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Modules with Serial Noetherian Endomorphism Rings

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In this paper we shall characterize the generators whose endomorphism rings are serial Noetherian. A module is *serial* when its submodules are linearly ordered with respect to inclusion. A ring Λ is said to be right serial if Λ_Λ is a direct sum of serial submodules, and Λ is serial if it is both left and right serial. By Warfield's structure theorem [7], a serial Noetherian ring Λ is the product of a serial Artinian ring and a finite number of serial prime Noetherian rings. We denote by Λ_{a} the Artinian component and Λ_{n} the Noetherian component of a serial Noetherian ring Λ; $\Lambda = \Lambda_{\mathrm{a}} \times \Lambda_{\mathrm{n}}$. Given a module M, by $\mathrm{add}(M)$ we understand the category of modules isomorphic to summands of finite direct sums of M. Our aim of this paper is to give a simple characterization for a module over a serial Artinian ring to have the serial endomorphism ring, and to prove the following theorem.

THEOREM. *Let Λ be a ring and G_Λ a generator. Then the endomorphism ring Γ of G is serial Noetherian if and only if*

(1) *Λ is serial Noetherian, and*

(2) *$G \in \mathrm{add}(\Lambda \oplus I \oplus I/\mathrm{soc}\, I)$, where I_Λ is the injective hull of the top of the Artinian component of Λ_{a}.*

Moreover, in this case, $\Gamma_{\mathrm{a}} = 0$ ($\Gamma_{\mathrm{n}} = 0$, respectively) if and only if $\Lambda_{\mathrm{a}} = 0$ ($\Lambda_{\mathrm{n}} = 0$, respectively).

To prove this, in view of Warfield's structure theorem, we have only to consider the case where the given ring is either serial Artinian or serial prime Noetherian. In the case of serial Artinian rings, in Section 1, we shall show the following theorem: *A module, say $M = \bigoplus M_i$ with M_i indecomposable, over a serial Artinian ring has the serial endomorphism ring if and only if for each connected component N of M, no summand M_i of N is isomorphic to a proper subfactor of any summand M_j of N* (Theorem 1).

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Here, a connected component N of M is a direct summand of M such that the endomorphism ring End N is connected (that is, indecomposable as a ring) and is a direct summand of the ring End M. Makino [4] gave a different characterization of such a module, from which he also proved the "if" part of Theorem 1. In Section 2, we shall consider a condition for a generator to have the right serial endomorphism ring, and prove the theorem for serial prime Noetherian rings.

All rings in this paper are associative with identity. Modules are usually assumed to be right modules, unless otherwise specified. When we refer to Noetherian rings, it always means both left and right Noetherian rings, and similarly for Artinian rings. Given a module M, morphisms operate from the left to M. The Jacobson radical, denoted by rad M, of a modul M is defined as the intersection of all maximal submodules if they exist. The socle and the factor $M/\text{rad}\, M$ (when rad M is defined) are denoted by soc M and top M, respectively, and $|M|$ stands for the composition length of M.

1. Serial Artinian Rings

In this section we characterize the modules with serial Artinian endomorphism rings. It is well known that, over a serial Artinian ring, every indecomposable left and every indecomposable right module is finitely generated serial, quasi-projective, and quasi-injective [3, 5]. A non-zero module X is called a *proper subfactor* of a module M if X is a proper submodule of M/N (that is, $X \subsetneq M/N$) for some non-zero submodule N of M. An indecomposable projective and an indecomposable injective module is obviously isomorphic to no proper subfactors of any indecomposable module. When we use the term "proper" for a morphism, it always means that the morphism is not an isomorphism.

LEMMA 1. *Assume that Λ is a serial Artinian ring. Then, for a non-projective indecomposable Λ-module X, the following statements are equivalent.*

(1) *X is isomorphic to $I/\text{soc}\, I$ for some indecomposable injective Λ-module I.*

(2) *X is isomorphic to no proper subfactors of any indecomposable projective Λ-module.*

(3) *X is isomorphic to no proper subfactors of any indecomposable Λ-module.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) is clear.

(1)$\Rightarrow$(2) Suppose on the contrary that the statement (2) does not hold. Then, since Λ is serial Artinian, there is an indecomposable projective Λ-module P such that $I/\text{soc}\, I$ is a proper factor of rad P. Clearly we have that $|I| \leqslant |\text{rad}\, P|$ and $\text{top}\, I \simeq \text{top}(\text{rad}\, P)$, which implies that I is a factor of rad P. Hence there is an epimorphism $f: \text{rad}\, P \to I$, and so a monomorphism $I \cong \text{rad}\, P/\text{Ker}\, f \subsetneq P/\text{Ker}\, f$. But, as I is injective, the embedding is splittable, which contradicts the indecomposability of $P/\text{Ker}\, f$.

(2)$\Rightarrow$(1) Let $u: P \to X$ be the projective cover. Then $X \simeq I/\text{soc}\, I$, where I stands for the factor module $P/\text{rad}(\text{Ker}\, u)$ and $\text{Ker}\, u \neq 0$. We claim that I is injective. Otherwise, let E be the injective hull of I and $v: P' \to E$ its projective cover. We have then a composed monomorphism $X \cong I/\text{soc}\, I \subsetneq \text{rad}\, E/\text{soc}\, E \cong \text{rad}\, P'/v^{-1}(\text{soc}\, E)$. But this implies that X is isomorphic to a proper subfactor of P', which is a contradiction.

Let M be a module whose endomorphism ring, say Γ, is semi-primary, and $\{e_i\}$ a complete set of orthogonal primitive idempotents of Γ. Then, Γ is serial if and only if both $\text{top}(\text{rad}\, e_i\Gamma)$ and $\text{top}(\text{rad}\, \Gamma e_i)$ are simple for all i. It amounts to the same to say that, in the category add M, the following two conditions hold for any indecomposable object X; (i) every non-zero morphism to X is splittable or there is the minimal right almost split map (sink map) $Y \to X$ with Y indecomposable, and (ii) every non-zero morphism from X is splittable or there is the minimal left almost split map (source map) $X \to Y$ with Y indecomposable. (See [1] for the minimal almost split maps, and [2, Theorem 3].) Moreover, in this case, given indecomposable modules X and Y in add M, a morphism $f: X \to Y$ is a sink map in add M if and only if it is a source map in add M. As mentioned in the introduction, the "if" part of the following theorem is already proved in [4]. Our proof is based on the notion of the minimal almost split maps in the representation theory of algebras, but does not use any recent results in the theory.

THEOREM 1. *Let Λ be a serial Artinian ring and $M = \bigoplus M_i$ a Λ-module with M_i indecomposable, and assume that* End M *is connected. Then,* End M *is serial Artinian if and only if no M_i is isomorphic to a proper subfactor of any M_j.*

Proof. We call here an indecomposable module in add M a good module when it is not isomorphic to a proper subfactor of any M_i, otherwise we call it a bad module.

First we suppose that all M_i are good modules. For an indecomposable object X of add M, we have to show the existence of the sink map to X and the source map from X in the category add M. But, by the duality principle in category theory, it suffices to show the existence of the sink map. (Note that the statement "all M_i are good" is self-dual.) Now assume that there are non-splittable maps to X in add M, and let $f: Y \to X$ be a morphism with the smallest length $|Y|$ among the non-isomorphisms from indecomposable modules to X in add M whose images are of the maximal length. We claim that f is a sink map. Let $g: W \to X$ be a non-isomorphism in add M with W indecomposable. We have to show the existence of a morphism $h: W \to Y$ such that $g = fh$. Let $p: P \to W$ be the projective cover of W in the category of Λ-modules. As $g(W) \subset f(Y)$ by the choice of f, there is a morphism $q: P \to Y$ such that $fq = gp$. In the case where $\operatorname{Ker} p \subset \operatorname{Ker} q$, we may take h as the composition of canonical maps $W \simeq P/\operatorname{Ker} p \to P/\operatorname{Ker} q \to Y$. In the other case, $\operatorname{Ker} q \subsetneqq \operatorname{Ker} p$ obviously, and so the composition $W \simeq P/\operatorname{Ker} p \xrightarrow{\bar{q}} Y/q(\operatorname{Ker} p)$ is monomorphic and $q(\operatorname{Ker} p) \neq 0$, where $\bar{q}$ is a canonical map induced from q. Since W is a good module by assumption, $\bar{q}$ must be an isomorphism and hence q is epimorphic. This implies that $f(Y) = g(W)$ and, by the choice of f, we have that $|Y| \leqq |W|$. Thus, by making use of the quasi-projectivity of W, we conclude that g factors through Y.

Conversely, we assume that End M is serial Artinian. Note that there are good modules, for example, the M_i with maximal length. To show that all M_i are good, we suppose the contrary. Since End M is connected, we have then a sink (= source) map $X \to Y$ or a source (= sink) map $Y \to X$ in add M, where X is bad indecomposable and Y is good indecomposable. Although we must imply a contradiction in any case, by the duality principle again we have only to consider the case of the sink map $f: X \to Y$. Here, it should be noted that the statement "End M is serial" is self-dual. Now let P be an indecomposable module in add M such that there is a proper epimorphism $u: P_0 \to X$ for some non-zero submodule $P_0 \subsetneqq P$. Let $f': X \to f(X)$ be a canonical map induced from f, and $i: f(X) \to Y$ and $j: P_0 \to P$ the inclusion maps. Let $\bar{j}: P_0/\operatorname{Ker} f'u \to P/\operatorname{Ker} f'u$ and $\overline{f'u}: P_0/\operatorname{Ker} f'u \simeq f(X)$ be canonical maps induced from j and $f'u$, respectively.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_0/\operatorname{Ker} f'u & \xrightarrow{\ \bar{j}\ } & P/\operatorname{Ker} f'u & \longrightarrow & P/P_0 & \longrightarrow & 0 \\
 & & {\scriptstyle \overline{f'u}}\downarrow\wr & & & & & & \\
0 & \longrightarrow & f(X) & \xrightarrow[\ i\ ]{} & Y & \longrightarrow & Y/f(X) & \longrightarrow & 0
\end{array}$$

The following two cases should happen: (i) $|Y/f(X)| < |P/P_0|$ or (ii) $|P/P_0| \leqq |Y/f(X)|$. In the case of (i), as Λ is serial Artinian, there is a monomorphism $s: Y \to P/\operatorname{Ker} f'u$ such that $si = \bar{j} \cdot \overline{f'u}^{-1}$. Since $\operatorname{Ker} f'u \neq 0$ and Y is good, s is an epimorphism and so an isomorphism. Hence, $|Y/f(X)| = |P/P_0|$, which is a contradiction. Finally, in the case of (ii), we have a morphism $\bar{h}$: $P/\text{Ker } f'u \to Y$ such that $\bar{h}j = i \cdot \overline{f'u}$. Hence, $hj = fu$ for the composition h: $P \to^{\text{can.}} P/\text{Ker } f'u \to^{\bar{h}} Y$. On the other hand, since f is a sink map, there is a morphism g: $P \to X$ such that $h = fg$. Therefore, we have that $h(P) = fg(P) \subset f(X) = fu(P_0) = h(P_0)$, so that $P = P_0 + \text{Ker } h = P_0$. This is also a contradiction. Thus we complete the proof.

COROLLARY 1. *Let Λ be a serial Artinian ring and G a generator. Then, the endomorphism ring of G is serial Artinian if and only if G belongs to* $\text{add}(\Lambda \oplus I \oplus I/\text{soc } I)$, *where the Λ-module I is the injective hull of the top of Λ. In particular, the endomorphism ring of a minimal fully faithful Λ-module is also serial Artinian (cf. Ringel and Tachikawa* [6, *Lemma* 5.6]).

Proof. A minimal fully faithful module G is by definition a basic module (i.e., any two different summands are not isomorphic) such that $G \in \text{add}(\Lambda \oplus I)$, $\Lambda \oplus I \in \text{add}(G)$. Hence, taking account of the fact that $\Lambda \in \text{add } G$, the corollary follows from Theorem 1 and Lemma 1.

In the case of serial Artinian rings, the theorem mentioned in the introduction is now an immediate consequence of Corollary 1. For, let G be a generator in $\text{add}(\Lambda \oplus I \oplus I/\text{soc } I)$ and e the projection from G to a summand of G being a projective generator. Then, since Λ and $e(\text{End } G)e$ are Morita equivalent, Λ is serial Artinian if so is End G.

COROLLARY 2. *For a serial self-injective Artinian ring Λ, the endomorphism ring of the right Λ-module $\Lambda \oplus \Lambda/\text{soc } \Lambda$ is also serial Artinian.*

2. SERIAL PRIME NOETHERIAN RINGS

The aim of this section is to prove the theorem in the introduction for serial prime Noetherian rings. But, until the theorem we shall proceed with our argument without any chain condition for rings.

LEMMA 2. *Let Λ be a right serial ring such that $\bigcap_{n \geq 0} \text{rad}^n \Lambda = 0$. Then every proper factor module of an indecomposable projective right Λ-module P has the finite composition length.*

Proof. It should be remembered that Λ is semi-perfect and so P is a summand of Λ. Now let I be a non-zero submodule of P. By assumption, there is a number n such that $\text{rad}^n P \subset I$, since P is serial. Then we have a composition series $P/\text{rad}^n P \supsetneq \text{rad } P/\text{rad}^n P \supsetneq \cdots \supsetneq \text{rad}^{n-1} P/\text{rad}^n P \supsetneq 0$. Since P/I is a factor of $P/\text{rad}^n P$, this proves the finiteness of the length of P/I.

LEMMA 3. *Let G_A be a generator. If the endomorphism ring of G is right serial, then A is right serial and every indecomposable summand of G is serial.*

Proof. Since G is a generator, there is a number m such that $G^{(m)} = A_A \oplus G'_A$ for a A-module G'. For the projection $e: G^{(m)} \to A$, $\operatorname{End}(G^{(m)})$ and $e \operatorname{End}(G^{(m)})e$ are right serial, and $A \simeq e \operatorname{End}(G^{(m)})e$. Thus, to show that every indecomposable summand of G is serial, we can assume that A_A is a direct summand of G, say $G = A \oplus G'$. Now, take an indecomposable summand X of G, and let $e_X: G \to X$ be the projection. Let $1 = \sum_{i=1}^{m} e_i$ be the sum of orthogonal primitive idempotents of A. Then, for any $x \neq 0$ and $y \neq 0$ belonging to X, it suffices to show that $x \in yA$ or $y \in xA$. For this, moreover, we can assume that $x = xe_i$ and $y = ye_j$ for some e_i and e_j. Let $f_i: e_iA \to X$ and $f_j: e_jA \to X$ be the morphisms such that $f_i(e_i) = x$ and $f_j(e_j) = y$, and let $\Gamma = \operatorname{End} G$. Then, $f_i = e_X f_i e_i$ and $f_j = e_X f_j e_j$ as elements of $e_X \Gamma$. From the assumption that $e_X \Gamma$ is serial, we have that $f_i \in f_j \Gamma$ or $f_j \in f_i \Gamma$. Hence we can assume that there is an element $\gamma = e_i \gamma e_j \in \Gamma$ such that $f_j = f_i \gamma$. By regarding the morphism $\gamma: e_j\Gamma \to e_i\Gamma$ as an element of e_iAe_j, we therefore have that $y = f_j(e_j) = (f_i\gamma)(e_j) = f_i(\gamma e_j) = f_i(e_i\gamma) = f_i(e_i)\gamma = x\gamma$, so that $y \in xA$ as desired.

LEMMA 4. *Let G_A be a generator and $\Gamma = \operatorname{End} G$. Assume that Γ is right serial such that $\bigcap_{n \geq 0} \operatorname{rad}^n \Gamma = 0$. Then G is a direct sum of a finitely generated projective generator and a module with finite length.*

Proof. Since Γ is semi-perfect, G is a direct sum of indecomposable modules with local endomorphism ring.

(a) First we show that every indecomposable summand X of G is finitely generated. For this, suppose that an indecomposable summand X of G is not finitely generated. Since A is semi-perfect by Lemma 3, there is a non-zero morphism $f_0: P_0 \to X$, where P_0 is indecomposable projective. Then $X_0 := f_0(P_0) \subsetneqq X$, because X is not finitely generated. Clearly X/X_0 is not finitely generated, and we can again take a non-zero morphism $f_1: P_1 \to X$ with indecomposable projective module P_1 such that $0 \neq (f_1(P_1) + X_0)/X_0 \subsetneqq X/X_0$. Hence, by Lemma 3, $X_0 \subsetneqq f_1(P_1) \subsetneqq X$. Let $X_1 = f_1(P_1)$ and $f_0: P_0 \to^{f'_0} X_0 \subsetneqq^{i} X_1 \subsetneqq^{j} X$. By the projectivity of P_0, there is a morphism $g_1: P_0 \to P_1$ such that $f'_1 g_1 = if'_0$, where $f_1: P_1 \to^{f'_1} X_1 \subsetneqq^{j} X$. Hence $f_0 = jif'_0 = jf'_1 g_1 = f_1 g_1$. Since f_1 and g_1 are not isomorphisms and since P_0 and P_1 are indecomposable projective, it follows that both f_1 and g_1 belong to $\operatorname{rad} \Gamma$, so that $f_0 \in \operatorname{rad}^2 \Gamma$. Similarly we know that $f_1 \in \operatorname{rad}^2 \Gamma$ and $f_1 = f_2 g_2$ for some f_2, g_2 in $\operatorname{rad} \Gamma$. Repeating this method, we should finally have that $f_0 \in \bigcap_{n \geq 0} \operatorname{rad}^n \Gamma$, a contradiction to the assumption for Γ.

(b) Let X be a non-projective indecomposable summand of G. Then X is finitely generated from (a), and serial from Lemma 3. Hence there is an epimorphism $f: P \to X$ with some indecomposable projective P. Since X is non-projective, $\operatorname{Ker} f$ is not zero, and so, by Lemma 2, $P/\operatorname{Ker} f$ has the finite length. Thus we know that the length of X is finite.

PROPOSITION 1. *Let G_Λ be a generator and Γ the endomorphism ring of G. Then the following statements are equivalent.*

(1) *Γ is right serial,* $\operatorname{soc}(\Gamma_\Gamma) = 0$, *and* $\bigcap_{n \geq 0} \operatorname{rad}^n \Gamma = 0$.

(2) (i) *Λ is right serial,* $\operatorname{soc}(\Lambda_\Lambda) = 0$, *and* $\bigcap_{n \geq 0} \operatorname{rad}^n \Lambda = 0$,

(ii) *G is finitely generated projective.*

Proof. The properties in (1) are invariant under the Morita equivalence. Hence the implication $(2) \Rightarrow (1)$ follows immediately, because Λ and Γ are Morita equivalent by (ii). By the same reason, to show the converse, it suffices to show that G is finitely generated projective. For this, we have only to show that $\operatorname{soc} G_\Lambda = 0$, taking account of Lemma 4. Now, on the contrary, let X be an indecomposable summand of G with $\operatorname{soc} X \neq 0$. Since Λ is semi-perfect and X has the simple socle from Lemma 3, there is an indecomposable projective P and a non-zero morphism from P to X factored through $\operatorname{soc}(X)$, say $f: P \to \operatorname{soc} X \subsetneq X$. Then $\operatorname{Ker} f = \operatorname{rad} P$ which is unique maximal in P. Thus we have that $f(\operatorname{rad} \Gamma) = 0$ in Γ, which implies that $f \in \operatorname{soc} \Gamma_\Gamma$. This contradicts the assumption that $\operatorname{soc} \Gamma_\Gamma = 0$.

THEOREM 2. *Let G_Λ be a generator. Then, the endomorphism ring of G is serial prime Noetherian if and only if Λ is serial prime Noetherian and G is finitely generated projective.*

Proof. If G is finitely generated projective, then Λ and Γ $(:= \operatorname{End} G)$ are Morita equivalent. Hence the "if" part is clear. Conversely, we assume that Γ is serial prime Noetherian. Then $\operatorname{soc} \Gamma_\Gamma = 0$ and $\bigcap_{n \geq 0} \operatorname{rad}^n \Gamma = 0$ by [7, Theorem 5.11]. It therefore follows from Proposition 1 that G is finitely generated projective, so that Λ and Γ are Morita equivalent. Thus Λ inherits the desired properties from Γ.

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