JOURNAL OF ALGEBRA 27, 1–10 (1973)

Q-Divisibility and Injectivity

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Communicated by P. M. Cohn

Received August 17, 1971

1. INTRODUCTION

Various types of "divisibility" have been considered in the literature [6, 12], all of which make injective modules "divisible." More recently, D. Wei [19] introduced a divisibility in terms of the maximal quotient ring Q of R. Thus an R-module M is divisible in the sense of Wei provided $\mathrm{Hom}_R(Q, N) \neq 0$ for each non-zero factor module N of M. Modifying the terminology slightly, we will call such an R-module *Q-divisible.* As noted in [19], all injective R-modules are Q-divisible and every R-module contains a unique maximal Q-divisible R-module.

Now the maximal quotient ring Q of R is the localization of R (in the sense of P. Gabriel [4]) corresponding to the (topologizing and idempotent) filter of dense left ideals of R, and the torsion class (in the sense of S. E. Dickson [2]) which corresponds to this filter is the class $\mathscr{T}$ of R-modules M for which $\mathrm{Hom}_R(M, E(R)) = 0$, $E(R)$ being the injective envelope of R. The connection between this torsion class, Q-divisible modules, and injective modules is the principal subject of this note, as well as some relationships between the torsion $\mathscr{T}$ and the usual torsion R-modules. The principal results, which occur in Section 3, are as follows:

(1) Every $\mathscr{T}$-torsion-free Q-divisible R-module is injective if and only if Q is a semisimple artinian ring.

* The second author held an NDEA Title IV fellowship at the University of Texas at Austin during the preparation of this work.

(2) Assume Q is a quasi-Frobenius ring; if M is a homomorphic image of an injective R-module and $M/T(M)$ is injective, then $T(M)$ is a direct summand of M.

Both (1) and (2) have interesting consequences in the case of classical quotient rings; in particular (1) yields a result of L. Levy: If R has a classical quotient ring K, then every divisible R-module is injective if and only if K is semisimple artinian. Moreover, (2) can be applied in case R has a classical quotient ring which is quasi-Frobenius (or semisimple artinian).

Since the hypotheses in (2) requires Q to be quasi-Frobenius, in Section 4 two characterizations of rings having Q quasi-Frobenius are given; as a by-product, rings for which Q is self-injective and semiprimary are characterized.

2. Preliminaries

All rings considered will have a unit and modules will be unital left modules. We begin by giving the usual remarks concerning definitions and notation.

Let M be an R-module, N a submodule of M. For $x \in M$ let $(N : x) = \{a \in R \mid ax \in N\}$. Then N is *dense* in M if for $x, y \in M$ with $x \neq 0$ we have $(N : y)x \neq 0$. A left ideal I is a *dense left ideal* of R if I is dense in R. We will denote the maximal (left) quotient ring of R by Q [9], while $Z(K)$ will denote the (left) singular ideal of a ring K.

Let $\mathscr{T}$ denote the class of all R-modules M for which $\mathrm{Hom}_R(M, E(R)) = 0$. The class $\mathscr{T}$ is then a hereditary torsion class [2], i.e., $\mathscr{T}$ is closed under submodules, factor modules, extensions, and direct sums and so every R-module M has a unique maximal $\mathscr{T}$-submodule, which is denoted by $T(M)$. The torsion class $\mathscr{T}$ has been considered in [5, 7, 14]. From [14], we have the following facts: For any R-module M, $T(M) = \{x \in M \mid (R : x) \text{ is dense in } R\}$ and $T(M/T(M)) = 0$; $T(M) = 0$ if and only if M is embeddable in a direct product of copies of $E(R)$. We call M $\mathscr{T}$-*torsion*-free whenever $T(M) = 0$ and M is *torsion*-free if whenever $ax = 0$ for $x \in M$, $a \in R$ regular, then $x = 0$.

The relationship between $\mathscr{T}$ and Q has been described in [10, p. 25]:

Proposition 2.1. *Let R be a ring with maximal quotient ring Q and define Q^* by $Q^*/R = T(E(R)/R)$. Then Q^* is a ring and there is ring isomorphism (fixing R) between Q^* and Q.*

We will identify Q with Q^* so that $Q = \{x \in E(R) \mid (R : x) \text{ is dense in } R\}$. Then Q is (left) self-injective if and only if $Q = E(R)$ [9, p. 95].

An R-module M is *Q-divisible* if $\mathrm{Hom}_R(Q, N) \neq 0$ for all non-zero factor modules N of M; M is a *Q_s-module* if M is a factor of a direct sum of copies of Q. An R-module M is *divisible* if $aM = M$ for all regular elements a of R. For an R-module M, let $q(M) = \sum \mathrm{Im} f$, where f ranges over $\mathrm{Hom}_R(Q, M)$. For each ordinal $i \geqslant 0$ define a submodule $q_i(M)$ by setting $q_0(M) = 0$ and, for $i \geqslant 1$, $q_i(M)/q_{i-1}(M))$ if $i - 1$ exists, while $q_i(M) = U_{j<i} q_j(M)$ otherwise. The least ordinal k for which $q_k(M) = q_{k-1}(M)$ is called the *q-length* of M.

The next proposition is now easily verified.

PROPOSITION 2.2. *Let M be an R-module with q-length $(M) = k$. Then:*

(a) *$q_k(M) =$ maximal Q-divisible submodule of M. Hence M is Q-divisible if and only if $q_k(M) = M$.*

(b) *If Q is also a classical quotient ring of R, then every Q_s-module is divisible and thus $q_k(M)$ is a divisible R-module.*

3. Q-DIVISIBLES, INJECTIVES, AND TORSION

The first result of this section characterizes those rings for which $\mathscr{T}$-torsion-free Q-divisible modules are injective.

THEOREM 3.1. *Let R be a ring with maximal quotient ring Q. Then the following are equivalent statements:*

(a) *Q is a semisimple artinian ring.*

(b) *$Z(R) = 0$ and R is finite-dimensional.*

(c) *Every $\mathscr{T}$-torsion-free Q-divisible R-module is injective.*

(d) *Every $\mathscr{T}$-torsion-free Q_s-module is injective.*

Proof. The equivalence of (a) and (b) is due to R. E. Johnson [8] while clearly (c) $\Rightarrow$ (d). To show that (b) $\Rightarrow$ (c), first note that, since $Z(R) = 0$, $Q = E(R)$ and since R is finite-dimensional, any direct sum of copies of Q is an injective R-module [3, Prop. 1]. Let M be Q-divisible with $T(M) = 0$. Then $q(M)$ is a Q_s-module. Since $Z(R) = 0$, $T(M)$ is the singular submodule of M [5] and the kernel of the map of $Q^{(I)}$ (= direct sum of copies of Q over some index set I) onto $q(M)$ can have no essential extension in $Q^{(I)}$ and so must be injective. Thus $q(M)$ is isomorphic to a direct summand of $Q^{(I)}$ and hence is injective. But, since M is Q-divisible, it follows that $M = q(M)$ so M is injective.

Now suppose (d) holds. If I is an essential left ideal of R then QI is an essential R-submodule of Q. Since QI is a Q-module, it is then a Q_s-module and thus QI is an injective R-module since $T(Q) = 0$. Thus $QI = Q$ for all essential left ideals I of Q. From this we then have $Z(R) = 0$ and so, by [18, Thm. 4.20] or [15, Thm. 1.6], Q is semisimple artinian, completing the proof.

As a consequence we obtain a result of L. Levy [11, Thm. 3.3].

COROLLARY 1. *Let R be a ring having a classical quotient ring K. Then every torsion-free divisible R-module is injective if and only if K is semisimple artinian.*

Proof. If K is semisimple artinian, then $K = Q$ and every torsion-free divisible R-module M is a K-module and thus an injective K-module. But this readily implies M is an injective R-module. Conversely, suppose every torsion-free divisible R-module is injective. Then K is injective and so $K = Q$. If M is $\mathscr{T}$-torsion-free, then M is isomorphic to a submodule of a direct product of copies of K and so M is also torsion-free. If, further, M is a Q_s-module then, by Proposition 2.2(b), M is divisible. Thus M is injective and hence K is semisimple artinian.

It should be noted that, by Theorem 3.1, the condition that $\mathscr{T}$-torsion-free Q-divisibles be injective does not imply that R must have a classical quotient ring.

The case when all Q-divisible R-modules are injective has been dealt with in [1]; for completeness we state the result:

THEOREM 3.2. *Let R be a ring with maximal quotient ring Q. Then every Q-divisible R-module is injective if and only if R is left hereditary and left noetherian.*

In the proof of Corollary 1 it was noted that, whenever Q is a classical quotient ring of R, every $\mathscr{T}$-torsion-free R-module is torsion-free. At this point it may be of interest to point out the class of rings R for which the $\mathscr{T}$-torsion R-modules and torsion R-modules coincide. In order to do this, and for later use, we recall that a ring K is a (right) *S-ring* if and only if $r_K(A) \neq 0$ for each proper left ideal A of K; S-rings have been characterized in the following manner:

PROPOSITION 3.3 [7, Thm. 3.2]. *For a ring K, the following conditions are equivalent:*

(a) *K is an S-ring.*

(b) *K contains a copy of each simple K-module.*

(c) *K has no proper dense left ideals.*

(d) *0 is the only $\mathscr{T}$-torsion K-module.*

It follows that, if K is an S-ring, then K is its own maximal quotient ring. Rings R for which Q is an S-ring have been considered in [18, Thm. 3.2], where the following result occurs, essentially.

THEOREM 3.4. *Let R be a ring with maximal quotient ring Q. Then the following statements are equivalent:*

(a) *Q is an S-ring.*

(b) *$QI = Q$ for each dense left ideal I of R.*

(c) *For any R-module M, $T(M)$ is the kernel of the canonical map $M \to Q \otimes_R M$.*

PROPOSITION 3.5. *Suppose R is a ring with maximal quotient ring Q. Then the $\mathscr{T}$-torsion R-modules and torsion R-modules coincide if and only if Q is a classical quotient ring of R and Q is an S-ring.*

Proof. If the two classes coincide, then any direct sum of torsion R-modules is torsion and so the torsion elements of any R-module form a submodule. Hence, by [11, Thm. 1.4], R has a classical quotient ring K and $R \subseteq K \subseteq Q$. Moreover, the filters associated with the two classes are identical and so every dense left ideal of R contains a regular element hence $KI = K$ for each dense left ideal I of R. If J is a dense left ideal of K, then $J \cap R$ is a dense left ideal of R and so $J = KJ = K$. Thus K is an S-ring and so K is its own maximal quotient ring; it follows that $K = Q$.

For the converse, if I is a dense left ideal of R, then $QI = Q$ so $1 = \sum_{j=1}^{n} q_j u_j$ for suitable $\{q_1, ..., q_n\} \subseteq Q$, $\{u_1, ..., u_n\} \subseteq I$. Then there exists $a \in R$ regular, $\{b_1, ..., b_n\} \subseteq R$ with $q_j = a^{-1} b_j$. Then $a = \sum_{j=1}^{n} b_j u_j \in I$, hence every dense left ideal contains a regular ideal. On the other hand, if $a \in R$ is regular, then Ra is dense: For let $x, y \in R$, with $x \neq 0$; then $Qa = Q$ so $y = c^{-1} da$, $c \in R$ regular, so $c \in (Ra : y)$, hence $(Ra : y)x \neq 0$. Thus the filters corresponding to the classes coincide, hence $\mathscr{T}$-torsion R-modules and torsion R-modules coincide.

Another result concerning the torsion R-modules of interest is:

PROPOSITION 3.6. *For a ring R the following statements are equivalent:*

(a) *Any direct product of torsion R-modules is torsion.*

(b) *0 is the only torsion R-module.*

(c) *Every regular element of R is invertible in R.*

Proof. Clearly (c) $\Rightarrow$ (b) $\Rightarrow$ (a) so assume (a). Then, since submodules of torsion R-modules are torsion, the torsion R-modules form a torsion class. By [7, Thm. 2.1], the associated filter of left ideals, which consists of the left ideals which contain regular elements, has a minimal element I. Moreover, $I^2 = I$, I is an ideal of R, and I is contained in every left ideal in the filter. Thus, if $a \in I$ is regular, then Ra is in the filter so $Ra = I$. Then $Ra = (Ra)^2$ implies $R = RaR = IR = I = Ra$ so $a^{-1} \in R$. Then, for any regular element, $b \in R$, $Ra \subseteq Rb$ implies $b^{-1} \in R$.

We now wish to use the properties of S-rings (in particular Theorem 3.4) to obtain certain instances when the $\mathscr{T}$-torsion submodule of an R-module M is a direct summand whenever M is close to being injective.

LEMMA 3.7. *Let R be a ring with maximal quotient ring Q and assume Q is an S-ring.*

(a) *If A and B are Q-modules such that B is a $\mathscr{T}$-torsion-free R-module, then every R-homomorphism f: $A \to B$ is a Q-homomorphism.*

(b) *If A is a $\mathscr{T}$-torsion free injective R-module, then A is an injective A-module.*

Proof. (a) Let f: $A \to B$ be an R-homomorphism, $q \in Q$ and $a \in A$. Then $(R : q)$ is dense in R and, for any $x \in (R : q)$, we have $xq \cdot f(a) = f(xqa) = x \cdot f(qa)$, hence $q \cdot f(a) - f(qa) \in T(B) = 0$. Thus f is also a Q-homomorphism.

(b) Let A be a $\mathscr{T}$-torsion-free injective R-module; then $Q \otimes_R A$ is a Q-module and, by Theorem 3.4(c), we can consider $A \subseteq Q \otimes_R A$. Let $x \in Q \otimes_R A$, $x \neq 0$; then $x = \sum_{j=1}^n q_j \otimes a_j$, $\{q_1, ..., q_n\} \subseteq Q$, $\{a_1, ..., a_n\} \subseteq A$. Then $\sum_{j=1}^n (R : q_j) = I$ is dense in R so, since $QI = Q$, $Ix \neq 0$. Thus, if $r \in I$ such that $0 \neq rx$, then $rx = \sum rq_i \otimes a_i = 1 \otimes \sum rq_i a_i \in A$. Hence A is essential in $Q \otimes_R A$. Since A is also injective, we have $Q \otimes_R A = A$ and so A is a Q-module. That A is an injective Q-module follows easily.

We now come to the second main result of this paper.

THEOREM 3.8. *Let R be a ring with maximal quotient ring Q such that Q is a quasi-Frobenius ring. If M is a Q_s-module such that $M/T(M)$ is an injective R-module, then $T(M)$ is a direct summand of M.*

Proof. Let $Q^{(I)}$ map onto M via an R-homomorphism g, for some index set I. Now Q is quasi-Frobenius hence an S-ring; thus $M/T(M)$ is an injective Q-module by Lemma 3.7(b), and so $M/T(M)$ is a projective Q-module. If h: $M \to M/T(M)$ is the natural map, then $f = h \circ g$: $Q^{(I)} \to M/T(M)$ is an epimorphism and, by Lemma 3.7(a), f is a Q-homomorphism. But then $K = \ker f$ is a direct summand of $Q^{(I)}$, say $Q^{(I)} = K \oplus L$ as Q-modules. Since this is also a splitting of R-modules, it can be verified in a straightforward manner that $M = T(M) \oplus g(L)$.

We now give some consequences of this result; note that, for any ring R, every injective module is a Q_s-module, hence every homomorphic image of an injective R-module is a Q_s-module.

COROLLARY 1. *Let R be a ring with a semisimple artinian classical quotient ring. If M is a homomorphic image of an injective R-module, then the torsion submodule of M is a direct summand of M.*

Proof. $T(M) =$ torsion submodule of M by Proposition 3.5, and $M/T(M)$ is injective by Theorem 3.1, Corollary 1.

The previous corollary generalizes Theorem 1.1 of [12]; we remark that F. L. Sandomierski has also generalized this last result to rings with $Z(R) = 0$ [15, Thm. 2.10].

COROLLARY 2. *Let R be a ring with a classical quotient ring which is quasi-Frobenius. If M is a homomorphic image of an injective R-module and $M/T(M)$ is injective, then $T(M)$ is a direct summand of M.*

Proof. Again use Proposition 3.5.

4. SELF-INJECTIVE MAXIMAL QUOTIENT RINGS

The hypothesis of Theorem 3.8 required that Q be quasi-Frobenius. In this section we obtain two characterizations of rings with this property. As a preliminary step we obtain a characterization of those rings for which Q is self-injective and semiprimary.

For a left ideal I of R, let $I^c =$ (*closure of* I) be defined by $I^c/I = T(R/I)$.

LEMMA 4.1. *Let R be a ring with maximal quotient ring Q such that Q is a S-ring.*

(a) *If J is a left ideal of Q, then $J = Q(J \cap R)$; if J is a two-siced ideal of Q, then $J^k = Q(J \cap R)^k$ for all $k \geqslant 1$.*

(b) *If I is a left ideal of R, then $I^c = QI \cap R$.*

Proof. (a) $Q(J \cap R) \subseteq J$, so let $x \in J$; then $(R : x)$ is dense in R, so $Q(R : x) = Q$. Thus $x \in Qx = Q(R : x)x \subseteq Q(J \cap R)$, hence $J = Q(J \cap R)$. If $JQ \subseteq J$, then $J^k = J^{k-1}Q(J \cap R) = J^{k-1}(J \cap R) = J^{k-2}Q(J \cap R)^2 = \cdots = J(J \cap R)^{k-1} = Q(J \cap R)^k$.

(b) Let $x \in QI \cap R$ so $x = \sum_{j=1}^{n} q_j u_j$, with $\{q_1, ..., q_n\} \subseteq Q$, $\{u_1, ..., u_n\} \subseteq I$. Then $T = \bigcap_{j=1}^{n} (R : q_j)$ is dense in R and so, for $r \in T$, we have

$$rx = \sum_{j=1}^{n} rq_j u_j \in I.$$

Thus $T \subseteq (I : x)$ so $(I : x)$ is dense in R and hence $x \in I^c$. If $y \in I^c$, then $(I : y)$ is dense in R and so $Q(I : y) = Q$. Then $Qy = Q(I : y)y \in QI$, hence $y \in QI \cap R$ and equality follows.

THEOREM 4.2. *Let R be a ring with maximal quotient ring Q. Then Q is a left self-injective semiprimary ring if and only if*

(a) $r_R(R : x) = 0$ *for all* $x \in E(R)$,

(b) $QI = Q$ *for each dense left ideal* I *of* Q,

(c) $Z(R)$ *is nilpotent.*

Proof. We first remark that condition (a) is equivalent to $R(:x)$ being a dense left ideal for all $x \in E(R)$; thus Q is left self-injective if and only if (a) holds. Now assume Q is self-injective and semiprimary. By the proof of [17, Thm. 3.4], Q is an S-ring so (b) holds. Now $Z(R) = Z(Q) \cap R$ and $Z(Q) = J(Q)$, the Jacobson radical of Q [16, Lemma 4.1]. Then $Z(Q)$ is nilpotent and so, by Lemma 4.1, $Z(R)$ is nilpotent. Conversely, assume (a), (b), (c) hold. Then Q is a self-injective S-ring by (a) and (b) and again, by Lemma 4.1, $J(Q)$ is nilpotent. Since Q is finite-dimensional, $Q/J(Q)$ is semisimple artinian by [17, Thm. 3.4] and so Q is semiprimary.

Using the previous result we can get the first characterization of rings with quasi-Frobenius maximal quotient ring.

THEOREM 4.3. *Let R be a ring with maximal quotient ring Q. Then Q is a quasi-Frobenius ring if and only if*

(a) $r_R(R : x) = 0$ *for all* $x \in E(R)$,

(b) $QI = Q$ *for each dense left ideal* I *of* Q,

(c) $Z = Z(R)$ *is nilpotent,*

(d) $R/(Z^k)^c$ *is finite-dimensional for* $k = 1, 2, 3, \ldots$.

Proof. Suppose conditions (a)–(d) hold. Then Q is a self-injective semiprimary ring. Now $Z = Z(Q) \cap R = J(Q) \cap R$ and, by Lemma 4.1,

$$(Z^k)^c = QZ^k \cap R = Q(J(Q) \cap R)^k \cap R = J(Q)^k \cap R.$$

If $\bar{R}_k = (R + J(Q)^k)/J(Q)^k$, then $\bar{R}_k$ is essential as an $\bar{R}_k$-module in $Q/J(Q)^k$.

Moreover, $R_k \approx R/(Z^k)^c$ and so $Q/J(Q)^k$ is finite-dimensional by (d). Then socle$(Q/J(Q)^k)$ is finitely generated for each $k \geqslant 1$ and, since $J(Q)$ is nilpotent, this gives a composition series for Q; thus Q is left artinian, hence quasi-Frobenius.

On the other hand, if Q is quasi-Frobenius, then R satisfies (a)–(c). As above, $R/(Z^k)^c$ is essential in $Q/J(Q)^k$, which is artinian, and so $R/(Z^k)^c$ is finite-dimensional for each $k \geqslant 1$.

Our other characterization is prompted by Theorem 2.6 of [13].

THEOREM 4.4. *Let R be a ring with maximal quotient ring Q. Then Q is a quasi-Frobenius ring if and only if*

(a) $r_R(R : x) = 0$ *for all* $x \in E(R)$,

(b) *R has* ACC *on annihilators of subsets of* $E(R)$.

Proof. If (a) and (b) hold, then Q is self-injective. Let A, B be annihilator left ideals in Q, $A = \ell_Q(X)$, $B = \ell_Q(Y)$ and suppose $A \subsetneq B$. Thus $\ell_R(X) \subseteq \ell_R(Y)$. Suppose $\ell_R(X) = \ell_R(Y)$; then, for $b \in B$, $(R : b)\, bY = 0$ so $(R : b)\, bX = 0$. Since $(R : b)$ is dense in R, $bX = 0$ and thus $b \in A$. It now follows from (b) that Q has ACC on left annihilators and so, by [3, Thm. 2], Q is quasi-Frobenius.

Conversely, if Q is quasi-Frobenius, then (a) holds and Q is an S-ring. Suppose $A = \ell_R(X)$ with $X \subseteq Q$. Since $(QA \cap R)X = 0$, we have $A = A^c$. Thus, if $B = \ell_R(Y)$ with $Y \subseteq Q$ and if $A \subseteq B$, then $QA = \ell_Q(X), QB = \ell_Q(Y)$. Moreover, if $QA = QB$, then, by Lemma 4.1, $A = A^c = QA \cap R = QB \cap R = B$ and so R has ACC on annihilators of subsets of $Q(=E(R))$.

We remark that as in [13] the following can be verified; a proof is omitted.

PROPOSITION 4.5. *Let R be a ring with maximal quotient ring Q. Then Q is self-injective and semiperfect if and only if*

(a) $r_R(R : x) = 0$ *for all* $x \in E(R)$,

(b) *R is finite-dimensional.*

REFERENCES

1. E. P. ARMENDARIZ, Q-divisible modules, *Canad. Math. Bull.* **14** (1971), 491–494.
2. S. E. DICKSON, A torsion theory for Abelian categories, *Trans. Amer. Math. Soc.* **121** (1966), 223–235.
3. C. FAITH, Rings with ascending condition on annihilators, *Nagoya Math. J.* **27** (1966), 179–191.
4. P. GABRIEL, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.

5. E. GENTILE, Singular submodule and injective hull, *Indag. Math.* **4** (1962), 426–433.
6. G. HELZER, On divisibility and injectivity, *Canad. J. Math.* **18** (1966), 901–919.
7. J. P. JANS, Some aspects of torsion, *Pacific J. Math.* **15** (1965), 1249–1259.
8. R. E. JOHNSON, Quotient rings with zero singular ideal, *Pacific J. Math.* **11** (1961), 1385–1392.
9. J. LAMBEK, "Lectures on Rings and Modules," Blaisdell, Waltham, Mass., 1966.
10. J. LAMBEK, "Torsion Theories, Additive Semantic, and Rings of Quotients" (Lecture Notes in Mathematics No. 177), Springer-Verlag, New York/Berlin, 1971.
11. L. LEVY, Torsion-free and divisible modules over non-integral domains, *Canad. J. Math.* **15** (1963), 132–151.
12. E. MATLIS, Divisible modules, *Proc. Amer. Math. Soc.* **11** (1960), 385–391.
13. A. C. MEWBORN AND C. N. WINTON, Orders in self injective semiperfect rings, *J. Algebra* **13** (1969), 4–9.
14. R. S. PIERCE, "Modules over Commutative Regular Rings," Memoirs American Mathematical Society 70, Providence, R.I., 1967.
15. F. L. SANDOMIERSKI, Semisimple maximal quotient rings, *Trans. Amer. Math. Soc.* **128** (1967), 112–120.
16. Y. UTUMI, On continuous rings and self-injective rings, *Trans. Amer. Math. Soc.* **118** (1965), 158–173.
17. Y. UTUMI, Self-injective rings, *J. Algebra* **6** (1967), 56–64.
18. C. L. WALKER AND E. A. WALKER, Quotient categories and rings of quotients, *Rocky Mt. J. Math.* **2** (1972), 513–555.
19. D. WEI, On the concept of torsion and divisibility for general rings, *Illinois J. Math.* **13** (1969), 414–431.