A \cup B) + X = I(B) + X$, and $I(B) \cap X = \{0\}$. This proves (i).

For (ii), note that X is complemented in $L_1(G)$ and therefore $I(B) \oplus X = I(A \cap B)$ is isomorphic to a complemented subspace of a second conjugate space, $L_1(G)^{**} \oplus L_1(G)^{**}$ in this case. So $I(A \cap B)$ is complemented in $I(A \cap B)^{**}$. By Gilbert [2], $I(A \cap B)^{\perp}$ is complemented in $L_\infty(G) = L_1^*(G)$. Hence, by the previous remark, $I(A \cap B)$ is complemented. ∎

This proposition will be used later to give other examples of ideals which are not complemented.

2. IDEALS WITH A DISCRETE HULL

The main result of this section is a characterization of the complemented ideals with a discrete hull. This is the natural generalization of the complemented ideals in $L_1(R)$. As was noted in the discrete examples in the last section, the result is complicated by the necessity of dealing with the cosets removed from and translated in the hull. There are two main steps to the theorem. First, we will give an obstruction criterion for the lift of Proposition 1.7; second, we will prove a decomposition result for discrete hulls.

2.1. PROPOSITION. *Let* Γ_1 *be a discrete subgroup of* Γ, A *an infinite set in* $\Omega(\Gamma_1)$, *and* $B \in \Omega_c(\Gamma)$ *such that*

(i) $A \cap B = \varnothing$,

(ii) *for each compact neighborhood* W *of* 0 *in* Γ, *there exists* $a \in A, b \in B$, *such that* $a - b \in W$.

Then $I(A \cup B)$ *is not complemented in* $I(B)$.

Proof. Suppose that $I(A \cup B)$ were complemented in $I(B)$ by a projection P. Then by Proposition 1.7, there is a subspace $X = \ker(P) \subset I(B)$ such that $C_\mu \pi|_X$ is an isomorphism onto $I(\Gamma_1 \backslash A) = I((\Gamma_1 \backslash A) \cup (B \cap \Gamma_1)) \subset L_1(G/H)$, where $H = \Gamma_1^\perp$ and μ is an idempotent measure on G/H with $\hat{\mu} = 1_A$. Because Γ_1 is discrete, G/H is compact and so $A \subset L_1(G/H)$ when A is considered as a set of continuous bounded functions on G/H.

For each $a \in A, \gamma \in \Gamma_1 \backslash A$, using multiplicative notation in Γ,

$$\int a(g+H)\,\bar{\gamma}(g+H)\,dm_{G/H}(g) = \int (a \cdot \bar{\gamma})(g+H)\,dm_{G/H}(g) = 0.$$

That is, $a \in I(\Gamma_1 \backslash A)$. Hence for each $a \in A$, there is an element $x(a) \in X$ such that $C_\mu \pi(x(a)) = a$ with $\|x(a)\|_1 \leqslant \|(C_\mu \pi|_X)^{-1}\|$. We will arrive at a contradiction by showing $\{x(a): a \in A\}$ is not relatively weakly compact, while $A \subset L_1(G/H)$ is clearly relatively weakly compact.

Because $x(a) \in I(B), \widehat{x(a)}|_B = 0$. But also $\widehat{x(a)}(a) = \hat{a}(a) = 1$ since $A \subset \Gamma_1$. Because the topology on Γ is the compact-open topology, for any compact neighborhood V of 0 in G, there is a neighborhood W of 0 in Γ such that $|\widehat{x(a)|_V}(\gamma) - \widehat{x(a)|_V}(\gamma')| < \varepsilon$ if $\gamma - \gamma' \in W, a \in A$. If $\{x(a): a \in A\}$ is in fact weakly relatively compact, then it would be uniformly integrable; thus, given any $\varepsilon_1 > 0$, there is a compact K such that $\|x(a)|_{K^c}\|_1 < \varepsilon_1$. Now let V above be a compact set containing K, let $\varepsilon_1 = \varepsilon = 1/4$. Then for some neighborhood W of 0,

$$|\widehat{x(a)|_V}(\gamma) - \widehat{x(a)|_V}(\gamma')| < \tfrac{1}{4}$$

for all $\gamma - \gamma' \in W, a \in A$. But by (ii), there is an $a \in A$ and $b \in B$ such that $a - b \in W$. Hence,

$$\begin{aligned}
\tfrac{1}{4} &> |\widehat{x(a)|_V}\,(a) - \widehat{x(a)|_V}\,(b)| \\
&\geqslant |\widehat{x(a)}(a) - \widehat{x(a)}(b)| - |\widehat{x(a)}(a) - \widehat{x(a)\,1_V}(a)| \\
&\quad - |\widehat{x(a)\,1_V}(b) - \widehat{x(a)}(b)| \\
&\geqslant 1 - 2\,\|x(a) - x(a)\,1_V\|_1 = 1 - 2(\tfrac{1}{4}) = \tfrac{1}{2}.
\end{aligned}$$

This contradiction completes the proof. ∎

2.2. LEMMA. *If $A \in \Omega_c(\Gamma)$ and A is discrete, then there are discrete closed subgroups $\Gamma_i, i = 1,..., n$, of Γ, $\{\gamma_i : i = 1,..., n\} \subset \Gamma$, and finite unions B_i of cosets of subgroups of Γ_i such that*

(i) $A = \bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i)$,

(ii) $(\gamma_i + (\Gamma_i \backslash B_i)) \cap (\gamma_j + (\Gamma_j \backslash B_j)) = \varnothing$ *for* $i \neq j, i, j = 1,..., n$.

Proof. We know that $A = \bigcup_{i=1}^{k} \rho_i + (\Lambda_i \backslash \bigcup_{j=1}^{k_i} \rho_{ij} + \Lambda_{ij})$, where Λ_i are closed subgroups of Γ, $\rho_i \in \Gamma$, $\rho_{ij} \in \Lambda_i$, and Λ_{ij} are clopen subgroups of Λ_i. Because A is discrete, all the subgroups Λ_i can be taken to be discrete. Hence, we can write $A = \bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i)$, where $\gamma_i \in \Gamma$, Γ_i are closed discrete subgroups of Γ, and B_i are finite unions of cosets of subgroups of Γ_i. Our task is to guarantee disjointness of the terms $\{\gamma_i + (\Gamma_i \backslash B_i): i = 1,..., n\}$.

First, let $\mathscr{S}$ consist of all sets of the form $\gamma_0 + (\Gamma_0 \backslash B_0)$, where $\gamma_0 \in \Gamma$, Γ_0 is a closed discrete subgroup of Γ, and B_0 is a finite union of cosets of subgroups in Γ_0. We claim that $\mathscr{S}$ is a semi-ring; that is,

(i) if $A, B \in \mathscr{S}$, then $A \cap B \in \mathscr{S}$,

(ii) if $A, B \in \mathscr{S}$, then $A \backslash B = \bigcup_{j=1}^{n} C_j$, where the pairwise disjoint sets $C_j \in \mathscr{S}$, $j = 1,..., n$.

To prove (i), let $\alpha_1 + (\Gamma_1 \backslash B_1)$ and $\alpha_2 + (\Gamma_2 \backslash B_2)$ be in $\mathscr{S}$. Without loss of generality $\alpha_1 + \Gamma_1 \cap \alpha_2 + \Gamma_2 \neq \varnothing$; so there is an $\alpha \in \Gamma$ such that

$$\begin{aligned}\alpha_1 + (\Gamma_1 \backslash B_1) \cap \alpha_2 + (\Gamma_2 \backslash B_2) &= \alpha + (\Gamma_1 \cap \Gamma_2) \backslash (\alpha_1 + B_1 \cup \alpha_2 + B_2) \\ &= \alpha + (\Gamma_1 \cap \Gamma_2) \backslash \alpha + B_3 = \alpha + (\Gamma_1 \cap \Gamma_2 \backslash B_3),\end{aligned}$$

where B_3 is a finite union of cosets in $\Gamma_1 \cap \Gamma_2$.

To prove (ii), first note that $\bigcup_{i=1}^{n} \gamma_i + \Gamma_i = \bigcup_{i=1}^{n} (\gamma_i + \Gamma_i \backslash \bigcup_{s=1}^{i-1} \gamma_s + \Gamma_s)$, a pairwise disjoint union of sets in $\mathscr{S}$ because each $\gamma_i + \Gamma_i \backslash \bigcup_{s=1}^{i-1} \gamma_s + \Gamma_s$ takes the form $\gamma_i + (\Gamma_i \backslash B_i)$ for some finite union B_i of cosets in Γ_i. Now

$$\begin{aligned}&(\gamma_1 + (\Gamma_1 \backslash B_1)) \backslash (\gamma_2 + (\Gamma_2 \backslash B_2)) \\ &\quad = (\gamma_1 + \Gamma_1 \backslash (\gamma_1 + B_1 \cup \gamma_2 + \Gamma_2)) \cup (\gamma_2 + B_2 \cap \gamma_1 + (\Gamma_1 \backslash B_1)).\end{aligned}$$

Since $B_2 \subset \Gamma_2$, $\gamma_1 + \Gamma_1 \backslash (\gamma_1 + B_1 \cup \gamma_2 + \Gamma_2)$ and $\gamma_2 + B_2 \backslash (\gamma_1 + \Gamma_1 \backslash B_1)$ are disjoint. Clearly, $\gamma_1 + \Gamma_1 \backslash (\gamma_1 + B_1 \cup \gamma_2 + \Gamma_2) \in \mathscr{S}$. But also, by the remark above, $\gamma_2 + B_2 = \bigcup_{j=1}^{n} C_j$, $C_j \in \mathscr{S}$, C_j pairwise disjoint, $i = 1,..., n$. Hence, $\gamma_2 + B_2 \cap \gamma_1 + (\Gamma_1 \backslash B_1)$ is also a disjoint union of sets in $\mathscr{S}$ by (i). This proves (ii).

It is a routine set theory argument to show that any finite union of elements of a semi-ring $\mathscr{S}$ is a finite union of disjoint elements in S. See [5, p. 33]. ∎

We are now ready to prove our characterization of complemented ideals with discrete hulls.

2.3. THEOREM. *Let $A \in \Omega_c(\Gamma)$ be discrete, and let $A = \bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i)$, where $\gamma_i \in \Gamma$, Γ_i is a closed discrete subgroup of Γ, and B_i is a union of cosets of subgroups of Γ_i. Assume $\gamma_i + (\Gamma_i \backslash B_i) \cap \gamma_j + (\Gamma_j \backslash B_j) = \varnothing$ if $i \neq j$,*

$i, j = 1, \ldots, n$. Then $I(A)$ is complemented if and only if there is a neighborhood W of $0 \in \Gamma$ such that for all $i \neq j$, $i, j = 1, \ldots, n$,

$$\gamma_i + (\Gamma_i \backslash B_i) + W \cap \gamma_j + (\Gamma_j \backslash B_j) + W = \varnothing.$$

Proof. Suppose that there is such a neighborhood W of $0 \in \Gamma$. Each of the ideals $I(\Gamma_i \backslash B_i)$ is complemented by Proposition 1.3. Say P_i is a projection onto $I(\Gamma_i \backslash B_i)$. Then define $\mathscr{P}_i(f) = \gamma_i P_i(\gamma_i^{-1} f)$ for all $f \in L_1(G)$. Then $\mathscr{P}_i$ is a projection onto $I(\gamma_i + (\Gamma_i \backslash B_i))$. Now apply Lemmas 1.4 and 1.5 inductively to show that $I(A)$ is complemented.

Conversely, suppose that no such W exists. Then there is some $i \neq j$ such that for all neighborhood W of $0 \in \Gamma$, $\gamma_i + (\Gamma_i \backslash B_i) \cap \bigcup_{j \neq i} \gamma_j + (\Gamma_j \backslash B_j) + W \neq \varnothing$. By Proposition 2.1, we know that $I(A)$ is not complemented in $I(\bigcup_{j \neq i} \gamma_j + (\Gamma_j \backslash B_j))$. Hence, $I(A)$ is not complemented in $L_1(G)$. ∎

As a corollary of this theorem, we get the characterization of complemented ideals in $L_1(R)$; see [1].

2.4. THEOREM. *If $A \in \Omega_c(R)$, then $I(A)$ is complemented if and only if $A = \bigcup_{i=1}^n \alpha_i Z + \beta_i \backslash F$, where $\alpha_i, \beta_i \in R$, F is finite, and $\{\alpha_i : i = 1, \ldots, n\}$ are pairwise rationally dependent.*

Proof. The only closed proper subgroups of R are of the form αZ, $\alpha \in R$; so $A = \bigcup_{i=1}^k \alpha_i Z + \beta_i \backslash F$. If the $\{\alpha_i : i = 1, \ldots, n\}$ are pairwise rationally dependent, then it is easy to write $\bigcup_{i=1}^n \alpha_i Z + \beta_i$ as a finite union of cosets of one subgroup θZ. Thus, there is an $\varepsilon > 0$ such that any two of these cosets is uniformly separated by a distance ε. By Theorem 2.3, $I(A)$ is complemented.

Conversely, if there are two rationally independent α_i, then in any decomposition of A into pairwise-disjoint sets of the form $a_i Z + b_i \backslash F_i$, F_i finite, there exists some α_s, α_t which are rationally independent. Thus, there is no neighborhood W of 0 such that

$$(a_s Z + b_s \backslash F_s) + W \cap (a_t Z + b_t \backslash F_t) + W = \varnothing.$$

Hence, again by Theorem 2.3, $I(A)$ is not complemented. ∎

Theorem 2.3 also applies to the two examples discussed in Section 1. Both the ideals $I(Z \times Z \cup \{0\} \times \sqrt{2}\, Z)$ and $I(\{0\} \times Z \cup \{0\} \times \sqrt{2}\, Z)$ are not complemented in $L_1(R^2)$.

3. A SUFFICIENT CONDITION FOR COMPLEMENTATION

In this section, we will develop an inductive procedure for building projections on ideals. At the end, we will need to know G is σ-compact. So we assume this now. We will explicitly point out where this is used. The procedure in full is rather technical, so we will begin with some special cases. The basic idea is simple. If $A = \bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i)$, then we build projections from $I(\bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i))$ onto $I(\bigcup_{i=1}^{k+1} \gamma_i + (\Gamma_i \backslash B_i))$ for $k = 0, 1, 2, \ldots, n-1$ by using lifts as in Proposition 1.7.

First, suppose that we wish to find a relative projection from $I(\bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i))$ onto $I(\bigcup_{i=1}^{k+1} \gamma_i + (\Gamma_i \backslash B_i))$ and $\Gamma_{k+1}^{\perp} = H_{k+1}$ is compact. Without loss of generality, we may assume $\gamma_{k+1} = 0$. In this case, $\phi(g) = 1$ for all $g \in G$ defines a Bruhat function for H_{k+1}. Let X be the subspace of $L_1(G/H)$ consisting of all functions $\phi(g) f(g + H_{k+1})$, where $f \in I([\Gamma_{k+1} \cap \bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i)] \cup B_{k+1})$. We claim that $X \subset I(\bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i))$. Indeed, if $\gamma \in \bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i)$ and $x \in X$, $x(g) = \phi(g) f(g + H_{k+1})$, then

$$\begin{aligned}
\hat{x}(\gamma) &= \int \bar{\gamma}(g)\, x(g)\, dm_G(g) \\
&= \int \bar{\gamma}(g)\, \phi(g)\, f(g + H_{k+1})\, dm_G(g) \\
&= \int_{G/H_{k+1}} \int_{H_{k+1}} \bar{\gamma}(g+h)\, \phi(g+h)\, f(g + H_{k+1})\, dm_{H_{k+1}^{(h)}}\, dm_{G/H_{k+1}^{(g)}} \\
&= \int_{G/H_{k+1}} f(g + H_{k+1}) \int_{H_{k+1}} \bar{\gamma}(g+h)\, dm_{H_{k+1}^{(h)}}\, dm_{G/H_{k+1}^{(g)}} \\
&= \begin{cases} \hat{f}(\gamma) & \text{if } \gamma \in \Gamma_{k+1}, \\ 0 & \text{if } \gamma \notin \Gamma_{k+1}. \end{cases}
\end{aligned}$$

Hence, $\hat{x}(\gamma) = 0$ if $f \in I([\Gamma_{k+1} \cap \bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i)] \cup B_{k+1})$. It follows from Proposition 1.7 that $I(\bigcup_{i=1}^{k+1} \gamma_i + (\Gamma_i \backslash B_i))$ is complemented in $I(\bigcup_{i=1}^{k} \gamma_i + (\Gamma_i \backslash B_i))$.

If G were compact, then $\Omega_c(\Gamma) = \Omega(\Gamma)$ and by Cohen's theorem, $I(A)$ is complemented for any $A \in \Omega_c(\Gamma)$. The above argument used inductively gives the following generalization of this fact:

3.1. PROPOSITION. *Suppose $A \in \Omega_c(\Gamma)$ and $A = \bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i)$, where each Γ_i has Γ/Γ_i discrete, $i = 1, \ldots, n$, and each $B_i \in \Omega(\Gamma_i)$. Then $I(A)$ is complemented in $L_1(G)$.*

As a second example, consider the ideal $I({}_0R \cup {}_{\pi/4}R) \subset L_1(R^2)$, where ${}_\theta R = \{(x, y) \in R^2 : y = \tan(\theta)\, x\}$. This is complemented by a projection similar to the one given for $I({}_0R \cup {}_{\pi/2}R)$ in Section 1, but the technique used there does not seem to extend to a spectrum composed of three lines in R^2. We develop instead in this section a completely different technique. In order to motivate the rather lengthy arguments used for this, we are going to show how to find a relative projection from $I({}_0R)$ onto $I({}_0R \cup {}_{\pi/4}R)$ by this new technique. This case, being unencumbered by other details, should provide some intuition for what follows. The success of this method depends on the following observations:

(i) as a subspace of $L_1(R^2/{}_{\pi/4}R^\perp)$, $I({}_0R \cap {}_{\pi/4}R) \sim L_1^0(R) = I(\{0\}) \sim (\sum_{n \in Z} X_n)_{l_1}$, where $X_n = \{f - \int_n^{n+1} f(t)\, dt\, 1_{[0,1]} : \operatorname{supp}(f) \subset [n, n+1]\}$;

(ii) if $\phi_k(x, y) = (1/k)\, 1_{[0,k]}((x-y)/\sqrt{2})$, $k \in Z^+$, and we define $X_{n,k}$ to be $\{\phi_k(x, y) f((x+y)/\sqrt{2}) : f \in X_n\}$, then $\lim_{k \to \infty} \|\pi_{{}_0R^\perp}|_{X_{n,k}}\| = 0$ for each $n \geqslant 1$;

(iii) there is a sequence of integers (k_n) and perturbations X_n' of X_{n,k_n} such that $X_n' \subset I({}_0R)$ for all $n \geqslant 1$ and $\pi_{{}_{\pi/4}R^\perp}$ restricted to the closed span of $\bigcup X_n'$ is an isomorphism onto $I({}_0R \cap {}_{\pi/4}R)$.

Our approach here is to imitate the compact case as nearly as possible. The difficulty with this is that the lifting needed depends on each part of the space and, thus, the resulting map is not given by a single Bruhat function lifting as was used previously. We now examine each of the observations (i), (ii), (iii) in more detail.

For (i), note that ${}_{\pi/4}R^\perp = {}_{-\pi/4}R$ and therefore $L_1(R^2/{}_{\pi/4}R^\perp)$ can be identified with $L_1(R)$ by composition with the map $(x, y) \mapsto (x+y)/\sqrt{2}$. Also, at the same time, $I({}_0R \cap {}_{\pi/4}R)$ gets identified with $L_1^0(R)$ by this map. But also, if $x_n \in X_n$, $n \in Z$, then

$$\Big\| \sum_{n \in Z} x_n \Big\|_1 = \sum_{k \in Z} \Big\| \sum_{n \in Z} x_n|_{[k,k+1]} \Big\|_1$$

$$= \sum_{k \in Z \setminus \{0\}} \|x_k|_{[k,k+1]}\|_1 + \Big\| x_0 - \sum_{n \in Z \setminus \{0\}} \int_n^{n+1} x_n(t)\, dt \cdot 1_{[0,1]} \Big\|_1 .$$

But $\int_0^1 x_0(t)\, dt = 0$, so this shows

$$\Big\| \sum_{n \in Z} x_n \Big\|_1 \geqslant (\tfrac{1}{2}) \sum_{k \in Z \setminus \{0\}} \|x_k\|_1 + (\tfrac{1}{2}) \|x_0\|_1 .$$

Since $L_1^0(R)$ can be identified with $\{\sum_{n \in Z} x_n : x_n \in X_n\}$, this shows $L_1^0(R) \sim (\sum_{n \in Z} X_n)_{l_1}$. The importance of having an l_1 sum here is that we can lift each piece X_n independently and still be assured that the resulting span is isomorphic to its image in $L_1(R^2/{}_{-\pi/4}R)$.

In (ii), we have defined liftings for each X_n which as $k \to \infty$ are close to being in the correct ideal $I({}_0R)$ because integrals along lines ${}_{\pi/2}R + (a, 0)$, $a \in R$, are close to being zero for large k. To see that $\|\pi_{{}_0R^\perp}|_{X_{n,k}}\| \to 0$ as $k \to \infty$, observe that $C_{n,k} = \operatorname{supp}(\phi_k) \cap \{(x, y): (x+y)/\sqrt{2} \in [0, 1] \cup [n, n+1]\}$ has the property that most vertical lines which intersect $C_{n,k}$ will intersect each of the two pieces of $C_{n,k}$ in a set of linear measure $\sqrt{2}$. Indeed, the proportion of these good vertical lines has the form $(k - c_n)/k$, where c_n depends only on n. On each of these good vertical lines, the integral of any $x \in X_{n,k}$ is zero. This shows that $\lim_{k\to\infty} \|\pi_{{}_0R^\perp}|_{X_{n,k}}\| = 0$.

Finally, we can show that for large enough k, there are perturbations of $X_{n,k}$ in $I({}_0R)$. Indeed, the map $Sx = x - \pi_{{}_0R^\perp}(x)\, 1_{Rx[0,1]}$, $x \in X_{n,k}$, is an isomorphism of $X_{n,k}$ into $I({}_0R)$ for k sufficiently large. In particular, $\|x - S(x)\|_1 \leqslant \|\pi_{{}_0R^\perp}(x)\|_1$. So choose k_n such that $\|\pi_{{}_0R^\perp}|X_{n,k_n}\| < \frac{1}{8}$. Then if $x_n \in X_{n,k_n}$, we have

$$\left\| \sum_{n\in Z} x_n - S\left(\sum_{n\in Z} x_n\right)\right\|_1 \leqslant \sum_{n\in Z} \|x_n - S(x_n)\|_1$$
$$\leqslant \left(\tfrac{1}{8}\right) \sum_{n\in Z} \|x_n\|_1 \leqslant \left(\tfrac{1}{4}\right) \left\| \sum_{n\in Z} x_n \right\|_1.$$

Let $X_n' = S(X_{n,k_n})$. Then clearly $X = \sum X_n' \subset I({}_0R)$ and by standard arguments $\pi_{{}_{\pi/4}R^\perp}|_X$ is an isomorphism of X onto $\pi_{{}_{\pi/4}R^\perp}(\sum X_{n,k_n}) = I({}_0R \cup {}_{\pi/4}R)$. Thus, by Proposition 1.7, we have $I({}_0R \cup {}_{\pi/4}R)$ complemented in $I({}_0R)$.

This very same method allows us to prove this proposition.

3.2. Proposition. *If $\Gamma_1, \ldots, \Gamma_n$ are hyperplanes in R^k, $k \geqslant 2$, then $I(\bigcup_{i=1}^n \Gamma_1)$ is complemented in $L_1(R^k)$.*

To prove this requires as much argument and notation as our general inductive procedure, and so we do not treat this case separately. However, it would be good to bear this case in mind (even with $n = 3$, $k = 2$) in the sequel.

We now begin to formulate these ideas in general. First, we have some perturbation results.

3.3. Definition. If $P: L_1(G) \to I(\bigcup_{i=1}^k \gamma_i + (\Gamma_i \backslash B_i))$ is a projection, we say that *P respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$ with bound M* if $\|(I - P)x\|_1 \leqslant M \sum_{i=1}^k \|C_{\mu_i} \pi_i(\overline{\gamma_i} x)\|_1$ for all $x \in L_1(G)$, where $\pi_i = \pi_{\Gamma_i^\perp}$ and $\hat{\mu}_i = 1_{\Gamma_i \backslash B_i}$.

The notation of this definition will be used throughout this section.

3.3. PROPOSITION. *Suppose that X is a subspace of $L_1(G)$ such that $C_{\mu_n}\pi_n|_X$ is an isomorphism onto $I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i)) \cup B_n) \subset L_1(G/H_n)$. Also assume that $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$ is complemented in $L_1(G)$ by a projection P which respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$, $i = 1,..., n-1$, with constant M. Let $0 < \varepsilon < 1/\|\mu_n\|$. Then if $\|C_{\mu_i}\pi_i \overline{\gamma_i}|_X\| < \varepsilon/2n \,\|(C_{\mu_n}\pi_n|_X)^{-1}\| \, M \, \|\mu_n\|$, for $i = 1,..., n-1$, the subspace $X_1 = PX$ of $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i B_i))$ is mapped isomorphically by $C_{\mu_n}\pi_n|_{X_1}$ onto $I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i)) \cup B_n)$. Moreover, $\|x_1 - (C_{\mu_n}\pi_n|_X)^{-1} C_{\mu_n}\pi_n x_1\|_1 < 2\varepsilon \,\|\mu_n\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\| \, \|x_1\|_1$ for all $x_1 \in X_1$, and $\|(C_{\mu_n}\pi_n|_{X_1})^{-1}\| \leqslant 2\|(C_{\mu_n}\pi_n|_X)^{-1}\| \, \|P\|$.*

Proof. If $x \in X$,

$$\|C_{\mu_n}\pi_n(x - Px)\|_1 \leqslant \|(I-P)\,x\|_1 \,\|\mu_n\| \leqslant \|\mu_n\| \, M \sum_{i=1}^{n-1} \|C_{\mu_i}\pi_i \overline{\gamma_i} x\|_1$$
$$\leqslant \varepsilon(n-1)\,\|x\|_1/2n \,\|(C_{\mu_n}\pi_n|_X)^{-1}\|.$$

Hence,

$$\|C_{\mu_n}\pi_n Px\|_1 \geqslant \|C_{\mu_n}\pi_n x\|_1 - \|C_{\mu_n}\pi_n(I-P)\,x\|_1$$
$$\geqslant (2n - \varepsilon(n-1))\,\|x\|_1/2n \,\|(C_{\mu_n}\pi_n|_X)^{-1}\|$$
$$\geqslant \|x\|_1/2 \,\|(C_{\mu_n}\pi_n|_X)^{-1}\|.$$

Hence, $C_{\mu_n}\pi_n$ maps $X_1 = PX$ isomorphically into $I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i)) \cup B_n)$. Also, since $\varepsilon < 1/\|\mu_n\|$, it follows by a standard perturbation argument that $\mathrm{range}(C_{\mu_n}\pi_n|_X) = \mathrm{range}(C_{\mu_n}\pi_n|_{PX})$. This proves our first assertion.

For the second, let $x_1 \in X$, $x_1 = Px$. Then

$$\|x_1 - (C_{\mu_n}\pi_n|_X)^{-1} C_{\mu_n}\pi_n x_1\|_1 = \|Px - (C_{\mu_n}\pi_n|_X)^{-1} C_{\mu_n}\pi_n Px\|_1$$
$$\leqslant \|Px - (C_{\mu_n}\pi_n|_X)^{-1} C_{\mu_n}\pi_n x\|_1 + \|(C_{\mu_n}\pi_n|_X)^{-1} C_{\mu_n}\pi_n(x - Px)\|_1$$
$$\leqslant \|Px - x\|_1 + \|\mu_n\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\| \, \|x - Px\|_1$$
$$\leqslant (1 + \|\mu_n\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\|)\,\varepsilon(n-1)\,\|x\|_1/2n \,\|(C_{\mu_n}\pi_n|_X)^{-1}\| \, \|\mu_n\|$$
$$\leqslant \varepsilon \,\|x\|_1 \leqslant 2\varepsilon \,\|\mu_n\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\| \, \|x_1\|_1.$$

Finally,

$$\|C_{\mu_n}\pi_n Px\|_1 \geqslant \|x\|_1/2 \,\|(C_{\mu_n}\pi_n|_X)^{-1}\| \geqslant \|Px\|_1/2 \,\|P\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\|.$$

Hence, $\|(C_{\mu_n}\pi_n|_{X_1})^{-1}\| \leqslant 2 \,\|P\| \, \|(C_{\mu_n}\pi_n|_X)^{-1}\|$. ∎

The next lemma shows that liftings of l_1 summands can be done independently with appropriate control of the terms.

3.4. PROPOSITION. *If* $I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i) \cup B_n) =$ closed span$(\sum Y_l)$ *is isomorphic to* $(\sum Y_l)_{l_1}$ *with isomorphism constant* C_1, *and there are subspaces* X_l *of* $L_1(G)$ *such that* $C_{\mu_n}\pi_n|_{X_l}$ *is an isomorphism onto* Y_l *for each* l *with* $\|(C_{\mu_n}\pi_n|_{X_l})^{-1}\| \leqslant C_2$ *and* $\|C_{\mu_i}\pi_i\bar{\gamma}_i|_{X_l}\| \leqslant \varepsilon/C_1^2 C_2 \|(C_{\mu_n}\pi_n|_{X_l})^{-1}\|$, *then* $X =$ closed span$(\sum X_l)$ *is a subspace of* $L_1(G)$ *such that* $C_{\mu_n}\pi_n(X) = I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i) \cup B_n)$, *while* $\|(C_{\mu_n}\pi|_X)^{-1}\| \leqslant C_1 C_2$ *and* $\|C_{\mu_i}\pi_i\bar{\gamma}_i|_X\| \leqslant \varepsilon \|\mu_n\|/\|(C_{\mu_n}\pi_n|_X)^{-1}\|$.

Proof. Suppose $x_l \in X_l$, $l = 1, 2, 3, \ldots$. Then

$$\begin{aligned}
\left\| C_{\mu_n}\pi_n \sum x_l \right\|_1 &= \left\| \sum C_{\mu_n}\pi_n x_l \right\|_1 \\
&\geqslant (1/C_1) \sum \|C_{\mu_n}\pi_n x_l\|_1 \\
&\geqslant (1/C_1) \sum \|(C_{\mu_n}\pi_n|_{X_l})^{-1}\|^{-1} \|x_l\|_1 \\
&\geqslant (1/C_1 C_2) \left\| \sum x_l \right\|_1 .
\end{aligned}$$

Thus, $C_{\mu_n}\pi_n|_X$ is an isomorphism onto $I(\Gamma_n \cap (\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i)) \cup B_n)$ with $\|(C_{\mu_n}\pi_n|_X)^{-1}\| \leqslant C_1 C_2$. Also,

$$\begin{aligned}
\left\| C_{\mu_i}\pi_i\bar{\gamma}_i \sum_l x_l \right\|_1 &\leqslant \sum_l \|C_{\mu_i}\pi_i\bar{\gamma}_i x_l\|_1 \\
&\leqslant \sum_l \varepsilon \|x_l\|_1/\|(C_{\mu_n}\pi_n|_{X_l})^{-1}\| \, C_1^2 C_2 \\
&\leqslant \sum_l \varepsilon \|C_{\mu_n}\pi_n x_l\|_1/C_1^2 C_2 \\
&\leqslant \varepsilon C_1 \left\| \sum C_{\mu_n}\pi_n x_l \right\|_1 \Big/ C_1^2 C_2 \\
&\leqslant \varepsilon \|C_{\mu_n}\pi_n|_X\| \left\| \sum x_l \right\|_1 \Big/ C_1 C_2 \\
&\leqslant \varepsilon \|C_{\mu_n}\pi_n|_X\| \left\| \sum x_l \right\|_1 \Big/ \|(C_{\mu_n}\pi_n|_X)^{-1}\|. \quad \blacksquare
\end{aligned}$$

Remark. X above is $C_1 C_2 \|\mu_n\|$-isomorphic to $(\sum X_l)_{l_1}$.

We now turn to the task of finding the lifts X_l. In the actual induction, these subspaces will be compactly supported, and this will be of technical importance in assuring that the induction can be completed. The next two measure-theoretic lemmas will be used in the induction to follow.

3.5. LEMMA. *Suppose that C is a compact subset of G and $1 > \varepsilon > 0$. Then there is a nonempty open set V with closure $\bar{V}$ being compact such that*

$$(1+\varepsilon)\, m_G(V) > m_G\left(\bigcup_{c\in C} c+V\right) \geqslant m_G(V)$$

$$\geqslant m_G\left(\bigcap_{c\in C} c+V\right) > (1-\varepsilon)\, m_G(V).$$

We omit the proof of this lemma since it says simply that a locally compact abelian group satisfies the Følner condition. See Rudin [15, p. 52], and Greenleaf [4]. In the next lemma, $m_H(A) = m_H(A \cap H)$ for a Borel subset A of G.

3.6. LEMMA. *Suppose $V \subset G$ is as in Lemma 3.5 and H is a closed subgroup of G. Then there is a Borel set $V_1 \subset V$ such that*

(i) $m_G(V_1) \geqslant (1 - 2\sqrt{\varepsilon})\, m_G(V)$,

(ii) $(1-\sqrt{\varepsilon})\, m_H(V-v) < m_H(\bigcap_{c\in C} c+V-v) \leqslant m_H(V-v) \leqslant m_H(\bigcup_{c\in C} c+V-v) < (1+\sqrt{\varepsilon})\, m_H(V-v)$ *for all* $v \in V_1$.

Proof. We may assume that $0 \in C$. Define $h, g, f \in L_1(G/H)$ by $h(x+H) = m_H(\bigcup_{c\in C} c+V-x)$, $g(x+H) = m_H(V-x)$, and $f(x+H) = m_H(\bigcap_{c\in C} c+V-x)$ for all $x \in G$. Note that $h \geqslant g \geqslant f$ and

$$(1+\varepsilon)\int g\, dm_{G/H} \geqslant \int h\, dm_{G/H} \geqslant \int g\, dm_{G/H} \geqslant \int f\, dm_{G/H}$$

$$\geqslant (1-\varepsilon)\int g\, dm_{G/H}.$$

Let $\nu \in M(G/H)$ be given by $d\nu = g\, dm_{G/H} / \int g\, dm_{G/H}$. If $g(x) = 0$, then we define the ratios $(f/g)(x)$, $(h/g)(x)$ to be 0. Let $1 > \rho > 0$ and define $A = \{x: (f/g)(x) > 1-\rho\}$, $B = \{x: (h/g)(x) < 1+\rho\}$. Then we have

$$(1-\varepsilon) \leqslant \int (f/g)\, d\nu \leqslant \nu(A) + (1-\rho)(1-\nu(A))$$

$$= \rho\nu(A) + (1-\rho).$$

So $\nu(A) \geqslant 1 - (\varepsilon/\rho)$. Also,

$$1+\varepsilon \geqslant \int h/g\, d\nu \geqslant (1+\rho)(1-\nu(B)) + \nu(B)$$

$$= 1 + \rho - \rho\nu(B),$$

so $\nu(B) \geqslant 1 - (\varepsilon/\rho)$. Let $\rho = \sqrt{\varepsilon}$ and let $V_1 = ((A \cap B) + H) \cap V$. We then have $m_G(V_1) = \int_{V_1+H} g\, dm_{G/H} = \int g\, dm_{G/H} \cdot \nu(V_1 + H) \geqslant (1 - 2\sqrt{\varepsilon}) \int g\, dm_{G/H} = (1 - 2\sqrt{\varepsilon})\, m_G(V)$. This inequality and the definitions A and B show that (i) and (ii) hold. ▮

We need only clarify one more point before construction of the lifts. Recall that the perturbation Proposition 3.3 depends on estimating $\| C_{\mu_i} \pi_i \bar{\gamma}_i |_X \|$. In order to keep control of this parameter, we need a relationship between m_G, m_{H_i}, and m_{H_n}, where $H_i = \Gamma_i^\perp$, $i = 1,\dots, n$. This will be provided by the assumption that

$$(\Gamma_i + \Gamma_n)/\Gamma_i \cap \Gamma_n = \Gamma_i/\Gamma_i \cap \Gamma_n \oplus \Gamma_n/\Gamma_i \cap \Gamma_n \tag{D}$$

for $i = 1,\dots, n-1$. To see this note that (D) implies with appropriate normalizations that

$$(H_i + H_n)/(H_i \cap H_n = H_i/H_i \cap H_n \oplus H_n/H_i \cap H_n$$

and hence

$$dm_{(H_i+H_n)/H_i \cap H_n} = dm_{H_i/H_i \cap H_n} \times dm_{H_n/H_i \cap H_n}.$$

Furthermore,

$$dm_G = dm_{H_i \cap H_n}\, dm_{H_i/H_i \cap H_n}\, dm_{H_n/H_i \cap H_n}\, dm_{G/H_i + H_n}.$$

Because $dm_{H_i} = dm_{H_i \cap H_n}\, dm_{H_i/H_i \cap H_n}$, this is the relationship that we require.

3.7. PROPOSITION. *Suppose Γ_i, $i = 1,\dots, n$, are closed subgroups of Γ such that* (D) *holds for $i = 1,\dots, n-1$. Let $\gamma_i \in \Gamma_i$, $i = 1,\dots, n$, Γ_{ij} clopen subgroups of Γ_i, $j = 1,\dots, k_i$, $i = 1,\dots, n$, and $\gamma_{ij} \in \Gamma_n \cap \Gamma_i$, for all i, j. Suppose X is a subspace of $I(\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash (\bigcup_{j=1}^{k_i} \gamma_{ij} + \Gamma_{ij})) \cup \bigcup_{j=1}^{k_n} \gamma_{nj} + \Gamma_{nj}) \subset L_1(G/H_n)$, and there is a compact set K such that for all $x \in X$,* supp$(x) \subset K$. *Then for all $\varepsilon > 0$, there is a subspace $Y \subset L_1(G)$ such that, if $\mu_i \in M(\Gamma_i)$ with*

$$\hat{\mu}_i = 1_{\Gamma_i \backslash \bigcup_{j=1}^{k_i} \gamma_{ij} + \Gamma_{ij}},$$

then $\| C_{\mu_i} \pi_i \overline{\gamma_i} |_Y \| < \varepsilon$, $i = 1,\dots, n-1$, and also $C_{\mu_n} \pi_n |_Y$ is an isometry onto X. Moreover, the support of Y is a compact subset of $K + H_n$. If H_n is not compact, then if $\delta > 0$ and $A \subset G$, A compact, the subspace Y can be chosen so that $\| y|_A \|_1 \leqslant \delta \| y \|_1$ for all $y \in Y$.

Proof. By the remarks at the beginning of this section, we may assume H_n is not compact. Let M be a compact subset of G such that $\pi_n(M) = K$.

For each i, j, let $H_{ij} = \Gamma_{ij}^{\perp}$. Then $H_{ij} \supset H_i$ and H_{ij}/H_i is compact; thus $M - M + \sum_{j=1}^{k_i} H_{ij}$ is a compact set in G/H_i, $i = 1,..., n-1$. For each $i = 1,..., n$, let M_i' be a compact subset of H_n, $0 \in M_i'$, such that $\pi_{n,i} M_i' = P_n(M - M + \sum_j H_{ij} \cap H_i + H_n)$, where $\pi_{n,i}: G \to G/H_i \cap H_n$ is the quotient map and $P_n: H_i + H_n/H_i \cap H_n \to H_n/H_i \cap H_n$ is the coordinate projection guaranteed to exist by (D). Let $C = (\sum_{i=1}^{n-1} M_i' + M_i' + M - M) \cap H_n$. Choose a V as in Lemma 3.5 with $G = H_n$.

Define $\Phi: X \to L_1(G)$ by $\Phi f(x) = f(x + H_n) 1_{M+V}(x)/m_{H_n}(M + V - x)$. Clearly, $\pi_n \Phi f = f$ for all $f \in X$. Define $Y = \Phi(X)$. This construction gives the desired subspace Y. Indeed, $C_{\mu_n} \pi_n|_Y = \pi_n|_Y$ and Φ is an isometry. So $C_{\mu_n} \pi_n|_Y$ is an isometry onto X. The final requirement of the theorem is easy to fulfill by taking V sufficiently large. The hard part is to estimate $\| C_{\mu_i} \pi_i \bar{\gamma}_i |_Y \|$.

Fix i, $1 \leqslant i \leqslant n-1$. Notice that $C_{\mu_i}(g) = \prod_{j=1}^{k_i} (\delta_0 - \bar{\gamma}_{ij} m_{H_{ij}}) * g$. To estimate the above norm, we are going to replace functions by ones close to them which are more easily dealt with in making estimates. Let F be a compact subset of G such that $\pi_i(F) = (M + V + \sum H_{ij})/H_i$ and $m_{H_i}(F - (m + v - y)) \geqslant \frac{1}{2}$ for all $m \in M$, $v \in V$, and $y \in \sum H_{ij}$. Let $f_y(x) = m_{H_i \cap H_n}(M + \bar{V} - (x + y))/m_{H_n}(V)$ for $x \in G$ and $y \in \sum H_{ij}$. Because M and $\bar{V}$ are compact, $\{f_y|_F : y \in \sum H_{ij}\}$ is a lattice bounded subset of $L_1(F, dm_G|_F)$; and thus, because $L_1(F, dm_G|_F)$ is a complete lattice, there is a lattice least upper bound $g_1 \in L_1(F, dm_G|_F)$ for $\{f_y|_F : y \in \sum H_{ij}\}$. Notice that $g_1(x) = g_1(x + h)$ for $x \in F$, $x + h \in F$, and $h \in \sum H_{ij}$. Define g by $g(x + H_i) = \int g_1(x + h)\, dm_{H_i}(h) / \int 1_F(x + h)\, dm_{H_i}(h)$ for all $x \in F + H_i$; and $g(x + H_i) = 0$ otherwise. Then $g \in L_1(G/H_i)$, and $g(x + y) = g(x)$ a.e. $[m_{G/H_i}]$ $y \in \sum H_{ij}/H_i$. Moreover, $g(x + H_i) \geqslant f_y(x)$ a.e. $[m_G]$ $y \in \sum H_{ij}$; and $g(x + H_i) \leqslant \sup_{y \in \sum H_{ij}} m_{H_n \cap H_i}(M + V - (x + y))/m_{H_n}(V)$ a.e. $[m_G]$ $x \in G$.

Choose V_1 as in Lemma 3.6 with $H = H_n \cap H_i$, $G = H_n$, and V and C as above. We want to estimate for $f \in X$,

$$\left\| \pi_i \bar{\gamma}_i \Phi f(x) - g(x) \int_{H_i + H_n/H_n} (\bar{\gamma}_i f)(x + y) \right.$$
$$\left. \times\, dm_{H_i + H_n/H_n}(y)\, 1_{V_1 + M + \sum_j H_{ij}}(x) \right\|_1$$

as a norm in $L_1(G/H_i)$. To do this we will need two estimates.

Claim 1. *With the notation above,*

$$\left| \sup_{z \in \sum_j H_{ij}} m_{H_i \cap H_n}(M + V - x - z) - m_{H_i \cap H_n}(M + V - x - y) \right|$$
$$< \varepsilon_1 m_{H_i \cap H_n}(M + V - x - y)$$

for all $x \in M + V_1 + \sum_j H_{ij}$, $y \in M + H_n - x \cap H_i$, where $\varepsilon_1 = \varepsilon_1(\varepsilon)$ *and* $\varepsilon_1 \to 0$ as $\varepsilon \to 0$.

Proof of Claim 1. Let $x = \sum_j h_{ij} + m + v$, where $m \in M$, $v \in V_1$, $h_{ij} \in H_{ij}$. If $y \in M + H_n - x \cap H_i$, then $y = m_1 + h_n - (\sum h_{ij} + m + v)$, where $m_1 \in M$, $h_n \in H_n$. We view $P_n: H_i + H_n \to H_n$ as defined on $H_i + H_n$ modulo $H_i \cap H_n$. Then $P_n y \in H_i \cap H_n$ and $P_n(y) = P_n(m_1 - m + h_n - v - \sum_j h_{ij}) = P_n(m_1 - m - \sum_j h_{ij}) + h_n - v$. Hence, $y - P_n y = m_1 - m - \sum_j h_{ij} + h' - P_n(m_1 - m - \sum_j h_{ij})$ for some some $h' \in H_i \cap H_n$. Also,

$$\begin{aligned} & m_{H_i \cap H_n}(M + V - (x+y)) \\ &\quad = m_{H_i \cap H_n}\left(M + V - v - m_1 + P_n\left(m_1 - m - \sum_j h_{ij}\right)\right) \\ &\quad = m_{H_i \cap H_n}\left(M - m_1 + P_n\left(m_1 - m - \sum h_{ij}\right) + V - v\right). \end{aligned}$$

Hence, by the choice of V and since $v \in V_1$,

$$\begin{aligned} & |m_{H_i \cap H_n}(M + V - (x+y)) - m_{H_i \cap H_n}(V - v)| \\ &\quad \leqslant \sqrt{\varepsilon}\, m_{H_i \cap H_n}(V - v) \\ &\quad \leqslant \sqrt{\varepsilon}(1 + \sqrt{\varepsilon})\, m_{H_i \cap H_n}(M + V - (x+y)). \end{aligned}$$

Similarly, if $z \in \sum H_{ij}$, then $M + V - x - z = M + V - m - v - \sum h_{ij} - \sum h'_{ij}$ for some $h'_{ij} \in H_{ij}$, $z = \sum_j h'_{ij}$. If we take $h'_{ij} = -h_{ij}$, all j, then $M + V - x - z = M - m + V - v$. So

$$|m_{H_i \cap H_n}(M + V - x - Z) - m_{H_i \cap H_n}(V - v)| < \sqrt{\varepsilon}\, m_{H_i \cap H_n}(V - v)$$

in this case. This shows that $\sup_{z \in \sum H_{ij}} m_{H_i \cap H_n}(M + V - x - z)$ is not 0 and is obtained by taking values of z with $M + V - x - z \cap H_i \cap H_n \neq \varnothing$. Note that this restriction on z forces us to have $z \in M'_i + M + v - x$ modulo $H_i \cap H_n$. Hence, as above, we can show

$$\begin{aligned} & \left| \sup_{z \in \sum_j H_{ij}} m_{H_i \cap H_n}(M + V - x - z) - m_{H_i \cap H_n}(V - v) \right| \\ &\quad < \sqrt{\varepsilon}\, m_{H_i \cap H_n}(V - x). \end{aligned}$$

It follows that

$$\begin{aligned} & \left| \sup_{z \in \sum_j H_{ij}} m_{H_i \cap H_n}(M + V - x - z) - m_{H_i \cap H_n}(M + V - x - y) \right| \\ &\quad < \varepsilon_1 m_{H_i \cap H_n}(M + V - (x+y)), \end{aligned}$$

where $\varepsilon_1 = 2\sqrt{\varepsilon}(1 + \sqrt{\varepsilon})$. ∎

Claim 2. *With the notation above,*

$$|m_{H_n}(M+V-(x+y))-m_{H_n}(V)|<\varepsilon m_{H_n}(V)$$

for all $x\in M+V+H_i+H_n$, $y\in(M+V-x+H_n)\cap H_i=(M-x+H_n)\cap H_i$.

Proof. Let $x=m+v+h_i+h_n$, $m\in M$, $v\in V$, $h_i\in H_i$, $h_n\in H_n$, and $y=m_1+v_1-m-v-h_i-h_n+h_n'$ for $m_1\in M$, $v_1\in V$, $h_n'\in H_n$. Again, $P_ny\in H_i\Delta H_n$ and $P_ny=P_n(m_1-m)+v_1-v+h_n'-h_n$. Thus $y=y-P_ny+h=m_1-m-P_n(m_1-m)-h_i+h$ for some $h\in H_i\cap H_n$. By the choice of V,

$$\begin{aligned}
&|m_{H_n}(M+V-(x+y))-m_{H_n}(V)|\\
&\quad=|m_{H_n}(M+V-(x+y-P_ny))-m_{H_n}(V)|\\
&\quad=|m_{H_n}(M+V-(m_1-P_n(m_1-m)+v+h_n+h))-m_{H_n}(V)|\\
&\quad=|m_{H_n}(M-m_1+P_n(m_1-m)+V)-m_{H_n}(V)|\\
&\quad<\varepsilon m_{H_n}(V).
\end{aligned}$$

Having these two estimates, Claim 1 and Claim 2, we can return to our proof of Proposition 3.7. Let $x\in G/H_i$. Then

$$\begin{aligned}
\pi_i\bar{\gamma}_i\Phi f(x)&=\int_{H_i}\bar{\gamma}_i(x+y)\,\Phi f(x+y)\,dm_{H_i}(y)\\
&=\int_{H_i/H_i\cap H_n}\int_{H_i\cap H_n}\bar{\gamma}_i(x+y+z)f(x+y+z+H_n)\,1_{M+V}(x+y+z)\\
&\quad\times[m_{H_n}(M+V-(x+y+z))]^{-1}\,dm_{H_i\cap H_n}(z)\,dm_{H_i/H_i\cap H_n}(y)\\
&=\int_{H_i/H_i\cap H_n}\bar{\gamma}(x+y)f(x+y+H_n)\,m_{H_i\cap H_n}(M+V-(x+y))\\
&\quad\times[m_{H_n}(M+V-(x+y))]^{-1}\,dm_{H_i/H_i\cap H_n}.
\end{aligned}$$

Note that $m_{H_i\cap H_n}(M+V-(x+y))\neq 0$ means that $x+y\in M+V+H_i\cap H_n$ and so $x\in M+V+H_i$. Identifying H_i+H_n/H_n with $H_i/H_i\cap H_n$, we can then write

$$\begin{aligned}
\pi_i\bar{\gamma}_i\Phi f(x)=g(x)&\int_{H_i+H_n/H_n}\bar{\gamma}_i(x+y)f(x+y+H_n)\\
&\times dm_{H_i+H_n/H_n}(y)\,1_{M+V_1+\Sigma_jH_{ij}}(x)+\mathscr{E}(x),
\end{aligned}$$

where the error term $\mathscr{E}(x)$ is given by

$$\mathscr{E}(x)=\int_{H_i/H_i\cap H_n} \bar{\gamma}_i(x+y)f(x+y+H_n)\{m_{H_i\cap H_n}(M+V-(x+y)) \times [m_{H_n}(M+V-(x+y))]^{-1} - g(x)\,1_{M+V}(x+y)\,1_{M+V_1+\sum_j H_{ij}}(x)\}\,dm_{H_i/H_i\cap H_n}(y).$$

Our Claims 1, 2 above show that on $M+V_1+\sum_j H_{ij}$, the error $\mathscr{E}(x)$ is bounded by

$$\varepsilon_2\left|\int_{H_i/H_i\cap H_n} \bar{\gamma}_i(x+y)f(x+y+H_n)\,m_{H_i\cap H_n}(M+V-(x+y)) \times [m_{H_n}(M+V-(x+y))]^{-1}\,dm_{H_i/H_i\cap H_n}(y)\right| = \varepsilon_2\,|\pi_i\bar{\gamma}_i\Phi f(x)|,$$

where $\varepsilon_2=\varepsilon_2(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence,

$$\|1_{M+V_1+\sum_j H_{ij}}\mathscr{E}(x)\|_{L_1(G/H_i)}\leqslant\varepsilon_2\,\|\pi_i\bar{\gamma}_i\Phi f\|_{L_1(G/H_i)}$$

for all $f\in X$. On $M+V+H_i\backslash M+V_1+\sum_j H_{ij}$, the error $\mathscr{E}(x)$ can be estimated in norm as follows:

$$\begin{aligned}
&\|\pi_i\bar{\gamma}_i\Phi f\,1_{M+V+H_i}-\pi_i\bar{\gamma}_i\Phi f\,1_{M+V_1+\sum_j H_{ij}}\|_{L_1(G/H_i)}\\
&\quad\leqslant\|\pi_i\bar{\gamma}_i\Phi f\,1_{M+V+H_i}-\pi_i\bar{\gamma}_i\Phi f\,1_{M+V_1+H_i}\|_{L_1(G/H_i)}\\
&\quad\leqslant\|\bar{\gamma}_i\Phi f\,1_{M+V}-\bar{\gamma}_i\Phi f\,1_{M+V_1+H_i\cap M+V}\|_{L_1(G)}\\
&\quad\leqslant\int_G |f(x+H_n)\ [m_{H_n}(M+V-x)]^{-1}\\
&\qquad\times(1_{M+V}(x)-1_{M+V_1+H_i\cap(M+V)}(x))\,|\,dm_G(x)\\
&\quad\leqslant\int_{G/H_n}|f(x+H_n)|\,[m_{H_n}(M+V-x)]^{-1}\\
&\qquad\times(m_{H_n}(M+V-x)-m_{H_n}(M+V_1-x))\,dm_{G/H_n}(x)\\
&\quad=\int_M|f(x+H_n)|\,[1-m_{H_n}(M+V_1-x)/m_{H_n}(M+V-x)]\,dm_{G/H_n}(x)\\
&\quad\leqslant\int_M|f(x+H_n)|\,[1-m_{H_n}(V_1)/(1+\varepsilon)\,m_{H_n}(V)]\,dm_{G/H_n}(x)\\
&\quad\leqslant\|f\|_{L_1(G/H_n)}(1-(1-2\sqrt{\varepsilon})/(1+\varepsilon))=\varepsilon_3\,\|f\|_{L_1(G/H_n)},
\end{aligned}$$

where $\varepsilon_3 = \varepsilon_3(\varepsilon) \to 0$ as $\varepsilon \to 0$. These estimates show that $\|\mathscr{E}\|_{L_1(G/H_i)} \leqslant \varepsilon_4 \|f\|_{L_1(G/H_n)}$, where $\varepsilon_4 = \varepsilon_4(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we may estimate $\|C_{\mu_i}\pi_i\bar{\gamma}_i\Phi f\|_{L_1(G/H_i)}$ by calculating

$$\mu_i * g(x) \int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y+H_n) \times dm_{H_i+H_n/H_n}(y)\, 1_{M+V_1+\sum_j H_{ij}}(x).$$

Because both g and $1_{M+V_1+\sum_j H_{ij}}$ are H_{ij} invariant for $j=1, 2, \ldots, k_i$ and because $\int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y)\, dm_{H_i+H_n/H_n}(y)$ is constant on cosets of $H_i + H_n$, we have

$$\begin{aligned}\mu_i * g(x) &\int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y+H_n) \\ &\times dm_{H_i+H_n/H_n}(y)\, 1_{M+V_1+\sum_j H_{ij}}(x) \\ &= g(x) \left(\nu_i * \int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y+H_n) \right. \\ &\quad \left. \times dm_{H_i+H_n/H_n}(y)\, 1_{M+V_1+\sum_j H_{ij}}(x)\right),\end{aligned}$$

where $\nu_i = \prod_{j=1}^{k_i} (\delta_0 - \bar{\gamma}_{ij}\nu_{ij})$ and $\nu_{ij} = m_{H_{ij}+H_n/H_i+H_n}$ for $j=1, \ldots, k_i$. Now observe that if $\pi: L_1(G/H_n) \to L_1(G/(H_n + H_i))$ is the canonical quotient map then

$$\nu_i * \int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y+H_n)\, dm_{H_i+H_n/H_n}(y) = \nu_i * \pi(\bar{\gamma}_i f).$$

But if $\gamma \in \Gamma_i \cap \Gamma_n$, then $\widehat{\pi\bar{\gamma}_i f}(\gamma) = \hat{f}(\gamma_i + \gamma)$. So if $\gamma \in \Gamma_i \cap \Gamma_n \backslash \bigcup_{j=1}^{k_i} \gamma_{ij} + \Gamma_{ij}$, then $\widehat{\pi\bar{\gamma}_i f}(\gamma) = 0$ for all $f \in X$. Hence, if $F = \nu_i * \pi\bar{\gamma}_i f$, $f \in X$, then $\hat{F}(\gamma) = 0$ for all $\gamma \in \Gamma_i \cap \Gamma_n$. This means that

$$\mu_i * g(x) \int_{H_i+H_n/H_n} \bar{\gamma}_i(x+y) f(x+y+H_n)\, dm_{H_i+H_n/H_n}(y)\, 1_{M+V_1+\sum_j H_{ij}}(x) = 0.$$

Therefore, $\|C_{\mu_i}\pi_i\bar{\gamma}_i|_Y\| \leqslant \varepsilon_5(\varepsilon)$ and $\varepsilon_5(\varepsilon)$ can be made arbitrarily small by a suitable choice of V. ∎

We now extend Proposition 3.7 to handle ideals where not all of the cosets intersect Γ_n. This is useful because it eliminates the hypothesis (D) when it is an unnecessary assumption.

3.8. PROPOSITION. *Suppose that Γ_i, $i=1,\dots,n$, Γ'_s, $s=1,\dots,k$, are closed subgroups of Γ. Let $\gamma_i, \gamma'_s \in \Gamma$, $i=1,\dots,n$, $s=1,\dots,k$. Let $B_i \in \Omega(\Gamma_i)$, $B'_s \in \Omega(\Gamma'_s)$ be finite unions of clopen subgroups of Γ_i and Γ'_s, respectively. Assume there is a neighborhood W of 0 in Γ such that $(\Gamma_n \backslash B_n) + W \cap \bigcup_{s=1}^k \gamma_s + (\Gamma_s \backslash B_s) = \varnothing$. Assume that* (D) *holds for (Γ_i, Γ_n), $i=1,\dots,n-1$, and X is a compactly supported subspace of $I(\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i) \cup B_n)$. Then for any $\varepsilon > 0$, there is a compactly supported subspace Y of $I(\bigcup_{s=1}^k \gamma'_s + (\Gamma'_s \backslash B'_s))$ such that, if $\mu_i \in M(\Gamma_i)$ with $\hat{\mu}_i = 1_{B_i}$, $i=1,\dots,n-1$, then $\| C_{\mu_i} \pi_i \bar{\gamma}_i |_Y \| < \varepsilon$, $i=1,\dots,n-1$, and also $C_{\mu_n} \pi_n |_Y$ is an isomorphism onto X. Moreover, if H_n is not compact, $\delta > 0$, and $A \subset G$, A compact, then the subspace Y can be chosen so that $\| y|_A \|_1 \leqslant \delta \| y \|_1$ for all $y \in Y$.*

Proof. Proposition 3.7 shows that we can find a subspace Y_1 of $L_1(G)$ fulfilling the requirements relative to $C_{\mu_1} \pi_i \bar{\gamma}_i$. Because $\bigcup_{s=1}^k \gamma'_s + (\Gamma'_s \backslash B'_s)$ is separated from $\Gamma_n \backslash B_n$, by the Lemma 1.6, there is a compactly supported $\mu \in M(G)$ such that $\hat{\mu} = 1$ on $\Gamma_n \backslash B_n$ and $\hat{\mu} = 0$ on $\bigcup_{s=1}^k \gamma'_s + (\Gamma'_s \backslash B'_s)$. Clearly, if $y \in Y_1$, then $\widehat{\mu * y} = \hat{y}$ on $\Gamma_n \backslash B_n$. We claim that $Y = \mu * Y_1$ is the subspace required above.

First, if $y \in Y_1$, then $C_{\mu_n} \pi_n (\mu * y) = C_{\mu_n} \pi_n y$. Also,

$$
\begin{aligned}
\|\mu_n\| \, \|\mu * y\|_1 &\geqslant \| C_{\mu_n} \pi_n (\mu * y) \|_1 \\
&= \| C_{\mu_n} \pi_n y \|_1 \\
&= \| y \|_1 \qquad (\text{since} \quad C_{\mu_n} \pi_n |_{Y_1} \ \text{is an isometry}), \\
&\geqslant \|\mu * y\|_1 / \|\mu\|.
\end{aligned}
$$

Hence, $C_{\mu_n} \pi_n |_Y$ is an isomorphism onto X. Because both Y_1 and μ have compact support, Y has compact support. Also, if $C \supset \operatorname{supp}(\mu)$, C compact, and if $\| y|_{A+C} \|_1 < \delta \| y \|_1$ for all $y \in Y_1$, then also $\| (\mu * y)|_A \|_1 \leqslant \delta \|\mu\| \, \|y\|_1 \leqslant \delta \|\mu\| \, \|\mu_n\| \, \|\mu * y\|_1$ for all $y \in Y_1$. So we need now only estimate $\| C_{\mu_i} \pi_i \bar{\gamma}_i |_Y \|$ to finish the proof.

It is convenient here to think of $f \in L_1(G/H_i)$ as a locally measurable function on G by $f(g) = f(g + H_i)$, $g \in G$. Observe that π_i maps $L_1(G)$ into the locally measurable functions on G that are constant on cosets of H_i, i.e., $\pi_i f = m_{H_i} * f$. Also, C_{μ_i} acts on these functions by convolution with the idempotent measure ν, where $\operatorname{supp}(\nu) \subset H_i$ and $\hat{\nu} = 1_{\Gamma_i \backslash B_i}$. Thus, $C_{\mu_i} \pi_i \bar{\gamma}_i (\mu * y) = \nu * m_{H_i} * (\bar{\gamma}_i \mu * y) = \mu * \nu * m_{H_i} \bar{\gamma} y$. Hence, in $L_1(G/H_i)$, if $y \in Y_1$,

$$
\begin{aligned}
\| C_{\mu_i} \pi_i (\bar{\gamma}_i \mu * y) \|_1 &= \| \mu * \nu * m_{H_i} (\bar{\gamma}_i y) \|_1 \\
&\leqslant \|\mu\| \, \| C_{\mu_i} \pi_i \bar{\gamma}_i y \|_1 \\
&\leqslant \varepsilon \, \|\mu\| \, \|\mu_i\| \, \|y\|_1 \\
&\leqslant \varepsilon \, \|\mu\| \, \|\mu_i\| \, \|\mu_n\| \, \|\mu * y\|_1 . \quad \blacksquare
\end{aligned}
$$

Remark. Here $\|(C_{\mu_n}\pi_n|_Y)^{-1}\| \leqslant \|\mu\|$.

We are now ready to formalize our induction procedure for constructing relative projections. We will need a few definitions.

3.9. DEFINITION. If $X \subset L_1(G)$ and X is isomorphic to $(\sum X_i)_{l_1}$, we say that this sum decomposition is *engulfing* if

(i) for any compactly supported $Y \subset X$, there exists some finite set F such that $Y \subset \sum_{i \in F} X_i$,

(ii) if E is finite and $\varepsilon > 0$, there is a $\delta > 0$ and K compact such that if $\|y|_K\|_1 < \delta \|y\|_1$, then $\|P_E y\| < \varepsilon \|y\|_1$, where P_E is the coordinate projection of X onto $\sum_{i \in E} X_i$.

3.10. DEFINITION. An operator P on $Y \subset L_1(G)$ is said to *preserve compactness* if, for any compactly supported subspace $X \subset Y$, PX is compactly supported.

In order to make the induction procedure work, we will need to prove more than complementation at each stage. In particular, we will need to know that the projection constructed will respect the associated ideals, preserve compactness, and that the ideals being considered have a decomposition into an engulfing l_1 sum of compactly supported subspaces. The induction advances from $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$ to $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$ if $I((\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i - \gamma_n + (\Gamma_i \backslash B_i)) \cup B_n)$ is complemented in $L_1(G/\Gamma_n^{\perp})$ by a projection of the same type and if (Γ_i, Γ_n) satisfy (D) for all i for which $\gamma_i + (\Gamma_i \backslash B_i)$ is not separated from $\gamma_n + (\Gamma_n \backslash B_n)$. We will also need to impose an additional restriction on G so that the l_1 sums will be countable. This will be guaranteed by assuming now that G is σ-compact.

To begin, let us note that $L_1(G)$ is isometric to an engulfing l_1 sum of $L_1(K_i)$, $i = 1, 2, 3, \ldots$, where the (K_i) are pairwise disjoint measurable sets with nonempty interior, compact closure, and satisfy $\bigcup_{i=1}^{\infty} K_i = G$. Also observe that if $B \in \Omega(\Gamma)$ and B is a finite union of clopen cosets in Γ, then $I(\Gamma \backslash B)$ is complemented by a projection which preserves compactness and $I(\Gamma \backslash B)$ is isomorphic to an engulfing l_1 sum of compactly supported subspaces. Indeed, let μ be an idempotent measure with $\hat{\mu} = 1_B$ and define $Pf = f - \mu * f$ for $f \in L_1(G)$. Because μ has compact support, P preserves compactness. To see that $I(\Gamma \backslash B)$ is isomorphic to an engulfing l_1 sum of compactly supported subspaces, note that we can choose sets K_i in the decomposition of $L_1(G)$ with $\mu * 1_{K_i} = 1_{K_i}$ and so $\mu * L_1(K_i) \subset L_1(K_i)$. This follows from the fact that the supp(μ) is contained in a compact subgroup for any idempotent measure μ. This gives $I(\Gamma \backslash B) = (\sum_{i=1}^{\infty} \mu * L_1(K_i))_{l_1}$, which is an engulfing l_1 sum. It easily follows then that $I(\gamma + (\Gamma \backslash B))$, $\gamma \in \Gamma$, is complemented in $L_1(G)$ by a projection which preserves compactness; and

$I(\gamma + (\Gamma \backslash B))$ has a decomposition as an engulfing l_1 sum of compactly supported subspaces. This completes the first step of the induction.

We are now ready for the inductive step. Suppose that $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$ is complemented in $L_1(G)$ by a projection P which preserves compactness and respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$, $i = 1,..., n-1$. Also, suppose $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$ is isomorphic to an engulfing l_1 sum of compactly supported subspaces. Finally, assume that (Γ_i, Γ_n) satisfies (D) for all $i = 1,..., n-1$ for which $\gamma_n + (\Gamma_n \backslash B_n)$ is not separated from $\gamma_i + (\Gamma_i \backslash B_i)$ and that $I(\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i - \gamma_n + (\Gamma_i \backslash B_i) \cup B_n)$ is isomorphic to an engulfing l_1 sum of compactly supported subspaces (this will be true in particular if this latter ideal is complemented by this procedure). We will show that $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$ is complemented by a projection which preserves compactness and respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$, $i = 1,..., n$, and $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$ is isomorphic to an engulfing l_1 sum of compactly supported subspaces.

By multiplication by γ_n and $\bar{\gamma}_n$, we may assume without loss of generality that $\gamma_n = 0$. We have $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i)) \sim (\sum_{l=1}^{\infty} X_l)_{l_1}$ and $I((\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i) \cup B_n) \sim (\sum_{p=1}^{\infty} Z_p)_{l_1}$ with the sums both being engulfing l_1 sums of compactly supported subspaces. There will be λ, ε_i, α_i in the sequel which will be chosen to satisfy certain constraints that are described as we proceed. First, consider X_1 and let $E_1 = \{1\}$. There is a finite F_1, $1 \in F_1$, such that $C_{\mu_n} \pi_n X_1 \subset \sum_{p \in F_1} Z_p$. Let P_1 be the coordinate projection of $L_1(G)$ onto X_1 and let K_1 be a compact subset of G and let $\delta_1 > 0$ such that if $y \in \sum X_i$ and $\|y|_{K_1}\| < \delta_1 \|y\|_1$, then $\|P_1 y\|_1 < \varepsilon_1 \|y\|_1$. By Proposition 3.8, there is a subspace Y_1 of $L_1(G)$ that $C_{\mu_n} \pi_n Y_1 = \sum_{p \in F_1} Z_p$, $\|y|_{K_1}\|_1 < \delta_1 \|y\|$ for all $y \in Y_1$, $\|C_{\mu_i} \pi_i \bar{\gamma}_i|_{Y_1}\| < \alpha_1$, $i = 1,..., n-1$, and also $\|(C_{\mu_n} \pi_n|_{Y_1})^{-1}\| \leqslant \lambda$. Let E_2 be a finite set of integers, $\{2\} \cup E_1 \subset E_2$, such that $PY_1 \subset \sum_{l \in E_2} X_l$. This completes the first cycle of the construction in the inductive step. We will do this once more for clarity and then state precisely what this procedure will produce.

There is a finite set F_2, $\{2\} \cup F_1 \subset F_2$, such that $C_{\mu_n} \pi_n \sum_{l \in E_2} X_l \subset \sum_{p \in F_2} Z_p$. Let P_2 be the projection onto $\sum_{i \in E_2} X_i$ and let K_2 be a compact subset of G, $K_2 \supset K_1$, and $\delta_2 > 0$ such that if $\|y|_{k_2}\|_1 < \delta_2 \|y\|_1$, $y \in \sum X_i$, then $\|P_2 y\| < \varepsilon_2 \|y\|$. By Proposition 3.8, there is a subspace $Y_2 \subset L_1(G)$ such that $C_{\mu_n} \pi_n Y_2 = \sum_{p \in F_2 \backslash F_1} Z_p$, $\|y|_{K_2}\|_1 < \delta_2 \|y\|_1$ for all $y \in Y_2$, $\|C_{\mu_i} \pi_i \bar{\gamma}_i|_{Y_2}\| < \alpha_2$, $i = 1, 2,..., n-1$, and $\|(C_{\mu_n} \pi_n|_{Y_2})^{-1}\| \leqslant \lambda$. Let E_3 be a finite set of integers with $\{3\} \cup E_2 \subset E_3$ and $PY_2 \subset \sum_{i \in E_3} X_i$.

In this way we get finite sets of integers $E_i \subset E_{i+1}$, $F_i \subset F_{i+1}$, $i \geqslant 1$, such that $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i = N$, compact subsets $K_i \subset K_{i+1} \subset G$, $i \geqslant 1$, sequence (δ_i), $\delta_i > \delta_{i+1} > 0$, $i \geqslant 1$, $\lim_{i \to \infty} \delta_i = 0$, and compactly supported subspaces $Y_i \subset L_1(G)$ such that

(a) $C_{\mu_n} \pi_n Y_j = \sum \{Z_p : p \in F_j \backslash F_{j-1}\}$,

(b) $\|(C_{\mu_n}\pi_n|_{Y_j})^{-1}\| \leqslant \lambda, j = 1, 2, \ldots,$

(c) $\|C_{\mu_i}\pi_i\bar{\gamma}_i|_{Y_j}\| \leqslant \alpha_j, i = 1, \ldots, n-1,$

(d) $\|y|_{K_j}\| < \delta_j \|y\|_1$ for all $y \in Y_j$, and if $y \in \sum X_j$, $\|y|_{K_j}\|_1 < \delta_j \|y\|_1$, then $\|P_j y\|_1 < \varepsilon_j \|y\|_1$, where P_j is the coordinate projection onto $\sum_{l \in E_j} X_l$,

(e) $PY_j \subset \sum_{l \in E_{j+1}} X_l$.

With the proper choice of the $(\alpha_j : j = 1, 2, 3, \ldots)$, by Propositions 3.3 and 3.4, $Y = P\sum Y_j$ is a subspace of $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$ such that $C_{\mu_n}\pi_n|_Y$ is an isomorphism onto $I(\Gamma_n \cap \bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i) \cup B_n)$. Hence, by Proposition 1.7, $R = (I - (C_{\mu_n}\pi_n|_Y)^{-1} C_{\mu_n}\pi_n) P$ is a projection onto $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$. We need to check that R preserves compactness and respects the ideals.

If W is a compactly supported subspace of $I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$, then $(C_{\mu_n}\pi_n|_Y)^{-1} C_{\mu_n}\pi_n W$ is compactly supported. Thus, if W_1 is a compactly supported subspace of $L_1(G)$, then $W = PW_1$ is a compactly supported subspace and, consequently, so is RW_1.

If $f \in L_1(G)$, then

$$\begin{aligned}
\|(I-R)f\|_1 &\leqslant \|(I-P)f\|_1 + \|(C_{\mu_n}\pi_n|_Y)^{-1} C_{\mu_n}\pi_n Pf\|_1 \\
&\leqslant \|(I-P)f\|_1 + \|(C_{\mu_n}\pi_n|_Y)^{-1}\| \, \|(C_{\mu_n}\pi_n Pf\|_1 \\
&\leqslant \|(I-P)f\|_1 + \|(C_{\mu_n}\pi_n|_Y)^{-1}\| \, (\|C_{\mu_n}\pi_n f\|_1 + \|C_{\mu_n}\pi_n(I-P)f\|_1) \\
&\leqslant M(1 + \|(C_{\mu_n}\pi_n|_Y)^{-1}\| \, \|\mu_n\|) \sum_{i=1}^{n-1} \|C_{\mu_i}\pi_i\bar{\gamma}_i f\| \\
&\quad + \|(C_{\mu_n}\pi_n|_Y)^{-1}\| \, \|C_{\mu_n}\pi_n f\|.
\end{aligned}$$

Thus, R respects the ideals $\gamma_i + (\Gamma_i \backslash B_i)$, $i = 1, \ldots, n$.

Finally, we need to show that $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$ is isomorphic to an engulfing l_1 sum of subspaces. We will first show that there are subspaces $X_i' \subset \sum_{l \in E_i} X_l$ such that $\sum_{i=1}^{\infty} X_i' + Y = I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$. Let $Q = I - R$, let R_l be the projection of $\sum_{l=1}^{\infty} X_l$ onto $\sum\{X_i : i \in E_l \backslash E_{l-1}\}$, and let S_l be the projection of the l_1 sum $\sum PY_i$ onto PY_l. If $y \in PY_l \subset \sum_{i \in E_{l+1}} X_i$, then $\|R_{l+1} y - y\| = \|P_l y\| < \varepsilon_l \|y\|$. Therefore, $\|QR_{l+1} y - y\| < \varepsilon_l \|Q\| \, \|y\|$. Thus, if $\varepsilon_l < \|Q\|^{-1} \|S_l\|^{-1}/2$, we would have $\|S_l QR_{l+1} y - y\| < \|y\|/2$ and consequently $S_l QR_{l+1}|_{PY_l}$ would be an isomorphism. Assume that ε_l has been so chosen.

Consider the projection $T_l : \sum_{i \in E_{l+1}} X_i \to PY_l$ given by $T_l = (S_l QR_{l+1}|_{PY_l})^{-1} S_l QR_{l+1}$. Let $X_1' = \sum_{i \in E_1} X_i$ and $X_{l+1}' = \ker(T_l) \cap \sum\{X_i : i \in E_{l+1} \backslash E_l\}$ and note that $PY_l \oplus X_{l+1}' \oplus \sum_{i \in E_l} X_i = \sum_{i \in E_{l+1}} X_i$ because $\ker(T_l) \supset \sum_{i \in E_l} X_i$. Clearly, $\sum_i X_i'$ is isomorphic to an l_1 sum and is a complement for $\sum PY_i$ in $\sum X_i$. So $\sum_i X_i' \oplus Y = I(\bigcup_{i=1}^{n-1} \gamma_i + (\Gamma_i \backslash B_i))$.

Now we must perturb $\sum X_i'$ to get a decomposition of $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$. However,

$$I\left(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i)\right) = R\left(\sum PY_j + \sum X_i'\right) = R\left(\sum X_i'\right) = \sum RX_i'.$$

Also, $R|_{X_i'}$ is an isomorphism and R preserves compactness, so $\sum RX_i'$ is an l_1 sum decomposition of $I(\bigcup_{i=1}^{n} \gamma_i + (\Gamma_i \backslash B_i))$ into compactly supported subspaces. We want to show that this is an engulfing l_1 sum. Before doing this, let us compute the estimates needed and how they restrict the choices of ε_i, α_i, δ_i, and λ.

Let C_1 be the l_1 constant of $\sum Z_p$. Let M be the bound for which P respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$, $i = 1,..., n-1$. Let $C_2 = \|\mu\|$, where μ is the separating measure in Proposition 3.8. Let C_4 be the l_1 constant of $\sum X_i$. In our construction $\lambda = C_2 = \|\mu\|$ by the remark following Proposition 3.8. We state the estimates needed below:

(1) As a subspace of $L_1(G)$, $\sum Y_l$ is $C_1 C_2 \|\mu_n\|$ isomorphic to $(\sum Y_l)_{l_1}$.

(2) $\|(C_{\mu_n} \pi_n|_{\sum Y_l})^{-1}\| \leqslant C_1 C_2$.

(3) $\|C_{\mu_i} \pi_i \bar{\gamma}_i|_{\sum Y_l}\| \leqslant \alpha_1 C_1^2 C_2^2 \|\mu_n\| / \|(C_{\mu_n} \pi_n|_{\sum Y_l})^{-1}\|$. Here and in the following we assume $\alpha_{j+1} < \alpha_j < \alpha_1$, $j \geqslant 1$; and so (3) comes from Proposition 3.4.

(4) $\|(P|_{\sum Y_l})^{-1}\| \leqslant 1/(M - M(n-1)\,\alpha_1 C_1 C_2 \|\mu_n\|)$. Indeed, for $y = \sum y_l \in \sum Y_l$, $y_l \in Y_l$,

$$\begin{aligned}
\|y - Py\|_1 &\leqslant M \sum_{i=1}^{n-1} \|C_{\mu_i} \pi_i \bar{\gamma}_i y\|_1, \\
&\leqslant M \sum_{i=1}^{n-1} \sum_l \|C_{\mu_i} \pi_i \bar{\gamma}_i y_l\|_1 \\
&\leqslant M \sum_{i=1}^{n-1} \sum_l \alpha_l \|y_l\|_1 \qquad \text{(by (c))} \\
&\leqslant M(n-1)\,\alpha_1 \sum \|y_l\|_1 \leqslant M(n-1)\,\alpha_1 C_1 C_2 \|\mu_n\| \, \|y\|_1.
\end{aligned}$$

Hence, $\|y\|_1 \leqslant \|y - Py\|_1 + \|Py\|_1 \leqslant M(n-1)\,\alpha_1 C_1 C_2 \|\mu_n\| \, \|y\|_1 + \|Py\|_1$ and $\|Py\|_1 \geqslant (1 - M(n-1)\,\alpha_1 C_1 C_2 \|\mu_n\|) \|y\|_1$.

(5) $\|(C_{\mu_n} \pi_n|_{\sum PY_l})^{-1}\| \leqslant 2\,\|(C_{\mu_n} \pi_n|_{\sum Y_l})^{-1}\| \, \|P\| \leqslant 2 C_1 C_2 \|P\|$ by Proposition 3.3.

(6) With $Q = I - R = (C_{\mu_n}|_{\sum PY_l})^{-1} C_{\mu_n} \pi_n$, $\|Q\| \leqslant 2\,\|\mu_n\|\, C_1 C_2 \|P\|$ by 5). Also $\|R\| = \|I - Q\| \leqslant 1 + 2\,\|\mu_n\|\, C_1 C_2 \|P\|$.

(7) The spaces $\sum PY_l$ and $(\sum PY_l)_{l_1}$ are C_3 isomorphic for some constant C_3 which we estimate as follows. Let $y_l \in Y_l$, $l \geqslant 1$. Then

$$\begin{aligned}\left\|\sum Py_l\right\|_1 &\geqslant \|(P|_{\sum Y_l})^{-1}\|^{-1}\left\|\sum y_l\right\|_1 \\ &\geqslant \|(P|_{\sum Y_l})^{-1}\|^{-1}\, C_1^{-1}C_2^{-1}\,\|\mu_n\|^{-1}\sum_l \|y_l\|_1 \\ &\geqslant \|(P|_{\sum Y_l})^{-1}\|^{-1}\, C_1^{-1}C_2^{-1}\,\|\mu_n\|^{-1}\,\|P\|^{-1}\sum_l \|Py_l\|_1.\end{aligned}$$

Hence, $C_3 \leqslant \|(P|_{\sum Y_l})^{-1}\|\, C_1C_2\,\|\mu_n\|\,\|P\|$. By 4), $C_3 \leqslant C_1C_2\,\|\mu_n\|\,\|P\|/(1 - M(n-1)\,\alpha_1C_1C_2\,\|\mu_n\|)$.

(8) $\|S_l\| \leqslant C_3$, where $S_l\colon \sum PY_j \to PY_l$ is the coordinate projection.

(9) $\|(R|_{\sum X_i'})^{-1}\| \leqslant \|I - \sum T_l\| \leqslant 1 + \|\sum T_l\| \leqslant 1 + C_4\sup_l\|T_l\| \leqslant 1 + C_4^2\,\|Q\|\,C_3\,\|(S_lQR_{l+1}|_{PY_l})^{-1}\|$; so $\|(R|_{\sum X_i'})^{-1}\| \leqslant 1 + 2C_4^2\,\|Q\|\,C_3$.

We see then that our requirements on (ε_i) and (α_i) could be, up to now, just that

(I) $\varepsilon_i < 1/4\,\|\mu_n\|\,C_1C_2\,\|P\|\,C_3$, $i \geqslant 1$, and

(II) $\alpha_1 < 1/2\,nM\,\|\mu_n\|^3\,C_1^2C_2^2$.

Indeed, with the requirement from (I) and (6) and (8), $\varepsilon_i < 1/2\,\|Q\|\sup_l\|S_l\|$, $i \geqslant 1$, and so $\varepsilon_l < \frac{1}{2}\,\|Q\|\,\|S_l\|$ as required earlier. Also (II) is exactly what is needed to apply Propositions 3.3 and 3.4 in both (3) and (5) and still satisfy the restriction in Proposition 3.3 that $\varepsilon < 1/\|\mu_n\|$ for the appropriate $\varepsilon > 0$. Moreover, (II) implies that $\alpha_1 < \frac{1}{2}(n-1)\,M\,\|\mu_n\|\,C_1C_2$ which is needed for (4) to be meaningful.

In order to show $\sum RX_i'$ is an engulfing l_1 sum, we first show that for any $\varepsilon > 0$ and any $m \geqslant 1$, there is a compact set K and $\delta > 0$ such that if $\|R\sum x_i|_K\|_1 \leqslant \delta\,\|R\sum x_i\|_1$, with $x_i \in X_i'$, $i \geqslant 1$, then $\|R\sum_{i=1}^m x_i\| < \varepsilon\,\|R\sum x_i\|$. Since $\sum RX_i' \sim (\sum RX_i')_{l_1}$, we need only show that for arbitrarily large n, there is a compact set K_n' and $\delta_n' > 0$ such that if $\|R\sum x_i|_{K_n'}\|_1 \leqslant \delta_n'\,\|R\sum x_i\|_1$, then $\|R\sum_{i=1}^n x_i\|_1 < \varepsilon_n'\,\|R\sum x_i\|_1$, with (ε_n') decreasing to 0. Indeed, if $l \leqslant n$, then there is a constant $C = \|R\|\,\|(R|_{\sum X_i'})^{-1}\|\,C_4$ independent of n, l such that $C\,\|R\sum_{i=1}^n X_i\|_1 \geqslant \|R\sum_{i=1}^l x_i\|_1$. Thus, if $\|R\sum x_i|_{K_n'}\|_1 \leqslant \delta_n'\,\|R\sum x_i\|_1$, then $\|R\sum_{i=1}^l x_i\|_1 \leqslant C\varepsilon_n'\,\|R\sum x_i\|_1$.

Fix $l \geqslant 1$. Let $x = R\sum_{i=1}^\infty x_i$, $x_i \in X_i'$, with $\|x|_{K_l}\|_1 < \delta_l\,\|x\|_1$; then also $\|x|_{K_s}\|_1 < \delta_s\,\|x\|_1$ for $s = 2,\dots, l$. Hence, for $s = 2,\dots, l$,

$$\begin{aligned}\|R_sx\|_1 &= \|(P_s - P_{s-1})\,x\|_1 \leqslant \|P_sx\|_1 + \|P_{s-1}x\|_1 \\ &\leqslant 2C_4\,\|P_lx\| < 2C_4\varepsilon_l\,\|x\|_1 \qquad \text{by (d).}\end{aligned}$$

Because $T_{s-1} = (S_{s-1}QR_s|_{PY_{s-1}})^{-1} S_{s-1}QR_s$, it follows that

$$\|T_{s-1}x\|_1 < 2C_4\varepsilon_l \|(S_{s-1}QR_s|_{PY_{s-1}})^{-1} S_{s-1}Q\| \, \|x\|_1.$$

Now $x = R\sum x_i = (I-Q)\sum x_i$ and $T_{s-1}x_i = 0$, $i \geqslant 1$; therefore, for $s = 2,\ldots, l$,

$$\left\|T_{s-1}Q\sum x_i\right\|_1 < 2C_4\varepsilon_l \|(S_{s-1}QR_s|_{PY_{s-1}})^{-1} S_{s-1}Q\| \, \|x\|_1.$$

If $s > l$, then

$$\begin{aligned}\left\|T_{s-1}Q\sum x_i|_{K_l}\right\|_1 &< \left\|T_{s-1}Q\sum x_i|_{K_l} - \left((P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right)\Big|_{K_l}\right\|_1 \\ &\quad + \left\|\left((P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right)\Big|_{K_l}\right\|_1 \\ &\leqslant \alpha_{s-1}M(n-1)\left\|(P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right\|_1 \\ &\quad + \delta_{s-1}\left\|(P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right\|_1 \\ &\leqslant (\alpha_{s-1}M(n-1)+\delta_{s-1})\left\|(P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right\|_1\end{aligned}$$

because (d) holds, $K_{s-1} \supset K_l$, and P respects ideals with bound M. Summing over s gives

$$\begin{aligned}\left\|\left(Q\sum x_i\right)\Big|_{K_l}\right\|_1 &\leqslant \sum_{s=2}^{\infty}\left\|\left(T_{s-1}Q\sum x_i\right)\Big|_{K_l}\right\|_1 \\ &\leqslant 2(l-1)\,C_4\varepsilon_l \max_{s=2,\ldots,l} \|(S_{s-1}QR_s|_{PY_{s-1}})^{-1} S_{s-1}Q\| \\ &\quad \times \|(R|_{\sum x_i'})^{-1}\| \left\|\sum x_i\right\|_1 \\ &\quad + \sum_{s=l+1}^{\infty} (\alpha_{s-1}M(n-1)+\delta_{s-1})\left\|(P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right\|_1 \\ &\leqslant 2(l-1)\,C_4\varepsilon_l \max_{s=2,\ldots,l} \|(S_{s-1}QR_s|_{PY_{s-1}})^{-1} S_{s-1}Q\| \\ &\quad \times \|(R|_{\sum x_i'})^{-1}\| \left\|\sum x_i\right\|_1 \\ &\quad + (\alpha_l M(n-1)+\delta_l)\,C_1C_2\|\mu_n\| \cdot \left\|\sum_{s=l+1}^{\infty} (P|_{Y_{s-1}})^{-1} T_{s-1}Q\sum x_i\right\|_1\end{aligned}$$

$$\leqslant 2(l-1)\,C_4\varepsilon_l \max_{s=2,\ldots,l} \|(S_{s-1}QR_s|_{PY_{s-1}})^{-1}\,S_{s-1}Q\|$$

$$\times \|(R|_{\Sigma X_i'})^{-1}\| \left\|\sum x_i\right\|_1$$

$$+ (\alpha_l M(n-1)+\delta_l)\,C_1C_2\,\|\mu_n\|\,\|(P|_{\Sigma Y_s})^{-1}\,Q\| \left\|\sum x_i\right\|_1.$$

Hence, we have the estimate

$$\left\|\sum x_i|_{K_l}\right\|_1 \leqslant \left\|R\sum x_i|_{K_l}\right\|_1 + \left\|Q\sum x_i|_{K_l}\right\|_1$$

$$< \delta_l \left\|R\sum x_i\right\|_1 + \left\|Q\sum x_i|_{K_l}\right\|_1$$

$$\leqslant \rho_l \left\|\sum x_i\right\|_1,$$

where

$$\rho_l = \delta_l\,\|R\| + 2(l-1)\,C_4\varepsilon_l \max_{s=2,\ldots,l} \|(S_{s-1}QR_s|_{PY})^{-1}\,S_{s-1}Q\|\,\|(R|_{\Sigma X_i'})^{-1}\|$$
$$+ (\alpha_l M(n-1)+\delta_l)\,C_1C_2\,\|\mu_n\|\,\|(P|_{\Sigma Y_s})^{-1}\,Q\|.$$

By (6), $\|R\| \leqslant 1 + 2\,\|\mu_n\|\,C_1C_2\,\|P\|$. By (9), $\|(R|_{\Sigma X_i'})^{-1}\| \leqslant 1 + 2C_3C_4^2\,\|Q\|$. By (4) and (6),

$$\|(P|_{\Sigma Y_s})^{-1}\,Q\| \leqslant 2\,\|\mu_n\|\,C_1C_2\,\|P\|/(1 - M(n-1)\,\alpha_1C_1C_2\,\|\mu_n\|).$$

And finally it is easy to see that $\|(S_{s-1}QR_s|_{PY})^{-1}\| \leqslant 2$ always. So $\max_{s=2,\ldots,l} \|(S_{s-1}QR_s)^{-1}\,S_{s-1}Q\| \leqslant 2C_3\,\|Q\| \leqslant 4\,\|\mu_n\|\,C_1C_2C_3\,\|P\|$ by (6) and (8). Hence,

$$\rho_l \leqslant \delta_l(1 + 2\,\|\mu_n\|\,C_1C_2\,\|P\|)$$
$$+ 8(l-1)\,\varepsilon_l\,\|\mu_n\|_1\,C_2C_3C_4\,\|P\|\,(1 + 4\,\|\mu_n\|\,C_1C_2C_3C_4^2\,\|P\|)$$
$$+ 2(\alpha_l M(n-1)+\delta_l)\,C_1^2C_2^2\,\|\mu_n\|^2\,\|P\|/(1 - M(n-1)\,\alpha_1C_1C_2\,\|\mu_n\|).$$

We now make the assumptions that

(III) $\lim_{l\to\infty} l\varepsilon_l = 0$ and $\lim_{l\to\infty} \alpha_l = 0$.

It is clear than (ε_l) and (α_l) can be chosen to satisfy (I), (II), (III) simultaneously.

But now we have that for any $t \geqslant 1$, there exists $l(t) \geqslant t$ such that $\rho_l(t) < \delta_t$. Hence,

$$\left\|P_{l(t)}\sum x_i\right\|_1 < \varepsilon_{l(t)} \left\|\sum x_i\right\|_1.$$

In short, if $\|R \sum x_i\|_1 < \delta_{l(t)} \|\sum x_i\|_1$, then

$$\left\| R \sum_{i=1}^{l(t)} x_i \right\|_1 \leqslant \|R\| \left\| P_{l(t)} \sum x_i \right\|_1$$
$$\leqslant \|R\| \, \varepsilon_{l(t)} \left\| \sum x_i \right\|_1$$
$$\leqslant \|R\| \, \|(R|_{\sum X_i'})^{-1}\| \, \varepsilon_{l(t)} \left\| R \sum x_i \right\|_1.$$

Since $\lim_{t \to \infty} \varepsilon_{l(t)} = 0$, this establishes (ii) in Definition 3.9 for the decomposition $\sum RX_i'$ of $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$.

Finally, for (i) of Definition 3.9, let W be a compactly supported subspace of $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$. Then $W \subset \sum_{i \in E_l} X_i$ for some l. Hence,

$$W = RW \subset R \sum_{i \in E_l} X_i = R\left(\sum_{i=1}^{l-1} PY_i + \sum_{i=1}^{l} X_i'\right) = \sum_{i=1}^{l} RX_i'.$$

This completes all of the details of proving that under our inductive hypotheses, the ideal $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$ is complemented by a projection R which preserves compactness, respects the ideals, and that $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$ is isomorphic to an engulfing l_1 sum of compactly supported spaces. We remark that in this induction we used the fact that G is σ-compact and we do not know if this restriction can be removed, although it seems likely.

Our inductive method gives a proof for complementation of $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$ in special cases.

3.11. THEOREM. *Suppose G is σ-compact and Γ_i, $i = 1,\dots, n$, are closed subgroups of Γ. Assume that the pairs $(\Gamma_j \cap \bigcap_{i=s}^n \Gamma_i, \Gamma_{s-1} \cap \bigcap_{i=s}^n \Gamma_i)$ for $1 \leqslant j \leqslant s-2$, $s = 3,\dots, n$, and the pairs (Γ_i, Γ_n), $i = 1,\dots, n-1$, satisfy* (D). *Then for all $\gamma_i \in \Gamma$, $i = 1,\dots, n$, and all finite unions B_i of cosets of clopen subgroups of Γ_i, $i + 1,\dots, n$, the ideal $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$ is complemented. Moreover, there is a complementing projection which preserves compactness and respects the ideals $I(\gamma_i + (\Gamma_i \backslash B_i))$.*

This is a much weaker result than our induction procedure allows because (D) is not required for pairs that have their associated cosets separated. We get this corollary which generalizes Proposition 3.2.

3.12. COROLLARY. *Suppose Γ_i is isomorphic to $R^{k(i)} \subset R^n$, where $1 \leqslant k(i) < n$. Let F_1 be a finite set. Then $I(\bigcup_{i=1}^n v_i + \Gamma_i \cup F_1)$ is complemented for any choice of vectors $v_i \subset R^n$, $i = 1,\dots, n$.*

Remark. We know generally that for σ-compact metric groups G, an ideal $I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i)) \subset L_1(G)$ is a $\mathscr{L}_1$ space. Also, the ideal is complemented if and only if it is isomorphic to $L_1[0, 1]$. This says something a posteriori about some particular ideals above which are (or are not) complemented. However, to use this information to prove the complementation theorems does not seem possible from what is known about the structure of these ideals.

4. Generalizations and Questions

In the last section, we formulated an inductive procedure to construct projections. However, the procedure appears to fail for examples such as Example 0.1 (v), the ideal $J = I(R \times Z \times \{0\} \cup \{0\} \times \sqrt{2}\, Z \times R \cup \{0\} \times R \times \{0\})$. We know that $I(R \times Z \times \{0\} \cup \{0\} \times R \times \{0\})$ is complemented by the inductive procedure. The difficulty in the next step is that $R \times Z \times \{0\} + \{0\} \times \sqrt{2}\, Z \times R$ is not a direct sum. But notice that $R \times R \times \{0\} + \{0\} \times \sqrt{2}\, Z \times R$ is direct modulo $\{0\} \times \sqrt{2}\, Z \times R$, and thus we can lift into $I(R \times R \times \{0\}) \subset I(R \times Z \times \{0\} \cup \{0\} \times R \times \{0\})$, i.e., there is a subspace X of $I(R \times R \times \{0\})$ such that $\pi_{(\{0\} \times \sqrt{2}\, Z \times R)^\perp}$ maps X isomorphically onto $I(\{0\} \times \sqrt{2}\, Z \times R \cap R \times R \times \{0\}) = I(\{0\} \times \sqrt{2}\, Z \times R \cap (R \times Z \times \{0\} \cup \{0\} \times R \times \{0\}))$. By Proposition 1.7, it follows that J is complemented. Moreover, we retain all the properties needed in this complementation of J to continue the inductive procedure; thus, we can continue building projections onto appropriate smaller ideals.

The basic idea of this example is that although a group Γ_i occurs in the representations of the hull such that Γ_i and Γ_n fail to satisfy (D), there is a larger group $\Gamma_i' \supset \Gamma_i$ such that Γ_1' and Γ_n satisfy (D) and $\Gamma_1' \cap \Gamma_n \subset \bigcup_{i=1}^{n-1} (\Gamma_i \backslash B_i) \cap \Gamma_n$. This could be modified to include more general coset forms. This leads to the following questions.

4.1. *Question.* Given two closed subgroups Γ_1 and Γ_2 in Γ failing (D), is there a (unique) minimal closed subgroup Γ_2', $\Gamma \supset \Gamma_2' \supset \Gamma_2$ such that Γ_1 and Γ_2' satisfy (D)?

Indeed, for the particular case above, we actually have this stronger question.

4.2. *Question.* Given two closed subgroups Γ_1, Γ_2 of Γ failing (D), is there a (unique) minimal closed subgroup H of Γ such that all the pairs, (H, Γ_1), (H, Γ_2), $(\Gamma_1, H + \Gamma_2)$, and $(\Gamma_2, H + \Gamma_1)$ all satisfy (D)?

In Example 0.1(v), H exists and is $\{0\} \times R \times \{0\}$. But even if there is no minimal object, we can ask the following question.

4.3. *Question.* Suppose Γ_1, Γ_2 are closed subgroups of Γ failing (D), but $I(A)$ is complemented for some $A, \Gamma \supset A \supset \Gamma_1 \cup \Gamma_2$. Is it true then that there exists some H as in Question 2 with $H \subset A$?

All these questions are related to the problem of discerning when, and how, the spectrum A of $I(A)$ contains parts which cover up the flaws in other parts, flaws that at first would seem to prevent $I(A)$ being complemented. These questions are also closely related to Proposition 2.1 and this generalization of it.

4.4. THEOREM. *If Γ_1, Γ_2 are closed subgroups of Γ such that* (D) *fails for* (Γ_1, Γ_2), *then $I(\Gamma_1 \cup \Gamma_2)$ is not complemented in $L_1(G)$.*

Proof. The failure of (D) means that for any compact neighborhoods L and W of 0 in Γ, there exists $\gamma_1 \in \Gamma_1 \backslash L + (\Gamma_1 \cap \Gamma_2)$, $\gamma_2 \in \Gamma_2 \backslash L$ such that $\gamma_1 - \gamma_2 \in W + (\Gamma_1 \cap \Gamma_2)$. Assume $I(\Gamma_1 \cup \Gamma_2)$ is complemented, so there exists a projection $P: I(\Gamma_2) \to I(\Gamma_1 \cup \Gamma_2)$. Let $X = \ker(P)$. Let $H = \Gamma_1^{\perp}$ and let $\pi: L_1(G) \to L_1(G/H)$ be the associated map. By Proposition 1.7, $\pi: X \to I(\Gamma_1 \cap \Gamma_2) \subset L_1(G/H)$ isomorphically.

Now let W_0 be a compact neighborhood of 0 in G/H with $m_{G/H}(W_0) = 1$. Consider $A = \{a_\gamma: \gamma \in \Gamma_1\}$, where $a_\gamma = \gamma 1_{W_0}$. Then $A \subset L_1(G/H)$, A is weakly relatively compact, $\hat{a}_\gamma(\gamma) = 1$ for all $\gamma \in \Gamma_1$. Denote the group $(\Gamma_1 \cap \Gamma_2)^{\perp}$ in G/H by H_1/H. Let $\pi_1: G/H \to G/H_1 = (G/H)/(H_1/H)$, and let ϕ_1 be a continuous Bruhat function for H_1/H, as discussed in the beginning of Section 1. Let $P_1: L_1(G/H) \to L_1(G/H)$ be the projection onto $I(\Gamma_1 \cap \Gamma_2)$ given by $P_1 x = x - \phi_1 \pi_1(x)$ from Proposition 1.3. For $\gamma \in \Gamma_1$, let $x_\gamma = (\pi|_X^{-1})(P_1 a_\gamma) = \pi|_X^{-1}(a_\gamma - \phi_1 \pi_1(a_\gamma))$. Then $\{x_\gamma: \gamma \in \Gamma_1\}$ is also weakly relatively compact in $L_1(G)$. Hence, there is a compact set K such that $\|x_\gamma|_{K^c}\|_1 < \frac{1}{4}$ for all $\gamma \in \Gamma_1$, by the uniform integrability of $\{x_\gamma: \gamma \in \Gamma_1\}$. Again, by the uniform integrability, there is a compact neighborhood W of 0 in Γ such that $|\widehat{x_\gamma|_K}(\gamma_1) - \widehat{x_\gamma|_K}(\gamma_2)| < \frac{1}{4}$ if $\gamma_1 - \gamma_2 \in W$, $\gamma \in \Gamma_1$. But notice also that if $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, then $\hat{x}_{\gamma_1}(\gamma_1) = \widehat{(a_{\gamma_1} - \phi_1 \pi_1(a_{\gamma_1}))}(\gamma_1) = \hat{a}_{\gamma_1}(\gamma_1) - \widehat{\phi_1 \pi_1(a_{\gamma_1})}(\gamma_1) = 1 - \widehat{\phi_1 \pi_1(a_{\gamma_1})}(\gamma_1)$. Also, $\hat{x}_{\gamma_1}(\gamma_2) = 0$ because $x_{\gamma_1} \in I(\Gamma_2)$. We claim that if $\varepsilon > 0$, there exists a compact set L such that for $\gamma_1 \notin L + \Gamma_1 \cap \Gamma_2$, then $|\widehat{\phi_1 \pi_1(a_{\gamma_1})}(\gamma_1)| < \frac{1}{8}$. Given this, choose $\gamma_1 \in \Gamma_1 \backslash L + (\Gamma_1 \cap \Gamma_2)$ and $\gamma_2' \in \Gamma_2 \backslash L$, $\gamma_1 - \gamma_2' \in W + \Gamma_1 \cap \Gamma_2$. Then $\gamma_1 - \gamma_2' - h \in W$ for some $h \in \Gamma_1 \cap \Gamma_2$. Let $\gamma_2 = \gamma_2' + h$. We have

$$\begin{aligned} \tfrac{7}{8} &< |\hat{x}_{\gamma_1}(\gamma_1) - \hat{x}_{\gamma_1}(\gamma_2)| \\ &\leqslant |\hat{x}_{\gamma_1}(\gamma_1) - \widehat{x_{\gamma_1}|_K}(\gamma_1)| + |\widehat{x_{\gamma_1}|_K}(\gamma_1) - \widehat{x_{\gamma_1}|_K}(\gamma_2)| \\ &\quad + |\widehat{x_{\gamma_1}|_K}(\gamma_2) - \hat{x}_{\gamma_1}(\gamma_2)| \\ &\leqslant \tfrac{3}{4}. \end{aligned}$$

This contradiction would complete the proof.

We have to estimate $\widehat{\phi_1 \pi_1(a_\gamma)}(\gamma)$, $\gamma \in \Gamma_1$. This can be done as follows:

$$|\widehat{\phi_1 \pi_1(a_\gamma)}(\gamma)| = \left| \int_{G/H} \phi_1 \pi_1(a_\gamma)(x) \overline{\gamma(x)}\, dm_{G/H}(x) \right|$$

$$= \left| \int_{(G/H)/(H_1/H)} \int_{H_1/H} \phi_1 \pi_1(a_\gamma)(x+k)\, \bar{\gamma}(x+k)\, dm_{H_1/H}(k)\, dm_{(G/H)/(H_1/H)}(x) \right|$$

$$= \left| \int_{(G/H)/H_1/H)} \int_{H_1/H} \phi_1(x+k) \int_{H_1/H} (\gamma 1_{W_0})(x+k+h)\, \bar{\gamma}(x+k) \right.$$

$$\left. \times\, dm_{H_1/H}(h)\, dm_{H_1/H}(k)\, dm_{(G/H)/(H_1/H)}(x) \right|$$

$$= \left| \int_{(G/H)/(H_1/H)} \psi_\gamma(x) \int_{H_1/H} (\gamma 1_{W_0})(x+h)\, dm_{H_1/H}(h)\, dm_{(G/H)/(H_1/H)}(x) \right|,$$

where $\psi_\gamma(x) = \int_{H_1/H} \phi_1(x+k)\, \bar{\gamma}(x+k)\, dm_{H_1/H}(k)$. Since $\psi_\gamma(x)$ is H_1/H-invariant this gives the estimate

$$|\widehat{\phi_1 \pi_1(a_\gamma)}(\gamma)| = \left| \int_{G/H} \psi_\gamma(x)(\gamma 1_{W_0})(x)\, dm_{G/H}(x) \right|$$

$$\leqslant \sup_{x \in W_0} |\psi_\gamma(x)| \int_{G/H} |\gamma 1_{W_0}(x)|\, dm_{G/H}(x)$$

$$= \sup_{x \in W_0} |\psi_\gamma(x)|.$$

We claim $\sup_{x \in W_0} |\psi_\gamma(x)| \to 0$ as $\gamma \to \infty$ in $\Gamma_1/\Gamma_1 \cap \Gamma_2$, which gives the estimate on $\widehat{\phi_1 \pi_1(a_\gamma)}(\gamma)$ that we wanted. Certainly, if x is fixed, $\psi_\gamma(x) \to 0$, $\gamma \to \infty$ in $\Gamma_1/\Gamma_1 \cap \Gamma_2$. Indeed $\psi_\gamma(x) = \widehat{{}_x\phi_1(\gamma)} \cdot \overline{\gamma(x)}$, where ${}_x\phi_1(h) = \phi_1(x+h)$, $h \in H_1/H$. Note ${}_x\phi_1 \in L_1(H_1/H)$ and $\widehat{H_1/H} = \Gamma_1/\Gamma_1 \cap \Gamma_2$. Since $|\psi_\gamma(x)| = |\widehat{{}_x\phi_1}(\gamma)|$, the Riemann–Lebesgue lemma proves the fact above.

Now recall that $\phi_1 = f * 1_M$, $f \in C_c(G/H)$ and M a measurable set. If $x_1, x_2 \in G/H$, then

$$|\widehat{{}_{x_1}\phi_1}(\gamma) - \widehat{{}_{x_2}\phi_2}(\gamma)|$$

$$= \left| \int_{H_1/H} [\phi_1(x_1+h) - \phi_2(x_2+h)]\, \bar{\gamma}(h)\, dm_{H_1/H}(h) \right|$$

$$\leqslant \|{}_{x_1}\phi_1 - {}_{x_2}\phi_1\|_{L_1(H_1/H)} \leqslant \|{}_{x_1}f - {}_{x_2}f\|_{L_1(G/H)}.$$

Since $f \in C_c(G)$, $\lim_{x_1 - x_2 \to 0} \|{}_{x_1}f - {}_{x_2}f\|_{L_1(G/H)} = 0$. Hence, $\lim_{x_1 - x_2 \to 0} |\widehat{{}_{x_1}\phi_1}(\gamma) - \widehat{{}_{x_2}\phi_1}(0)| = 0$ uniformly in γ. Because W_0 is compact, and because $|\widehat{{}_x\phi_1}(\gamma)| \to 0$ as $\gamma \to \infty$ in $\Gamma_1/\Gamma_1 \cap \Gamma_2$ for any fixed $x \in W_0$, this shows that we can make $|\psi_\gamma(x)| = |\widehat{{}_x\phi_1}(\gamma)|$ uniformly small on W_0 by letting $\gamma \to \infty$ in $\Gamma_1/\Gamma_1 \cap \Gamma_2$. ∎

Remark. In Reiter [11], the construction of the Bruhat functions shows that we could assume at the outset in the above that $\phi_1|_{W_0 + H_1/H}$ is compactly supported. This would be enough to make the last part of the argument work too without our special form for ϕ_1.

Almost all that we have done here is based upon the relationship between the pairs of subgroups in the representation of the hull of the ideal. However, we know that we cannot handle some ideals in this way. In particular, this is the case for Example 0.1(iii). Each pair of subgroups satisfy (D). So the inductive procedure must fail when $\tan(\theta)$ is irrational because the ideal $I({}_\theta R \cap (Z \times R \cup R \times Z))$ is not a l_1 sum of compactly supported subspaces. We have no direct method of reaching this conclusion. It is only by Proposition 1.8 and the nature of the inductive procedure in Section 3 that we see it is true. Unfortunately, Proposition 1.8 is of limited usefulness because of the strength of the assumptions. In particular, we cannot decide when an ideal $J = I(\bigcup_{i=1}^n \gamma_i + (\Gamma_i \backslash B_i))$ is complemented in $L_1(R^2)$ if $n \geqslant 5$. But if $n \leqslant 4$, then Proposition 2.1 and Proposition 3.1 can be used to show that if J is complemented, and the cosets are not separated, then there are two pairs of subgroups in the representation which satisfy (D), i.e., we have without loss of generality some $I(\bigcup_{i=1}^2 \gamma_i + (\Gamma_i \backslash B_i))$ and $I(\bigcup_{i=3}^4 \gamma_i + (\Gamma_i \backslash B_i))$ which are complemented. Then we apply Proposition 1.9 or use the inductive procedure of Section 3 to complete the construction of a projection on J. The result is that, for any complemented ideal J in $L_1(R^2)$, with no more than four terms $\gamma_i + (\Gamma_i \backslash B_i)$ in the representation of its hull, the ideal can be shown to be complemented by the inductive procedure.

Indeed, as far as we know, every complemented ideal in $L_1(G)$ may be complemented by the modified inductive procedure discussed, by way of Example 0.1(v), at the beginning of this section. But this would not give a nice criterion on A for $I(A)$ to be complemented.

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