European Journal of Combinatorics 31 (2010) 106–119

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Dualities in full homomorphisms

Richard N. Ball [a], Jaroslav Nešetřil [b,c], Aleš Pultr [b,c]

[a] *Department of Mathematics, University of Denver, Denver, CO 80208, USA*

[b] *Department of Applied Mathematics, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25, Czech Republic*

[c] *ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25, Czech Republic*

ARTICLE INFO

Article history:
Received 21 November 2008
Accepted 10 April 2009
Available online 21 June 2009

ABSTRACT

In this paper we study dualities of graphs and, more generally, relational structures with respect to full homomorphisms, that is, mappings that are both edge- and non-edge-preserving. The research was motivated, a.o., by results from logic (concerning first order definability) and Constraint Satisfaction Problems. We prove that for any finite set of objects $\mathcal{B}$ (finite relational structures) there is a finite duality with $\mathcal{B}$ to the left. It appears that the surprising richness of these dualities leads to interesting problems of Ramsey type; this is what we explicitly analyze in the simplest case of graphs.

0. Introduction

We will illustrate the motivation and the type of results to be presented on the simple example of finite binary relations (which may be thought of as directed graphs). Given such relations $G = (X, R)$ and $G' = (X', R')$ a mapping $f : X \to X'$ is said to be a *homomorphism* $G \to G'$ if

$$(x, y) \in R \Rightarrow (f(x), f(y)) \in R'.$$

Homomorphisms capture many combinatorial properties of graphs and relations, see [8]. Thus for instance the k-colorability of a graph can be reformulated as the existence of a homomorphism $G \to K_n$ where K_n is the complete symmetric graph without loops.

The classes

$$\{G \mid \text{there is a homomorphism } f : G \to B\} \qquad (*)$$

E-mail address: nesetril@kam.mff.cuni.cz (R.N. Ball).

0195-6698/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2009.04.004

with a fixed B are of particular interest (one often speaks of B-colorings). More generally, one can consider n-ary relations R, R' and the homomorphisms satisfying

$$(x_1, \ldots, x_n) \in R \Rightarrow (f(x_1), \ldots f(x_n)) \in R' \qquad (**)$$

or the relational structures $(R_t)_{t\in T}$, $(R'_t)_{t\in T}$ (see 1.2 below), and the homomorphisms f satisfying $(**)$ for the relations R_t, R'_t, $t \in T$. The set $(*)$ represents the Constraint Satisfaction Problem (briefly, CSP — see, e.g. [7,9] and literature quoted there). This is why we use the notation

$$\mathbf{CSP}(B) \quad \text{for } \{G \mid \text{there is a homomorphism } f : G \to B\}.$$

The class $\mathbf{CSP}(B)$ can be represented in a complementary way by *forbidding* homomorphisms, namely as

$$\mathbf{Forb}(\mathcal{A}) = \{G \mid \text{there is no homomorphism } f : A \to G \text{ with } A \in \mathcal{A}\}$$

(it suffices to take $\mathcal{A} = \{A \mid \text{there is no homomorphism } f : A \to B\}$). We are interested in the cases where such a $\mathcal{A}$ can be chosen *finite*.

If we have such a finite $\mathcal{A}$ we speak of a *finite duality* (first defined in [15])

$$\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B).$$

Finite dualities in binary relations (graphs) and standard homomorphisms do not always exist (consider for example the class of all 3-colorable graphs: the set of minimal forbidden relations is necessarily infinite and coincides with that of the so called 4-critical graphs).

They have been recently studied from the logical point of view, and also in the optimization (mostly CSP) context. The following has been proved (as a combination of results of [2,16]):

Theorem. *Let B be a finite binary relation. Then the following statements are equivalent.*

(i) *The class $\mathbf{CSP}(B)$ is first order definable;*
(ii) *B has finite duality; explicitly, there exists a finite set $\mathcal{A}$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$;*
(iii) *$\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$ for a finite set $\mathcal{A}$ of finite oriented trees.*

Similar theorems hold more generally. Finite dualities for finite relational structures are well characterized, and it can be shown that they are abundant (see e.g. [5,11,16,17], or [10]).

The constraints can be sometimes determined by another choice of special maps like for instance the *full homomorphisms* satisfying

$$(x_1, \ldots, x_n) \in R \Leftrightarrow (f(x_1), \ldots, f(x_n)) \in R'$$

(these are what we will be concerned with in this paper; it should be noted that the CSP for full homomorphisms of binary relations has been discussed in the already mentioned [7], and in [4]).

Let us note that the problem of finite dualities of algebras and their homomorphisms has a negative answer: there is no non-trivial one [12]. Therefore it may come as a surprise that for the full homomorphisms of relational systems, on the contrary, the answer is always positive (although homomorphisms of algebras seem to be structurally similar: for instance if f is a one–one mapping, then the requirement to be a homomorphism is equivalent to

$$x = \alpha_i(x_1, \ldots, x_{n_i}) \Leftrightarrow f(x) = \alpha'_i(f(x_1), \ldots, f(x_{n_i})).$$

That is, we will prove (Theorem 3.3) that for every finite system of relational objects (sets with relational structures of a finite type) $\mathcal{B}$ there is a finite system $\mathcal{A}$ of relational objects such that

$$\mathbf{Forb}_{\mathsf{full}}(\mathcal{A}) = \mathbf{CSP}_{\mathsf{full}}(\mathcal{B})$$

(where the subscript "full" emphasizes that we are concerned with full homomorphisms).

For binary relations such a result appeared already in [4] and [7] (in [7] one forbids subgraphs instead of homomorphisms, but these two types of prohibition are closely related).

The paper is organized as follows. In Section 1 we review the basic definitions. We treat the problems in a fairly general categorical setting; this also explains our detailed exposition in this introduction. In Section 2 we consider the dualities still in the abstract way, and in Section 3 we prove our main result (Theorem 3.3). In Sections 4 and 5 we deal with the binary relations and then with the even more special classes of undirected graphs; in particular we have here a procedure that produces (albeit not very effectively) finite "left-hand sides" to the **CSP** (B)'s.

1. Preliminaries

1.1

We will be concerned with very special categories of a combinatorial nature. In particular, we will typically assume the following properties.

(bi-LocFin) The category is *bi-locally finite*, that is, for any object A there are (up to isomorphism) only finitely many monomorphisms $B \to A$ and only finitely many epimorphisms $A \to B$.

(wFac) The category has a *weak (epi-mono) factorization*, that is, every morphism f can be written as $f = m \cdot e$ with m a monomorphism and e an epimorphism.

(Ch) The category has *choice*, that is, every epimorphism is a retraction.

Monomorphisms $B \to A$, or just the B in such monomorphisms, will be sometimes referred as *subobjects* of A.

Only basic facts and notions from category theory (monomorphisms, epimorphisms, retractions and coretractions, products) are assumed; see, for instance, the opening chapters of [13].

1.2

An *n-ary relation* on a set X is a subset $R \subseteq X^n$, and a mapping $f : X \to Y$ is a *homomorphism* with respect to R, S if

$$(x_1, \ldots, x_n) \in R \quad \Rightarrow \quad (f(x_1), \ldots, f(x_n)) \in S.$$

The mappings with the (much) stronger property

$$(x_1, \ldots, x_n) \in R \quad \Leftrightarrow \quad (f(x_1), \ldots, f(x_n)) \in S$$

will be called *full homomorphisms*.

A (finite) *type* is a finite collection $\Delta = (n_t)_{t\in T}$ of natural numbers, and a *relational structure* of type Δ on X is a collection $R = (R_t)_{t\in T}$ where the R_t are n_t-ary relation on X; (X, R) is then referred to as a *relational object*. A *(full) homomorphism* $f : (X, R = (R_t)_{t\in T}) \to (Y, S = (S_t)_{t\in T})$ is a mapping that is a (full) homomorphism with respect to R_t, S_t for each $t \in T$.

The category of all relational objects of type Δ and full homomorphisms will be denoted by

$$\mathbf{Rel}_{\mathsf{full}}(\Delta).$$

The category of undirected graphs (resp. connected undirected graphs) with full homomorphisms will be viewed as a full subcategory of $\mathbf{Rel}_{\mathsf{full}}((2))$; that is, the set of edges is represented as a symmetric antireflexive binary relation. It will be denoted by

$$\mathbf{Graph}_{\mathsf{full}} \quad \text{resp. } \mathbf{ConnGraph}_{\mathsf{full}}.$$

Note that the mentioned categories satisfy all the properties from 1.1.

1.3

With a category $\mathcal{C}$ we will associate the preordered class $\widetilde{\mathcal{C}} = (\widetilde{C}, \to)$ of the objects from $\mathcal{C}$ with the preorder

$$A \to B \quad \equiv_{\mathrm{df}} \quad \exists f : A \to B \text{ in } \mathcal{C}.$$

Thus, for a set $\mathcal{A}$ of objects of $\mathcal{C}$ we have the increasing and decreasing sets

$$\uparrow \mathcal{A} \equiv_{\mathrm{df}} \{C \in \mathcal{C} \mid \exists A \in \mathcal{A} \quad A \to C\}, \qquad \downarrow \mathcal{A} \equiv_{\mathrm{df}} \{C \in \mathcal{C} \mid \exists A \in \mathcal{A} \quad C \to A\}.$$

We will write

$$A \sim B \quad \text{if } A \to B \text{ and } B \to A$$

and speak of $\sim$-*equivalence classes* or simply of *equivalence classes*.

The fact that there is no $f : A \to B$ will be indicated by

$$A \nrightarrow B.$$

1.4

An object A of a category $\mathcal{C}$ is said to be a *core* if each $f : A \to A$ is an isomorphism.

Lemma. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac), and (Ch). Then*
1. *the sets $\mathcal{C}(A, B)$ of morphisms $A \to B$ are (up to isomorphism) finite, and*
2. *an object A in $\mathcal{C}$ is a core iff there is no proper (that is, non-isomorphic) retraction out of A.*

Proof. 1 is trivial.

2: If A is a core and $r : A \to B$ is a retraction, with $r \cdot m : B \to B$ identical, then we have that $m \cdot r : A \to A$ is an isomorphism and hence also r.

Now suppose that $f : A \to A$ is not an isomorphism. If the e in the decomposition $f = me$ (m monic and e epic) is not an isomorphism then we have found a proper retraction with source A. So suppose that e is an isomorphism, so that f is a monomorphism. By 1 there are integers $n, k > 0$ such that f^{n+k} is equivalent to f^n, say $f^n h = f^{n+k}$ for an isomorphism h. Since f^n is a monomorphism, $f^k = 1$. But then f is both the left factor of an epimorphism and the right factor of a monomorphism, and hence it is itself both. And in a category with (Ch), this implies that f is an isomorphism. □

Proposition 1.5. *If a category $\mathcal{C}$ satisfies (bi-LocFin), (wFac), and (Ch) then each $\sim$-equivalence class contains (up to isomorphism) exactly one core.*

Proof. If two cores A and B are equivalent then they are, trivially, isomorphic.

Now let A be any object. Consider the class $\mathcal{M}$ of all the coretractions $m : A_m \to A$ and (pre)order it by $m \prec n$ iff there is an f such that $m = nf$. By (bi-LocFin), $\mathcal{M}$ is, up to isomorphism, finite and hence there is an $m \in \mathcal{M}$ minimal in $\prec$. Then A_m cannot admit a proper retraction $A_m \to B$, for such a B would be smaller in $\prec$, and hence it is a core by 1.4. □

Proposition 1.6. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac) and (Ch). Then*
1. *if A is a core then every $A \to B$ is a monomorphism, and*
2. *for every A and every property $\mathcal{P}$ satisfied by A there exists an $A_0 \to A$ minimal in $\to$ such that it still satisfies $\mathcal{P}$.*

Proof. 1. Set, by (wFac),

$$f = (A \xrightarrow{e} C \xrightarrow{m} B).$$

By (Ch) e is a retraction an by 1.4 it is an isomorphism.

2. By 1 and (bi-LocFin) we have, in $\to$, under each object only finitely many $\sim$-classes. Hence we have minimal objects with any property $\mathcal{Q}(B)$ that is satisfied by some object (here: $\mathcal{Q}(B) \equiv$"$B \to A$ and $\mathcal{P}(B)$"). □

Remark 1.6.1. Note that in the categories from 1.2, monomorphisms are precisely the embeddings of induced objects. Thus, searching for objects smaller then a given one can be restricted to its subobjects.

2. Dualities and Ramsey lists

In this section, $\mathcal{C}$ is a fixed category. Starting with 2.3.1 we will require some special properties; this will be then explicitly stated.

2.1

Let $\mathcal{A}$ be a subclass of the class of objects of $\mathcal{C}$. Write

$$X \to \mathcal{A} \quad \text{for } \exists A \in \mathcal{A}, X \to A,$$
$$\mathcal{A} \to X \quad \text{for } \exists A \in \mathcal{A}, A \to X,$$
$$X \nrightarrow \mathcal{A} \quad \text{for } \forall A \in \mathcal{A}, X \nrightarrow A,$$
$$\mathcal{A} \nrightarrow X \quad \text{for } \forall A \in \mathcal{A}, A \nrightarrow X.$$

Set

$$\mathbf{Forb}(\mathcal{A}) = \{X \mid \mathcal{A} \nrightarrow X\}, \qquad \mathbf{CSP}(\mathcal{B}) = \{X \mid X \to \mathcal{B}\} \quad \text{and} \quad \mathcal{N}(\mathcal{A}) = \{X \mid X \nrightarrow \mathcal{A}\}.$$

A *finite duality* in $\mathcal{C}$ is a couple $\mathcal{A}, \mathcal{B}$ of *finite* subsets of objects of $\mathcal{C}$ such that

$$\mathcal{A} \nrightarrow X \quad \text{iff } X \to \mathcal{B}, \qquad \text{that is, } \mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B}).$$

Proposition 2.2. *We have*

$$\mathcal{N}(\mathcal{B}) \nrightarrow X \quad \textit{iff } X \to \mathcal{B}$$

and

$$\mathcal{A} \nrightarrow X \quad \textit{iff } X \to \mathbf{Forb}(\mathcal{A}).$$

In other words,

$$\mathbf{Forb}(\mathcal{N}(\mathcal{B})) = \mathbf{CSP}(\mathcal{B}) \quad \textit{and} \quad \mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathbf{Forb}(\mathcal{A})).$$

Proof. The desired condition $\mathcal{A} \nrightarrow X$ iff $X \to \mathcal{B}$ coincides in the general setting of a preordered class $(P, \leq)$ with the equality

$$P \setminus \uparrow \mathcal{A} = \downarrow \mathcal{B}.$$

Now we have $\mathbf{Forb}(\mathcal{A}) = P \setminus \mathcal{A}$ and $\mathcal{N}(\mathcal{B}) = P \setminus \downarrow \mathcal{B}$. Thus $P \setminus (\uparrow \mathcal{N}(\mathcal{B})) = P \setminus (P \setminus \downarrow \mathcal{B}) = \downarrow \mathcal{B}$, and $P \setminus \uparrow \mathcal{A} = \downarrow (P \setminus \uparrow \mathcal{A}) = \downarrow \mathbf{Forb}(\mathcal{A})$. □

2.3

An object A will be called *critical* with respect to a class of objects $\mathcal{B}$ if

- it is a core,
- $A \nrightarrow \mathcal{B}$, and
- if $A' \to A \nrightarrow A'$ then $A' \to \mathcal{B}$.

Thus, since we can restrict ourselves to cores, by 1.6 the third condition amounts to requiring that for every proper subobject A' of A there is a $B \in \mathcal{B}$ with $A' \to B$.

Set

$$\mathcal{N}_0(\mathcal{B}) = \{X \in \mathcal{N}(\mathcal{B}) \mid X \text{ critical w.r.t. } \mathcal{B}\}.$$

We have

Proposition 2.3.1. *If $\mathcal{C}$ is a category satisfying (bi-LocFin), (wFa) and (Ch), then*

$$\mathcal{N}_0(\mathcal{B}) \nrightarrow X \quad \textit{iff } X \to \mathcal{B}.$$

Proof. Use 2.2 and 1.6.1: there is an $A \in \mathcal{N}(\mathcal{B})$ with $A \to X$ iff there is such an A in $\mathcal{N}_0(B)$. □

2.4

The Propositions in 2.2 and 2.3.1 are not necessarily finite dualities, since neither $\mathbf{Forb}(\mathcal{A})$ nor $\mathcal{N}(\mathcal{A})$ nor $\mathcal{N}_0(A)$ is necessarily finite just because $\mathcal{A}$ is finite. However, we will see that in the categories we are interested in, a finite $\mathcal{B}$ can always be extended to a finite duality $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$. This leads to the following definition.

2.5

A collection of cores $\mathcal{A} = \{A_1, \ldots, A_n\}$ is said to be a *Ramsey list*, or, briefly, to be *Ramsey*, if there is a finite set $\mathcal{F}$ of objects of $\mathcal{C}$ such that for each core X that is not isomorphic with an object from $\mathcal{F}$, some of the A_i is isomorphic to a subobject of X. (The reader can consult [14,6] for general background of Ramsey theory.)

Proposition 2.5.1. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac) and (Ch). Then a finite $\mathcal{A}$ is Ramsey iff there is a finite duality*

$$\mathcal{A} \nrightarrow X \quad \text{iff } X \to \mathcal{B}.$$

Proof. If there is such a duality then it suffices to take for $\mathcal{F}$ the set of all subobjects of the elements of $\mathcal{B}$.

On the other hand, if $\mathcal{A}$ is Ramsey then $\mathcal{A} \nrightarrow X$ iff

$$X \to \mathbf{Forb}(\mathcal{A}) = X \to \{X \mid \mathcal{A} \nrightarrow X\} = X \to \{X \mid \mathcal{A} \nrightarrow X \text{ and } X \in \mathcal{F}\}. \quad \square$$

Note. Recall that a down-set in a preordered set $(X, \leq)$ is an $M \subseteq X$ such that $x \leq y \in M$ implies $x \in M$. Proposition 2.5.1 can be reformulated by stating that $\mathcal{A}$ is Ramsey iff the downset $\mathbf{Forb}(A)$ in $(C, \to)$ is finitely generated.

3. The category of relational systems

3.1. Convention

In this section we will deal with the finite dualities in $\mathbf{Rel}_{\text{full}}(\Delta)$. Just to avoid too many indices we will present the proof in 4.3 as if for one n-ary relation. If one reads n_t for n and R_t for every relation constituting the relational system, and if one does everything simultaneously, one obtains correctly the general result.

3.2

If $B = (X, R)$ is an object of $\mathbf{Rel}_{\text{full}}(\Delta)$ write $X = X_B$, $R = R_B$.

Proposition. *Let $\mathcal{B}$ be a finite set of objects of $\mathbf{Rel}_{\text{full}}(\Delta)$. Let $\Delta = (n_t)_{t\in T}$ and let $m > \max_t n_t$. Then, with possibly finitely many exceptions, every A critical with respect to $\mathcal{B}$ can be embedded into an object of $\mathbf{Rel}_{\text{full}}(\Delta)$ carried by X^m where*

$$X = X_B \cup \{\omega\}$$

for some $B \in \mathcal{B}$ and $\omega \notin X_B$.

Proof. Consider an A critical with respect to $\mathcal{B}$. For every $a \in A$ there is a $B_a \in \mathcal{B}$ such that $A\backslash\{a\} \to B_a$. If A is sufficiently large, there are distinct $a_1, \dots, a_m$ such that the B_{a_i} coincide. Denote $B = B_{a_i}$ the common value.

Since A is a core, it suffices to find a full homomorphism from A into an object as stated.

Recall the convention 3.1. For every $i = 1, \dots, m$ there is a full homomorphism

$$f_i : A \setminus \{a_i\} \to B.$$

Set

$$X_{B_i^+} = X \ (= X_B \cup \{\omega\}) \quad \text{and} \quad X_{A_i^+} = X_A$$

and define

$$f_i^+ : X_{A_i^+} \to X_{B_i^+}$$

by setting $f_i^+(x) = f_i(x)$ if $x \neq a_i$, and $f_i^+(a_i) = \omega$.

Now put

$$(y_1, \dots, y_n) \in R_{B_i^+} \text{ iff either } (y_1, \dots, y_n) \in R_S \text{ or at least one of the } y_j\text{'s is } \omega.$$

Further define the relation for A_i^+ by

$$(x_1, \dots, x_n) \in R_{A_i^+} \quad \text{iff } (f_i^+(x_1), \dots, f_i^+(x_n)) \in R_{B_i^+},$$

thus making each

$$f_i^+ : A_i^+ \to B_i^+$$

a full homomorphism. Furthermore, it is obvious that the maps

$$\widetilde{f_i} : A \to B_i^+$$

defined by the same formula are homomorphisms, albeit not full, and hence we have a homomorphism

$$f : A \to \prod_{i=1}^{m} B_i^+$$

defined by requiring $p_i \cdot f = \widetilde{f_i}$ for the natural projections.

Now this f is full. Indeed, let $(f(x_1), \dots, f(x_n))$ be in the relation of the product. Then for every i,

$$(f_i^+(x_1), \dots, f_i^+(x_n)) = (p_i f(x_1), \dots, p_i f(x_n)) \in R_{B_i^+}.$$

Since $m > n$ there exists an i such that none of the x_j's is a_i, hence

$$(f_i^+(x_1), \dots, f_i^+(x_n)) = (f_i(x_1), \dots, f_i(x_n)) \in R_B.$$

Since f_i is full, the statement follows. □

3.3

Thus, $\mathcal{N}_0(\mathcal{B})$ is finite and we obtain as an immediate consequence

Theorem. *In* $\mathbf{Rel}_{\mathrm{full}}(\Delta)$ *there exists for every finite set of objects* $\mathcal{B}$ *a finite system of objects* $\mathcal{A}$ *and a finite duality*

$$\mathcal{A} \nrightarrow X \quad \textit{iff } X \to \mathcal{B}.$$

3.4

Let us briefly discuss the *inverse problem*: given a finite $\mathcal{A}$, does there exists a finite $\mathcal{B}$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$? The answer is in general negative. For instance, in connected graphs there are only four such $\mathcal{A}$ containing less then three objects — see 5.3.1 and 5.4.

Nevertheless, we can isolate a necessary condition. The key to this is a definition of an "unavoidable" set of "complete systems".

Let $(X, <)$ be a linearly ordered set. Let $(a_1, \dots, a_k)$, $(b_1, \dots, b_k)$ be two k-tuples of elements of X. We say that these tuples are *equivalent* if there exists a monotone (with respect to $<$) mapping $\iota : \{a_1, \dots, a_k\} \longrightarrow \{b_1, \dots, b_k\}$ such that $\iota(a_i) = b_i$ for every $i = 1, \dots, k$. This equivalence will be denoted by $\sim$. The equivalence classes of $\sim$ are called *types* (of the arity k). A type σ' is the mirror image of σ if σ' corresponds to the tuple $(a_k, \dots, a_1)$.

Let Σ be a set of order types (a *type-set*). By K_n^{Σ} we denote the following relational object (X, R): $X = \{1, \dots, n\}$ and the relation structure consists of all tuples of X with a type $\sigma \in \Sigma$ (with respect to natural ordering of X). K_n^{Σ} is called a *complete* object (with type set Σ).

The type-set Σ and the complete object K_n^{Σ} are said to be *trivial* if (for every n) the object K_n^{Σ} is full homomorphism equivalent to the singleton complete object K_1^{Σ}. Note that there are many trivial type-sets ($2^{|T|}$ in $\mathbf{Rel}_{\mathrm{full}}(\Delta)$ with $|\Delta| = |T|$).

Lemma. *Let* Σ, Σ' *be sets of types. Then* $K_m^{\Sigma} \longrightarrow K_n^{\Sigma'}$ *iff one of the following possibilities occur:*

(i) $\Sigma = \Sigma'$ *is a trivial type-set;*
(ii) $m \le n$ *and either* $\Sigma = \Sigma'$ *or* Σ' *is the mirror image of* Σ.

Proof. This follows by observing that from any non-trivial type-set Σ we can reconstruct the ordering of X (for any complete object K_X^{Σ} on X). □

Finally, we say that a set Ξ of type sets is a *majorizing set* (in $\mathbf{Rel}_{\text{full}}(\Delta)$) if for every non-trivial type-set Σ (of relations in $\mathbf{Rel}_{\text{full}}(\Delta)$) there exists a set $\Sigma' \in \Xi$ such that either $\Sigma = \Sigma'$ or Σ' is the mirror image of Σ.

We have the following

Proposition. *For a finite set $\mathcal{A}$ of objects of $\mathbf{Rel}_{\text{full}}(\Delta)$ the following holds:*

(i) *If there exists $B \in \mathbf{Rel}_{\text{full}}(\Delta)$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$; then $\mathcal{A}$ contains a set of non-trivial complete objects with majorizing set of set-types.*

(ii) *For every set $\mathcal{A}$ with majorizing order types there exists a finite set $\mathcal{A}'$ of non-trivial objects such that $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}') = \mathbf{CSP}(B)$ for some B.*

Proof. (i): Suppose to the contrary. This equivalently means that there exists a type set Σ distinct from all the non-trivial set-types of all complete (arbitrarily ordered) objects in $\mathcal{A}$. As any subobject of any complete object K_n^{Σ} is again a complete object with the same set-type we obtain that, using preceding lemma, that there is no finite duality with $\mathcal{B}$.

(ii): Let Ξ be a majorizing set of set-types. Let n be the maximal order (universum size) of an object in $\mathcal{A}$. Assume $\mathbf{Forb}(\mathcal{A})$ is non-empty and let $B \in \mathbf{Forb}(\mathcal{A})$. Put $\mathcal{A}' = \mathcal{N}_0(\mathcal{B})$. $\mathcal{A}'$ is a finite set by 3.3 and clearly $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}') = \mathbf{CSP}(B)$. □

Remark. We may choose B as the disjoint union of nontrivial complete objects K_{n-1}^{Σ} for $K_n^{\Sigma} \in \mathcal{A}$ together with the trivial forbidden objects in $\mathcal{A}$. Then the complete objects in $\mathbf{Forb}(\mathcal{A})$ and $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}')$ coincide. The structure of the non-complete Ramsey lists is more complex and it will be investigated in the next sections.

On the other side, by iterating Ramsey's theorem we see easily that every large object of $\mathbf{Rel}_{\text{full}}(\Delta)$ contains a large complete subsystem. The condition (i) of the Lemma is responsible for the difficulty in characterizing Ramsey lists. Let us finally remark that the properties of classes $\mathbf{Forb}(\mathcal{A})$ are closely related to the intensively studied Ramsey-type problems, particularly to Erdős–Hajnal problem; see [1].

4. One binary relation

The proof of Proposition 3.2 presents a finite system of objects containing the desirable $\mathcal{N}_0(\mathcal{B})$. It is, however, very large; listing the actual $\mathcal{N}_0(\mathcal{B})$ seems to be in general very hard.

In this section we will consider the simple (but important) case of one binary relation. Here, the listing is more feasible. In the next paragraph we will then discuss Ramsey lists in classical graphs and provide several concrete examples.

It should be noted that the case of symmetric graphs (not necessarily connected) has been studied in [4]. Among other results the authors have proved that $\mathbf{Forb}(B)$ can consist of graphs A with $|A| \leq |B| + 1$ which (together with other data) is a good start. Allowing the disconnectedness is essential, though: see 5.11 below.

4.1

We will write $\mathbf{Rel}_{\text{full}}$ for $\mathbf{Rel}_{\text{full}}((2))$. The objects of $\mathbf{Rel}_{\text{full}}$ can be interpreted as oriented graphs with possible loops.

4.2. The object B+

Let B be an object of $\mathbf{Rel}_{\text{full}}$. Choose two distinct elements $\omega, \omega' \notin X_B \times \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ and set

$$X_{B+} = (X_B \times \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}) \cup \{\omega, \omega'\},$$

$$\begin{aligned} R_{B+} = {} & \{(xu, yv) \mid xR_By, u, v \subseteq \{0, 1\}\} \cup \{(\omega', \omega')\} \\ & \cup \{(x\{0\}, \omega), (x\{0\}, \omega') \mid x \in X_B\} \\ & \cup \{(\omega, x\{1\}), (\omega', x\{1\}) \mid x \in X_B\} \\ & \cup \{(x\{0, 1\}, \omega), (\omega, x\{0, 1\}), (x\{0, 1\}, \omega'), (\omega', x\{0, 1\}) \mid x \in X_B\}. \end{aligned}$$

Proposition 4.3. *Let $A \in \mathcal{N}_0(\mathcal{B})$ in $\mathbf{Rel}_{\mathsf{full}}$. Then there is a $B \in \mathcal{B}$ such that $A \to B+$ (and A is isomorphic to a subobject of $B+$).*

Proof. Choose an $a \in X_A$ and consider the object C carried by $X_A \setminus \{a\}$, with the relation inherited from A. Then, as A is a core, C is in $\to$ strictly smaller than A and hence there is a $B \in \mathcal{B}$ and a morphism

$$f : C \to B.$$

Define a mapping

$$g : A \to B+$$

by setting

$$g(a) = \begin{cases} \omega' & \text{if } aR_B a, \\ \omega & \text{otherwise,} \end{cases}$$

and for $x \in C$,

$$g(x) = \begin{cases} f(x)\emptyset & \text{if } x \notin aR_A \cup R_A a, \\ f(x)\{0\} & \text{if } x \in R_A a \setminus aR_A, \\ f(x)\{1\} & \text{if } x \in aR_A \setminus R_A a, \\ f(x)\{01\} & \text{if } x \in R_A a \cap aR_A. \end{cases}$$

Let $xR_A y$. If $x, y \neq a$ then $f(x)R_B f(y)$ and hence $g(x)R_{B+}g(y)$. If $xR_A a$ then $g(x)$ is $f(x)\{0\}$ or $f(x)\{0, 1\}$, in both cases $\cdots R_{B+}\omega = g(a)$ or $\cdots R_{B+}\omega' = g(a)$. Similarly for $aR_A y$.

Now let $g(x)R_{B+}g(y)$. If $g(x), g(y) \neq \omega, \omega'$ then $x, y \neq a$ and $f(x)uR_{B+}f(y)v$, hence $f(x)R_B f(y)$ and finally $xR_A y$. Let $g(x) = \omega$ or $g(x) = \omega'$ (so that $x = a$) and $g(y) \neq \omega, \omega'$. Then $g(y) = zu$ with $0 \in u$, and $xR_A a$. Similarly if $g(x) = \omega$ or $g(y) = \omega'$ and $g(x) \neq \omega, \omega'$. When $g(x) = zu$ with $1 \in u$, and $aR_A x$. The only remaining case is $g(x) = g(y) = \omega'$; then $x = y = a$ and $aR_A a$. □

4.4

The object $B+$ thus constructed can be applied to determining the Ramsey lists of finite $\mathcal{B}$ in categories such as

- $\mathbf{Graph}_{\mathsf{full}}$ of classical graphs, that is, symmetric antireflexive (X, R),
- $\mathbf{ConnGraph}_{\mathsf{full}}$ of connected classical graphs,
- $\mathbf{OrGraph}_{\mathsf{full}}$ of oriented graphs, that is, antisymmetric antireflexive (X, R),
- $\mathbf{Tour}_{\mathsf{full}}$ of tournaments, that is, antisymmetric antireflexive (X, R) in which for any two distinct x, y either xRy or yRx,
- $\mathbf{Poset}_{\mathsf{full}}$ of posets, that is, transitive antisymmetric (X, R),

and their variants with xRx allowed.

In fact, we typically do not even need to search the whole of the $B+$ since (unlike $B+$ itself) the images $g[A]$ stay in the category in question. Thus,

- in the antireflexive cases we can drop the ω',
- in the symmetric case we can make do with $X_B \times \{\emptyset, 2\}$ instead of the whole of $X_B \times \mathfrak{P}(2)$,
- in the antisymmetric cases the $X_B \times \{\emptyset, \{0\}, \{1\}\}$ will do.

The object $B+$ from 4.2 typically does not stay in the category $\mathcal{C}$ in question but this does not impede the validity of the reasoning in 4.4 — with one exception. This concerns $\mathbf{ConnGraph}_{\mathsf{full}}$: while the properties of the whole of $B+$ are not relevant, it is essential that the object $C = A \setminus \{a\}$ does stay in $\mathcal{C}$. Now unlike all the other categories above, $\mathbf{ConnGraph}_{\mathsf{full}}$ does not have the property that every subset of an object carries an object. But luckily enough, in every connected A with more than one vertex there is an a such that $A \setminus \{a\}$ is connected. Thus, we can use the proof of 4.3 again, only the $a \in A$ cannot be chosen arbitrarily.

Consequently we have

Proposition 4.4.1. *Let $\mathcal{C}$ be any of the categories from 4.1. Let $A \in \mathcal{N}_0(\mathcal{B})$ in $\mathcal{C}$. Then there is a $B \in \mathcal{B}$ such that A is isomorphic to a subobject of $B+$.*

4.5. Note

Already in 3.3 (resp. 3.2) we had a finite collection of objects containing all the elements of $\mathcal{N}_0(\mathcal{B})$ as subobjects. Thus, one can say that we could list $\mathcal{N}_0(\mathcal{B})$ by means of a finite search; but of course the number of cases and individual checkings is prohibitive and one can seldom expect satisfactory results obtained by brute force. The mentioned result from [4] (stating that the size of each $A \in \mathcal{N}_0(B)$ is at most $|B| + 1$ in the symmetric not necessarily connected case) makes the search easier, but even there the existence of an efficient search procedure is an open problem.

5. Ramsey lists in symmetric graphs

5.1

First, observe that in the cases of $\mathbf{Graph}_{\mathsf{full}}$ and $\mathbf{ConnGraph}_{\mathsf{full}}$ the $B+$ from 4.2 and 4.3 can be a core to the $B+'$ defined as follows.

Choose an element $\omega \notin B \times \{0, 1\}$ and set

$$X_{B+'} = (B \times \{0, 1\}) \cup \{\omega\},$$
$$R_{B+'} = \{(xi, yj) \mid xR_B y, i, j = 0, 1\} \cup \{(x1, \omega), (\omega, x1) \mid x \in X_B\}.$$

5.1.1

Now we can find all the elements of $\mathcal{N}_0(\mathcal{B})$ in among the subgraphs of the $B+$ with $B \in \mathcal{B}$. Such a search is not very effective, and requires a lot of checking. For simple $\mathcal{B}$'s, however, it does yield the lists fairly smoothly.

A more effective procedure remains an open problem.

5.1.2

Note that in our case an object is a core iff

$$Rx = Ry \quad \Rightarrow \quad x = y.$$

5.2. Some particular graphs

We will use the following symbols for particular graphs (here, "ij" indicates that "both (i, j) and (j, i) are in the relation")

- $K_n = (\{0, 1, \dots, n-1\}, \{ij \mid i \neq j\})$ is the complete graph with n vertices,
- P_n is the n-path $(\{0, 1, \dots, n\}, \{01, 12, \dots, (n-1)n\})$,
- C_n is the n-cycle $(\{0, 1, \dots, n-1\}, \{01, 12, \dots, (n-1)0\})$,
- $Y = (\{0, 1, 2, 3\}, \{01, 12, 23, 13\})$,
- $T = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 25\})$,
- $A = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 45, 14\})$,
- and $B = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 45, 14, 05\})$.

Lemma 5.3. *Every Ramsey list in* $\mathbf{ConnGraph}_{\mathsf{full}}$ *contains a complete graph K_n and a path P_m.*

Proof. Each complete graph is a core. Hence some of the A_i has to exclude a complete graph K_k. Thus, $A_i \to K_k$ and hence $A_i = K_n$ since all subgraphs of a complete graph are complete.

Similarly with the paths, where all *connected* subgraphs of paths are paths, and the only one that is not a core is P_2. □

Corollary 5.3.1. *In* $\mathbf{ConnGraph}_{\mathsf{full}}$*, the only one-element Ramsey lists are $\{K_1\}$ (=$\{P_0\}$) and $\{K_2\}$ (=$\{P_1\}$).*

Proposition 5.4. *There are only two two-element Ramsey lists in* $\mathbf{ConnGraph}_{\mathsf{full}}$*, namely $\{K_3, P_3\}$ and $\{K_3, P_4\}$.*

Proof. By 5.3, a two-element list is a $\{K_n, P_m\}$ with $n, m \geq 3$. Consider the graphs

$$S_k = (\{a, b_i, c_i \mid i = 1, \ldots, k\}, \{ab_i, ac_i, b_ic_i \mid i = 1, \ldots, k\})$$

where $a, b_1, c_1, b_2, c_2, \ldots$ are distinct elements. S_n are cores and infinitely many, and if $n \geq 3$ and $m \geq 4$ we have $K_n, P_m \not\to S_k$. Thus, $\{K_3, P_3\}$ and $\{K_3, P_4\}$ are the only alternatives left. The first is dual to $\{P_1\}$ and the second to $\{P_3, A\}$ which is easy to check. □

5.5

While by 4.3 for every finite $\mathcal{B}$ there is a finite $\mathcal{A}$ such that $\mathcal{A} \not\to X$ iff $X \to \mathcal{B}$, the reverse does not hold, and indeed the finite $\mathcal{A}$ for which we can have a finite $\mathcal{B}$ to form a duality are rare.

Still, we have infinitely many three-element Ramsey lists.

Proposition. *We have the duality in* **ConnGraph**$_{\text{full}}$

$$\{K_{n+1}, P_3, Y\} \not\to X \quad \textit{iff } X \to K_n.$$

Proof. Let $M \to K_n+$ be a minimal (core) such that $M \not\to K_n$. Define M_i, $i = 0, 1$ by setting

$$M_i \times \{i\} = M \cap (K_n \times \{i\})$$

(thus, the set of vertices of M is $(M_0 \times \{0\}) \cup (M_1 \times \{1\}) \cup \{\omega\}$).

I. Let $M_0 = \emptyset$. Then $M_1 = K_n$ and $M \cong K_{n+1}$ (else $M \cong K_k$ with $k \leq n$ and $M \to K_n$).

If $M_0 \neq \emptyset$ then $M_1 \neq \emptyset$ as well, by connectedness.

II. Let $M_0 = \{x\}$. Then we cannot have $M_0 \cap M_1 = \emptyset$ since otherwise $x \sim \omega$ and M is not a core. Thus, $x \in M_1$ and by connectedness there has to be another $y \in M_1 \setminus \{x\}$ and there is $x0, y1, x1, \omega$ isomorphic to Y.

III. Let $|M_0| \geq 2$. If there exist distinct x, y, z with $x, y \in M_0$ and $z \in M_1$ we have $x0, y0, z1, \omega$ isomorphic to Y.

Thus, we are left with $M_0 = \{x, y\} \supseteq M_1 \neq \emptyset$, $x \neq y$, say, $x \in M_1$. Then we have the path $x0, y0, x1, \omega$. □

Lemma 5.6. *Every connected graph that contains C_4, that does not contain C_3, and that is a core contains A or B (recall 5.2).*

Proof. Represent the 4-cycle as $(\{1, 2, 3, 4\}, \{12, 23, 34, 41\})$. One of the vertices 1, 3, say 1, has to be connected with an x not connected with the other, and to avoid a triangle, it cannot be connected with 2 and 4 either. Similarly we can assume (by symmetry) a y connected just with 2. We cannot have $x = y$ in which case there would be a triangle. Now if x and y are not connected we have A, if they are we have B. □

Lemma 5.7. *Every tree that is a core is either a path or contains T.*

Proof. If it is not a path then there is a vertex x with degree at least three. If two of its neighbors were leaves, they would be equivalent, and our tree would not be a core. □

Proposition 5.8. *For paths we have the dualities*

$$\{P_4, C_3, A, C_5\} \not\to X \quad \textit{iff } X \to P_3,$$

and for $n \geq 4$,

$$\{P_{n+1}, T, C_3, A, B, C_5, \ldots, C_{n+2}\} \not\to X \quad \textit{iff } X \to P_n.$$

Proof. For a core X, $X \not\to P_n$ if and only if it either contains a cycle C_k with $k \neq 4$, or C_4 extended to A or B (recall 5.6), or is a tree that cannot be mapped into P_n. Since B contains P_4 it is not minimal in the case of P_3 (which accounts for its absence there).

It remains to determine the acyclic minimal $X \not\to P_n$. There is, of course, P_{n+1}, and the only remaining candidate is T, by lemma. Now T cannot be embedded into P_3+, but it can be embedded into any P_n+ with $n > 3$. □

5.9

By exactly the same reasoning we obtain

Proposition. *For cycles we have the dualities*

$$\{P_4, C_3, A\} \nrightarrow X \quad \text{iff } X \to C_5,$$

and for $n \geq 6$,

$$\{P_{n-1}, T, C_3, A, B, C_5, \ldots, C_{n-1}\} \nrightarrow X \quad \text{iff } X \to C_n.$$

Remarks 5.10. 1. Note the similarities of the "left duals" of the paths and the cycles. Compare for instance the dualities

$$\{P_5, T, C_3, A, B, C_5, C_6\} \nrightarrow X \quad \text{iff } X \to P_4$$

and

$$\{P_6, T, C_3, A, B, C_5, C_6\} \nrightarrow X \quad \text{iff } X \to C_7.$$

2. In the cycles we have started with the C_5 (anomalous by the absence of T) and proceeded with the more regular C_n, $n \geq 6$, in analogy with the equally anomalous P_3 proceeded by the equally regular P_n, $n \geq 4$.

We have the extra cases of $n = 3, 4$. Now C_3 has been dealt with in 5.5, since $C_3 = K_3$, and we could say that C_4 is of no interest since it is not a core. This is, however, just trying to escape the tedious analysis of $X \to A$ and $X \to B$: indeed, in all the formulas above, A is really the way to treat (and prohibit) the four-cycles (see 5.6), and should be viewed as such.

3. The duality of $X \to C_5$ appeared as one of the characteristics of monochromes in exact Gallai cliques in [3].

5.11. Another example

Tedious checking of the subgraphs of $A+$ (A from 5.2) yields the duality

$$\{P_4, C_3, C_5, E\} \nrightarrow X \quad \text{iff } X \to A.$$

E stands for "exotic". It is

$$(\{0, 1, 2, 3, 4, 5, 6, 7\}, \{01, 12, 23, 34, 45, 14, 17, 26, 46, 67\}),$$

a relatively complex graph (in this context).

Remark 5.11.1. This seems to contradict the result of [4], as $|E| = |A| + 2$. But it should not be forgotten that our examples concern $\mathbf{ConnGraph}_{\text{full}}$ while the mentioned result speaks of general, not only connected obstruction graphs.

5.12

In the larger category $\mathbf{Graph}_{\text{full}}$ the system $\mathcal{N}_0(K_n)$ is simpler than that of 5.5. It contains an element smaller than both Y and P_3, namely

$$P_0 + P_1,$$

where $G + H$ indicates (and will indicate in the sequel) the categorical sum (here, the disjoint union) of the two graphs.

Consequently we obtain

Proposition. *In* $\mathbf{Graph}_{\text{full}}$ *we have the dualities*

$$\{K_{n+1}, P_0 + P_1\} \nrightarrow X \quad \text{iff } X \to K_n.$$

Thus, in contrast with Proposition 5.4, if we consider disconnected graphs, there are infinitely many two-element proper Ramsey lists.

5.13. *Duals of paths in* **Graph**$_{\text{full}}$

While admitting disconnected graphs simplified the dual Ramsey lists of the complete graphs, in the case of the paths the situation gets rather more complex. Let us see what happens.

The $\ldots, T, C_3, A, B, C_5, \ldots, C_{n+2}$ part of the Ramsey list from 5.8 remains intact: each proper subgraph of any of the graphs, connected or not, can be mapped into P_n (for the case with $n \geq 6$; for the shorter paths, the $P_0 + P_1 + P_1$ contained in T has to be discussed separately). Thus, we have to analyze the (possibly disconnected) $M \subseteq P_{n+1}$ minimal with respect to the property $M \not\to P_n$.

We have the following obvious observations:

5.13.1

- *both of the endpoints of P_{n+1} are in A, and no two of the vertices in $P_{n+1} \setminus A$ are neighbors (else we obtain a subgraph of P_n),*
- *none of the resulting connected intervals is isomorphic to P_2 (else the resulting A could be mapped into P_n),*
- *at most one of the resulting connected intervals consists of a single point,*
- *and the connected intervals constituting A can be arbitrarily permuted.*

Denote by

$$\mathcal{S}(n)$$

the collection of the (isomorphism types of) the $M \subseteq P_n$ minimal with respect to the property $M \not\to P_{n-1}$ (such M's will be represented by means of sums of paths), and by

$$\mathcal{S}_0(n) \quad \text{resp. } \mathcal{S}_1(n)$$

the sets of the elements of $\mathcal{S}(n)$ containing resp. not containing the summand P_0.

Further denote by

$$\mathcal{S}^{\square}(n)$$

the collection of the $M \subseteq P_n$ minimal with respect to the combined property

$$M \not\to P_{n-1} \quad \text{and} \quad M \text{ has not } P_0 \text{ for a summand.}$$

Note that $\mathcal{S}^{\square}(n)$ is typically bigger than $\mathcal{S}_1(n)$: for instance we have

$$P_3 \in \mathcal{S}^{\square}(3), \qquad P_5 \in \mathcal{S}^{\square}(5)$$

but not in $\mathcal{S}_1(3)$ resp. $\mathcal{S}_1(5)$.

From 5.13.1 we easily infer that (if n is sufficiently large)

$$\mathcal{S}(n) = (P_0 + \mathcal{S}^{\square}(n-2)) \cup (P_1 + \mathcal{S}_1(n-3)),$$
$$\mathcal{S}^{\square}(n) = (P_1 + \mathcal{S}^{\square}(n-3)) \cup (P_3 + \mathcal{S}^{\square}(n-5)) \cup (P_5 + \mathcal{S}^{\square}(n-7))$$

(where $P + \mathcal{S}$ stands for $\{P + S \mid S \in \mathcal{S}\}$).

Note. In the second formula one stops with the third summand since all the P_k with $k \geq 6$ already contain non-trivial sums without P_0. In fact, it seems that for n sufficiently large one obtains all the cases already in the first summand (the other two containing just repetitions).

As examples we can now compute the $\mathcal{S}(n)$ for small n (kG indicates $\overbrace{G + \cdots + G}^{n\text{-times}}$). An easy checking yields:

$$\mathcal{S}(1) = \{P_1\} = \mathcal{S}_1(1) = \mathcal{S}^{\square}(1), \qquad \mathcal{S}_0(n) = \emptyset,$$
$$\mathcal{S}(2) = \emptyset = \mathcal{S}_0(2) = \mathcal{S}_1(2) = \mathcal{S}^{\square}(2),$$
$$\mathcal{S}(3) = \{P_0 + P_1\} = \mathcal{S}_0(3), \qquad \mathcal{S}_1(3) = \emptyset, \qquad \mathcal{S}^{\square}(3) = \{P_3\},$$
$$\mathcal{S}(4) = \{2P_1\} = \mathcal{S}_1(4) = \mathcal{S}^{\square}(4), \qquad \mathcal{S}_0(4) = \emptyset,$$
$$\mathcal{S}(5) = \{P_0 + P_3\} = \mathcal{S}_0(5), \qquad \mathcal{S}_1(5) = \emptyset \qquad \mathcal{S}^{\square}(5) = \{P_5\},$$
$$\mathcal{S}(6) = \{P_0 + 2P_1\} = \mathcal{S}_0(6), \qquad \mathcal{S}_1(6) = \emptyset, \qquad \mathcal{S}^{\square}(6) = \{P_1 + P_3\},$$
$$\mathcal{S}(7) = \{P_0 + P_5,\ 3P_1\}, \qquad \mathcal{S}_0(7) = \{P_0 + P_5\}, \qquad \mathcal{S}_1(7) = \mathcal{S}^{\square}(7) = \{3P_1\}.$$

Further we can proceed by the formulas above

$$\mathcal{S}(8) = \{P_0 + P_1 + P_3\}, \qquad \mathcal{S}^{\square}(8) = \{P_1 + P_5, 2P_3\},$$
$$\mathcal{S}(9) = \{P_0 + 3P_1\}, \qquad \mathcal{S}^{\square}(9) = \{2P_1 + P_3\},$$
$$\mathcal{S}(10) = \{P_0 + P_1 + P_5, P_0 + 2P_3, 4P_1\}, \qquad \mathcal{S}^{\square}(10) = \{4P_1, P_3 + P_5\},$$
$$\mathcal{S}(11) = \{P_0 + 2P_1 + P_3\}, \qquad \mathcal{S}^{\square}(11) = \{2P_1 + P_5, P_1 + 2P_3\},$$
$$\mathcal{S}(12) = \{P_0 + 4P_1, P_0 + P_3 + P_5\}, \qquad \mathcal{S}^{\square}(12) = \{3P_1 + P_3, 2P_5\},$$
$$\mathcal{S}(13) = \{P_0 + 2P_1 + P_5, P_0 + P_1 + 2P_3, 5P_1\}, \qquad \mathcal{S}^{\square}(13) = \{5P_1, P_1 + P_5, 3P_3\},$$
$$\mathcal{S}(14) = \{P_0 + 3P_1 + P_3, P_0 + 2P_5\}, \qquad \mathcal{S}^{\square}(14) = \{3P_1 + P_5, 2P_1 + 2P_3\},$$
$$\mathcal{S}(15) = \{P_0 + 5P_1, P_0 + P_1 + P_3 + P_5, P_0 + 3P_3\}$$

etc. Thus, the resulting Ramsey lists corresponding to the paths do not seem to be more transparent than those in the connected case.

Note. After this paper was written we learned that some related results for graphs were independently obtained by P. Hell and its collaborators. See [7] for a survey of these results.

Acknowledgements

The last two authors would like to express their thanks for support by the project 1M0545 of the Ministry of Education of the Czech Republic.

References

[1] N. Alon, J. Pach, J. Solymosi, Ramsey-type theorems with forbidden subgraphs, Combinatorica 21 (2) (2001) 155–170.
[2] A. Atserias, On digraph coloring problems and treewidths duality, in: 20th IEEE Symposium on Logic in Computer Science, LICS, 2005, pp. 106–115.
[3] R.N. Ball, A. Pultr, P. Vojtěchovský, Colored graphs without colorful cycles, Combinatorica 27 (4) (2007) 407–427.
[4] T. Feder, P. Hell, On realizations of point determining graphs and obstructions to full homomorphisms, Discrete Math. 308 (9) (2008) 1639–1652.
[5] J. Foniok, J. Nešetřil, C. Tardif, Generalized dualities and maximal finite antichains in the homomorphism order of relational structures, European J. Combin. 29 (4) (2008) 881–899.
[6] R.L. Graham, J. Spencer, B.L. Rothschild, Ramsey Theory, Wiley, New York, 1980.
[7] P. Hell, From graph colouring to constraint satisfaction: There and back again, in: Topics in Discrete Mathematics, @6. Algorithms and Combinatorics, Springer Verlag, 2006.
[8] P. Hell, J. Nešetřil, Graphs and Homomorphisms, Oxford University Press, Oxford, 2004.
[9] P. Hell, J. Nešetřil, Colouring, constraint satisfaction, and complexity, in: Comp. Sci. Rev., vols. 2, 3, University Press, 2008, pp. 143–164.
[10] P. Komárek, Some new good characterizations for directed graphs, Čas. Pěst. Mat. 109 (1984) 348–354.
[11] G. Kun, J. Nešetřil, Forbidden lifts (NP and CSP for combinatorists), European J. Combin. 29 (4) (2008) 930–945.
[12] G. Kun, J. Nešetřil, Density and dualities for algrebras (submitted for publication).
[13] S. Mac Lane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971.
[14] J. Nešetřil, Ramsey theory, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Elsevier, 1995, pp. 1331–1403.
[15] J. Nešetřil, A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22 (1978) 287–300.
[16] J. Nešetřil, C. Tardif, Duality theorems for finite structures (characterizing gaps and good characterizations), J. Combin. Theory B 80 (2000) 80–97.
[17] J. Nešetřil, C. Tardif, Short answers to exponentially long questions: Extremal aspects of homomorphism duality, SIAM J. Discrete Math. 19 (4) (2005) 914–920.