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# Diversity of $p$-adic analytic manifolds

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**Abstract**

The problem of the cardinality of the set of non-homeomorphic $p$-adic manifolds is solved. It is proved that there exist $2^{\aleph_1}$ pairwise non-homeomorphic non-metrizable one-dimensional $p$-adic analytic manifolds of weight $\aleph_1$. This contrasts with the single isomorphism class of metrizable manifolds of the same weight. Further, we prove that for $p > 2$, there are $2^{\aleph_1}$ pairwise non-isomorphic non-metrizable manifolds of weight $\aleph_1$, which are homeomorphic.

To demonstrate the wide variety of non-metrizable $p$-adic manifolds, and contrast with the situation for real analytic manifolds, we construct a range of 'pathological' non-metrizable $p$-adic manifolds.



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## 1. Introduction

The central aims of this paper are to demonstrate the huge variety of non-metrizable $p$-adic analytic manifolds in comparison with the paucity of metrizable $p$-adic analytic manifolds; and to contrast the topological behaviour of $p$-adic analytic manifolds with that of real manifolds. (Here and below, $p$ is a prime number).

Roughly speaking, a $p$-adic analytic manifold of dimension $n$, is a space locally homeomorphic to Cantor spaces $(p^\omega)^n$, which are joined together by smooth maps. They are the analogues for the complete field of $p$-adic numbers, of real analytic manifolds for the field of real numbers, and complex analytic manifolds for the field of complex numbers.

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Analytic manifolds over various fields have been intensively studied in the context of Lie algebras and Lie groups. Most modern introductions to Lie groups and algebras (especially those with an algebraic slant) develop the general theory for arbitrary complete fields, specialising to the three most important cases, real, complex and $p$-adic, at a later stage. Thus, for further information on $p$-adic analytic manifolds, the reader is referred to the books of Bourbaki [2] and Serre [8] on Lie algebras and Lie groups. The brief exposition below of analytic manifolds over complete fields is taken from the latter text.

*Complete fields*

Let $k$ be a field. An absolute value on $k$ is a function $|\cdot|: k \to [0,\infty)$ satisfying (1) $|x| = 0$ if and only if $x = 0$, (2) $|xy| = |x||y|$, (3) $|1| = 1$, and (4) $|x + y| \leqslant |x| + |y|$. Every absolute value induces a metric on $k$, given by $d(x, y) = |x - y|$. The absolute value is called complete if $d$ is complete. The reals, $\mathbb{R}$, and complex numbers, $\mathbb{C}$, with their usual absolute values are complete fields.

An important family of complete fields (one for each prime $p$) is the family of $p$-adic numbers, denoted $\mathbb{Q}_p$. Fix a prime $p$, and let $\mathbb{Q}$ be the field of rational numbers. For any $a \in \mathbb{Q}$, $a \neq 0$, write $a = p^n \cdot (r/s)$ where $r$ and $s$ are integers prime to $p$, and define $|a|_p = 1/p^n$. Then $\mathbb{Q}_p$ is the completion of $(\mathbb{Q}, |\cdot|_p)$; its absolute value is also denoted $|\cdot|_p$. Define $\mathbb{Z}_p$, the $p$-adic integers, to be $\{a \in \mathbb{Q}_p\colon |a|_p \leqslant 1\}$. Then $\mathbb{Z}_p$ is a compact, open subring of $\mathbb{Q}_p$. It is naturally homeomorphic to $p^\omega$ (and so homeomorphic to the Cantor set).

*Analytic functions*

Let $k$ be a field with complete absolute value, $|\cdot|$. For any $x \in k$, and $r \in (0,\infty)$, define the open disc of radius $r$ about $x$ to be $D(x,r) = \{y \in k\colon |x - y| < r\}$. A function $f: U \to k$, where $U$ is an open subset of $k$, is *analytic*, if for each point $x$ in $U$, there is an $r > 0$, so that $f$ can be represented as a power series convergent in $D(x,r)$. A function $f: U \to k^n$, where $U$ is an open subset of $k^m$, is *analytic*, if the coordinate functions of $f$ are analytic.

*k analytic manifolds*

In the following, $k$ is a complete field, and $n$ is a fixed integer $\geqslant 1$. Let $X$ be a space. A chart on $X$ is a pair $(U,\phi)$ consisting of an open subset, $U$, of $X$, and a homeomorphism, $\phi$, of $U$ onto $\phi(U)$ an open subset of $k^n$. A pair of charts, $(U,\phi)$ and $(V,\psi)$, are compatible if, setting $W = U \cap V$, the functions $\psi \circ (\phi^{-1}|\phi(W))$ and $\phi \circ (\psi^{-1}|\psi(W))$ are analytic. An atlas, $A$, is a family of charts which cover $X$ and whose members are mutually compatible. The full atlas, denoted $A(X)$, generated by $A$, is the family of all charts on $X$ compatible with every chart in $A$. The full atlas is an atlas. Two atlases $A$ and $A'$ on $X$, are compatible if $A(X) = A'(X)$. Compatibility of atlases is an equivalence relation, and a space $X$ with an equivalence class of atlases is called a *k analytic manifold, of dimension n*. When $k = \mathbb{R}$, then $X$ is a real analytic manifold; when $k = \mathbb{C}$, then $X$ is a complex analytic manifold; and when $k = \mathbb{Q}_p$, then $X$ is a $p$-adic analytic manifold.

*Morphisms*

Let $A_i$ be an atlas for the $k$ analytic manifold $X_i$ $(i = 1, 2)$. A function $f : X_1 \to X_2$ is a *morphism* or *analytic function* if $f$ is continuous and, whenever $(U_i, \phi_i) \in A_i$ $(i = 1, 2)$ then, setting $W = U_1 \cap f^{-1}(U_2)$, the map $\phi_2 \circ f \circ {\phi_1}^{-1}$ is analytic when restricted to $\phi_1(W)$. Isomorphisms are defined in the natural way.

*Open submanifolds*

Let $X$ be a $k$ analytic manifold with full atlas $A(X)$, and let $U$ be an open subset of $X$. Define $A_U = \{(V, \psi)\colon V \subseteq U\}$. Then $A_U$ is an atlas on $U$, and $U$ with this $k$ analytic structure is called an open submanifold of $X$.

*Topology of $p$-adic analytic manifolds*

Recall that the $p$-adic integers, $\mathbb{Z}_p$, form a compact and open subring of $\mathbb{Q}_p$. Thus for every point $x$ of a $p$-adic analytic manifold, $X$, of dimension $n$, there is a chart $(U, \phi)$ such that $x$ is in $U$, and $\phi$ is a homeomorphism of $U$ with $\mathbb{Z}_p^n$. In particular, Hausdorff $p$-adic analytic manifolds have inductive dimension 0, and are locally compact and locally metrizable. The structure of metrizable $p$-adic manifolds is completely understood:

**Theorem 1.1** (Serre [8,9]). *Let $X$ be a $p$-adic analytic manifold of dimension $n$. Then $X$ is metrizable if and only if it is isomorphic to a disjoint sum of copies of $\mathbb{Z}_p^n$, say $X = \bigoplus_{\lambda\in\Lambda} \mathbb{Z}_p^n$.*

*Suppose $X$ is metrizable, and define*

$$\mathit{class}(X) = \min\left\{|\Lambda|\colon X \text{ is isomorphic to } \bigoplus_{\lambda\in\Lambda} \mathbb{Z}_p^n\right\}.$$

*Then* (1) *the cardinal number class$(X)$ classifies $X$ up to isomorphism, and* (2) *the image of class$(\cdot)$ is $\{1, 2, \ldots, p-1\} \cup \{\kappa\colon \kappa$ is an infinite cardinal$\}$.*

Note that for $X$ a metrizable $p$-adic analytic manifold, $w(X) = \mathrm{class}(X).\aleph_0$, and $X$ is compact if and only if $\mathrm{class}(X)$ is finite. Note also that if $X$ and $Y$ are non-compact metrizable $p$-adic analytic manifolds, of the same dimension, then $X$ and $Y$ are homeomorphic if and only if they are isomorphic.

In [9] Serre gives an example, which he attributes to George Bergman, of a non-metrizable $p$-adic analytic manifold. The example is essentially the $p$-adic analogue of the long ray.

*Constructing $p$-adic manifolds*

For later convenience, we state three simple lemmas which will aid us in the construction of non-metrizable $p$-adic manifolds. The first is immediate from the definitions.

**Lemma 1.2.** *Let $X$ be a topological space. Suppose $X = \bigcup_{\lambda\in\Lambda} U_\lambda$, where*

(1) *each $U_\lambda$ is open in $X$,*

(2) *on each $U_\lambda$ there is a $p$-adic analytic manifold structure, and*

(3) *for all* $\lambda, \mu$, *the* $p$*-adic analytic manifold structures on* $U_\lambda \cap U_\mu$, *induced by the* $p$*-adic analytic manifold structure on* $U_\lambda$ *and* $U_\mu$, *agree.*

*Then* $X$ *has a unique* $p$*-adic analytic manifold structure, with the* $U_\lambda$*s as open submanifolds.*

**Lemma 1.3.** *Let* $M$ *be a one dimensional* $p$*-adic analytic manifold isomorphic to* $\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$, *and suppose* $N$ *is the one point compactification of* $M$. *Then* $N$ *admits the structure of a one dimensional* $p$*-adic analytic manifold, such that* $N$ *is isomorphic to* $\mathbb{Z}_p$, *and* $M$ *is an open submanifold of* $N$.

**Proof.** Clear from the fact that $\mathbb{Z}_p \setminus \{0\}$ is isomorphic to $\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$. □

**Lemma 1.4.** *If* $X$ *is a* $\sigma$*-compact non-compact* $p$*-adic manifold of dimension* $m$, *then* $X$ *is isomorphic to* $\bigoplus_{n\in\omega}(\mathbb{Z}_p^m)_n$.

**Proof.** A $\sigma$-compact $p$-adic analytic manifold is separable metrizable. So the claim follows from Serre's result. □

## 2. Homeomorphism and isomorphism classes

In contrast to the single isomorphism class of one dimensional metrizable $p$-adic analytic manifolds of weight $\aleph_1$, there are the maximal possible number, $2^{\aleph_1}$ many, pairwise non-homeomorphic one dimensional non-metrizable $p$-adic analytic manifolds of weight $\aleph_1$.

In addition, there is a family of $2^{\aleph_1}$ pairwise non-isomorphic one dimensional $p$-adic analytic manifolds of weight $\aleph_1$, which are all mutually homeomorphic. Recall that homeomorphic non-compact metrizable $p$-adic analytic manifolds, of the same dimension, are isomorphic. Also recall S. Donaldson's famous result that there are $2^{\aleph_0}$ pairwise non-isomorphic (real!) analytic structures on $\mathbb{R}^4$.

Both constructions are based on an idea of Nyikos [4].

**Definition 2.1.** A space $X$ is of type I if it is the union of an $\omega_1$-sequence $\{U_\alpha\}_{\alpha<\omega_1}$ of open subspaces such that $\overline{U_\beta} \subseteq U_\alpha$ whenever $\beta < \alpha$, $U_\lambda = \bigcup_{\beta<\lambda} U_\beta$ for $\lambda$ limit, and such that $\overline{U_\alpha}$ is Lindelöf for all $\alpha$.

Let $X$ be a type I space. Any $\omega_1$-sequence $\{U_\alpha\}_{\alpha<\omega_1}$ witnessing that $X$ is type I, is called a canonical sequence for $X$.

It is easy to see that every type I manifold (real or $p$-adic analytic) is of weight $\leqslant \aleph_1$. Any two canonical sequences agree on a closed unbounded set of indices:

**Lemma 2.2.** *If a type* I *space* $X$ *has two canonical sequences* $\Sigma = \{X_\alpha\}_{\alpha<\omega_1}$ *and* $\Sigma' = \{X'_\alpha\}_{\alpha<\omega_1}$, *then the set* $C = \{\alpha \in \omega_1\colon\ X_\alpha = X'_\alpha\}$ *is closed unbounded in* $\omega_1$.

It is well known that $\omega_1$ can be partitioned into $\omega_1$ stationary sets. Fix such a partition $\{A_\alpha\}_{\alpha\in\omega_1}$ (so $\bigcup_{\alpha\in\omega_1} A_\alpha = \omega_1$, the $A_\alpha$'s are pairwise disjoint, and every $A_\alpha$ has non-empty intersection with every closed unbounded subset of $\omega_1$).

**Theorem 2.3.** *There are $2^{\aleph_1}$ pairwise non-homeomorphic one dimensional $p$-adic analytic manifolds of weight $\aleph_1$.*

**Proof.** For each $\sigma : \omega_1 \to \{0,1\}$, we will define a type I one dimensional $p$-adic analytic manifold $M^\sigma$ with a canonical sequence $\{M^\sigma_\alpha\}_{\alpha<\omega_1}$ of open submanifolds such that

$$M^\sigma_\alpha = \mathbb{Z}_p \quad \text{for } \alpha \text{ successor}, \qquad M^\sigma_\lambda = \bigoplus_{n\in\omega} (\mathbb{Z}_p)_n \quad \text{for } \lambda \text{ limit}, \quad \text{and},$$

$$\left|\partial M^\sigma_\lambda\right| = \begin{cases} 1 & \text{if } \sigma(\xi) = 0, \\ p & \text{if } \sigma(\xi) = 1, \end{cases} \qquad \text{for } \lambda \text{ limit},$$

where $\xi$ is unique such that $\alpha \in A_\xi$.

Assuming, for the moment, we have these $p$-adic manifolds, suppose $\sigma \neq \tau$, $\sigma, \tau : \omega_1 \to \{0,1\}$. We claim $M^\sigma$ is not homeomorphic to $M^\tau$. Well, suppose for a contradiction, that there is a homeomorphism $\psi : M^\sigma \to M^\tau$. Since $\sigma \neq \tau$, there is $\xi \in \omega_1$ such that (with out loss of generality) $0 = \sigma(\xi) \neq \tau(\xi) = 1$. Now $C = \{\alpha \in \omega_1\colon \alpha \text{ is a limit and } M^\tau_\alpha = \psi(M^\sigma_\alpha)\}$ is closed unbounded. Hence we can pick $\alpha \in A_\xi \cap C$. Then $\psi$ carries $\partial M^\sigma_\alpha$ onto $\partial M^\tau_\alpha$, but, by construction, the two boundaries have differing cardinalities.

It remains to show that we can construct $M^\sigma$ with canonical sequence of open submanifolds $\{M^\sigma_\alpha\}_{\alpha\in\omega_1}$, as above. In fact, once we have defined the $M^\sigma_\alpha$s then we can define $M^\sigma$ as the union of the $M^\sigma_\alpha$s with the direct limit topology and unique one dimensional $p$-adic analytic structure given by Lemma 1.2.

We proceed by transfinite recursion on $\alpha < \omega_1$. There are three cases. Suppose, first, that $\alpha$ is a limit. Let $M^\sigma_\alpha$ be the space with underlying set $\bigcup_{\beta<\alpha} M^\sigma_\beta$, direct limit topology, and unique one dimensional $p$-adic analytic structure guaranteed by Lemma 1.2. Then $M^\sigma_\alpha$ must be isomorphic with $\bigoplus_{n\in\omega} (\mathbb{Z}_p)_n$ (by Lemma 1.4).

Now suppose, $\alpha = \beta + 1$, and $\beta$ is a successor. Then $M^\sigma_\beta = \mathbb{Z}_p$. Define $M^\sigma_\alpha = \bigoplus_{i=1}^p (\mathbb{Z}_p)_i$, and identify $M^\sigma_\beta$ with the first copy of $\mathbb{Z}_p$. Note that $M^\sigma_\alpha$ is isomorphic to $\mathbb{Z}_p$, and all the other hypotheses are satisfied.

The last and most interesting case is when $\alpha = \lambda + 1$, where $\lambda$ is a limit. We know that $M^\sigma_\lambda = \bigoplus_{n\in\omega} (\mathbb{Z}_p)_n$. If $\sigma(\xi) = 0$, then let $M^\sigma_\alpha$ be the one point compactification of $M^\sigma_\lambda$, with analytic structure of Lemma 1.3. If $\sigma(\xi) = 1$, then write $M^\sigma_\lambda = \bigoplus_{i=1}^p (\bigoplus_{n\in\omega} (\mathbb{Z}_p)_n)_i$. One point compactify each of the copies of $\bigoplus_{n\in\omega} (\mathbb{Z}_p)_n$, to yield $M^\sigma_\alpha$, with analytic structure of Lemma 1.3. Observe that $M^\sigma_\alpha$ is isomorphic to $p$ copies of $\mathbb{Z}_p$, and so is isomorphic to $\mathbb{Z}_p$, and, since $M^\sigma_\alpha$ is closed and open in $M^\sigma$, the desired boundary properties of $M^\sigma_\lambda$ hold. □

**Theorem 2.4.** *For $p \geqslant 3$. There are $2^{\aleph_1}$ pairwise non-isomorphic one dimensional $p$-adic analytic manifolds of weight $\aleph_1$, which are all homeomorphic.*

**Proof.** For each $\sigma:\omega_1 \to \{0,1\}$, we will define a type I one dimensional $p$-adic analytic manifold $N^\sigma$ with a canonical sequence $\{N^\sigma_\alpha\}_{\alpha<\omega_1}$ of open submanifolds such that for $\alpha=\lambda+1$, $\lambda$ limit

$$N^\sigma_\alpha = \begin{cases} \mathbb{Z}_p & \text{if } \sigma(\xi)=0, \\ \mathbb{Z}_p \oplus \mathbb{Z}_p & \text{if } \sigma(\xi)=1, \end{cases}$$

where $\xi$ is unique so that $\alpha \in A_\xi$. The manifold $N^\sigma$ will have the direct limit topology and analytic structure induced by the $N^\sigma_\alpha$s. We will also ensure that $N^\sigma_{\lambda+1}$ is the closure of $N^\sigma_\lambda$, for all limit $\lambda$, and that the topology and underlying sets of the $N^\sigma_\alpha$s do not depend on $\sigma$. The last condition, of course, ensures that all the manifolds $N^\sigma$ are homeomorphic.

Similarly to the proof of Theorem 2.3, if $\sigma,\tau:\omega_1\to\{0,1\}$, $\sigma\neq\tau$, but $\psi$ is an isomorphism of $N^\sigma$ with $N^\tau$, then for some limit $\lambda$, we have $\psi(N^\sigma_\lambda)=N^\tau_\lambda$. Hence $\psi(N^\sigma_{\lambda+1})=\psi(\overline{N^\sigma_\lambda})=\overline{N^\tau_\lambda}=N^\tau_{\lambda+1}$. So $N^\sigma_{\lambda+1}$ and $N^\tau_{\lambda+1}$ are isomorphic, when, by construction they are non-isomorphic.

Fix $\sigma$, and let us construct the $N^\sigma_\alpha$s by transfinite recursion on $\alpha<\omega_1$. Four cases arise. Suppose first that $\alpha=\beta+1$, where $\beta$ is a successor. Then define $N^\sigma_\alpha=N^\sigma_\alpha\oplus\mathbb{Z}_p$. Next suppose that $\alpha=\lambda$, $\lambda$ a limit. Define $N^\sigma_\alpha=\bigcup_{\beta<\lambda}N^\sigma_\beta$ with the direct limit topology and unique one dimensional analytic structure guaranteed by Lemma 1.2.

The final two cases are when $\alpha=\lambda+1$, for $\lambda$ a limit. Note that $N^\sigma_\lambda=\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$. If $\sigma(\xi)=0$, then embed $N^\sigma_\lambda$ in $N^\sigma_{\lambda+1}=\mathbb{Z}_p$ by the standard embedding of Lemma 1.3. On the other hand, if $\sigma(\xi)=1$, then write $N^\sigma_\lambda=(\mathbb{Z}_p)_0\oplus\bigoplus_{n\geqslant1}(\mathbb{Z}_p)_n$. Embed $\bigoplus_{n\geqslant1}(\mathbb{Z}_p)_n$ in $\mathbb{Z}_p$ by the standard embedding of Lemma 1.3, and $N^\sigma_\lambda$ in $N^\sigma_{\lambda+1}=(\mathbb{Z}_p)_0\oplus\mathbb{Z}_p$ in the natural manner.

Observe that in both the two preceding cases, $N^\sigma_{\lambda+1}$ is the one point compactification of $N^\sigma_\lambda$. Thus both the analytic and topological conditions are satisfied by the construction. □

## 3. Comparison with real manifolds

Probably the two most important results on the general topology of non-metrizable real manifolds are the following.

**Theorem 3.1** (Rudin and Zenor [7], Rudin [6]). *It is consistent and independent that every perfectly normal real manifold is metrizable.*

**Theorem 3.2** (Reed and Zenor [5]). *Every* (*perfectly*) *normal Moore real manifold is metrizable.*

Two other, minor but attractive, results concerning real manifolds are (1) a real manifold with regular $G_\delta$ diagonal is metrizable, and (2) a real manifold with $G^*_\delta$ diagonal is a Moore space. From the first of these, it follows that submetrizable real manifolds are metrizable.

All these results depend crucially on the local connectedness of real manifolds, and not at all on the manifold structure. This is exposed by Examples 3.5–3.7 below which are counter-examples to the natural $p$-adic analogues of the positive results for real manifolds above.

Our first example, however, demonstrates how to adapt the techniques for constructing pathological real manifolds to the construction of $p$-adic analytic manifolds with similar properties.

**Lemma 3.3.** *Let $\tau, \sigma$ be topologies on a set $X$, such that $\sigma \subseteq \tau$, $(X, \sigma)$ is hereditarily separable and for every countable subset $B$ of $X$, $|\overline{B^{\sigma}} \setminus \overline{B^{\tau}}| \leqslant \aleph_0$. Then $(X, \tau)$ is hereditarily separable.*

**Proof.** Take any $Y$ contained in $X$. As $(X, \sigma)$ is hereditarily separable, there is a countable subset $A$ of $Y$ such that $\overline{A^{\sigma}} \supseteq Y$. Let $D = (\overline{A^{\sigma}} \setminus \overline{A^{\tau}}) \cap Y$, and $B = A \cup D$ (note that $B$ is countable). Then $\overline{B^{\tau}} \supseteq \overline{A^{\tau}} \cup D \supseteq \overline{A^{\sigma}} \supseteq Y$. Thus, $B$ is $\tau$-dense in $Y$. $\square$

**Example 3.4** (CH). There is a space $X$ which is Hausdorff, submetrizable, hereditarily separable, but not Lindelöf, which has the structure of a one dimensional $p$-adic analytic manifold.

*Construction.* Let $X = (\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n) \oplus \mathbb{Z}_p$. Write $um$ for the ultra-metric topology on $X$. Assume *CH* and let $\{x_\alpha\}_{\omega\leqslant\alpha<\omega_1}$ enumerate $\mathbb{Z}_p$. Define $X_\omega = \bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$, $Y_\alpha = \{x_\beta\colon \beta < \alpha\}$ and let $X_\alpha = X_\omega \cup Y_\alpha$. Let $\{B_\alpha\}_{\alpha<\omega_1}$ enumerate all countably infinite subsets of $X$. Define $\mathcal{B}_\beta = \{B_\gamma\colon \gamma < \beta,\ B_\gamma \subseteq X_\beta,\ x_\beta \in \overline{B_\gamma^{um}}\}$ and enumerate each $\mathcal{B}_\beta$ as $\{C_{(\beta,n)}\colon n \in \omega\}$, where each $B \in \mathcal{B}_\beta$ appears infinitely often in the enumeration of $\mathcal{B}_\beta$.

We will construct a sequence $\{\tau_\alpha\}_{\omega\leqslant\alpha<\omega_1}$ of topologies and a sequence $\{A_\alpha\}_{\omega\leqslant\alpha<\omega_1}$ of atlases, where $\tau_\alpha$ is a topology on $X_\alpha$, $A_\alpha$ is an atlas for a one dimensional $p$-adic analytic manifold structure on $(X_\alpha, \tau_\alpha)$ such that

(1) $\tau_\alpha$ is a 0-dimensional topology refining the ultra-metric topology on $X_\alpha$.
(2) If $\beta < \alpha$, then $(X_\beta, \tau_\beta, A_\beta)$ is isomorphic to $\bigoplus_{n\in\omega}\mathbb{Z}_p$, and is an open submanifold of $(X_\alpha, \tau_\alpha, A_\alpha)$.
(3) If $\beta < \alpha$ and $B \in \mathcal{B}_\beta$, then $x_\beta \in \overline{B^{\tau_{\alpha+1}}}$.

Suppose we have such $\tau_\alpha$s and $A_\alpha$s. Let $X$ have topology with basis $\bigcup_{\alpha\in\omega_1}\tau_\alpha$. Then, by Lemma 1.2, $X$ has a unique one dimensional $p$-adic manifold structure, with atlas $A$. The topology $\tau$ on $X$, refines the usual ultra-metric topology, and hence is Hausdorff and submetrizable. The sets $X_\alpha$ are all open in $X$, and no countable subcollection covers. Hence $X$ is not Lindelöf.

It remains to show that $(X, \tau)$ is hereditarily separable. We do this by checking that $(X, \tau)$ satisfies the hypotheses of Lemma 3.3. So take any countably infinite subset $B$ of $X$. Then $B = B_\gamma$, for some $\gamma$. As $B_\gamma$ is countable, there is a $\beta_0 > \gamma$ such that $B_\gamma \subseteq X_{\beta_0}$. We aim to show that $\overline{B_\gamma^{um}} \setminus \overline{B_\gamma^{\tau}}$ is contained in $Y_{\beta_0+1}$. If $x_\beta \in \overline{B_\gamma^{um}}$, $\beta > \beta_0 > \gamma$, then we need to show $x_\beta \in \overline{B_\gamma^{\tau}}$. For that, pick any $\alpha > \beta + 1$. Then $B_\gamma \in \mathcal{B}_\beta$ and by (3), $x_\beta \in \overline{B_\gamma^{\tau_{\beta+1}}} \subseteq \overline{B_\gamma^{\tau}}$. Suppose $x \in X_\omega$, and $x \in \overline{B_\gamma^{um}}$. Then $X_\omega$ is an open subspace of $(X, \tau)$ and has the $um$ topology. Hence $x \in \overline{B_\gamma^{\tau}}$.

To complete the proof we need to construct the topologies $\tau_\alpha$ and atlases $A_\alpha$ by transfinite recursion.

*Case*. $\alpha = \beta + 1$.

Pick a sequence $\{y_n\}_{n\in\omega}$ closed and discrete in $X_\beta$, such that $y_n \in C_{\beta,n}$ and the distances between $x_\beta$ and $y_n$ forms a sequence converging to zero (this is possible because the $C_{\beta,n}$s enumerate $\mathcal{B}_\beta$ and if $B$ is in $\mathcal{B}_\beta$, then $x_\beta \in \overline{B^{um}}$). Then, for each $n$, pick pairwise disjoint $U_{\beta,n}$, open in the ultra-metric topology, so that $y_n \in U_{\beta,n}$. Choose an isomorphism of $X_\beta$ with $\bigoplus_{n\in\omega}(\mathbb{Z}_p)'_n \oplus \bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$ such that $y_n \in (\mathbb{Z}_p)_n \subseteq U_{\beta,n}$.

Define $V_{\beta,k} = \{x_\beta\} \cup \bigcup_{n\geqslant k}(\mathbb{Z}_p)_n$, for $k \in \omega$. A basis for $(X_\alpha, \tau_\alpha)$ is $\tau_\beta \cup \{V_{\beta,k}\}_{k\in\omega}$. Thus $(X_\alpha, \tau_\alpha)$ is homeomorphic to $\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$ and the one point compactification of $\bigoplus_{n\in\omega}(\mathbb{Z}_p)_n$. Using Lemma 1.3, we give $X_\alpha = \bigoplus_{n\in\omega}(\mathbb{Z}_p)'_n \oplus \bigoplus_{n\in\omega}(\mathbb{Z}_p)_n \cup \{x_\beta\}$, the manifold structure, so that $X_\alpha$ is isomorphic to $\bigoplus_{n\in\omega}(\mathbb{Z}_p)'_n \oplus \mathbb{Z}_p$, and $X_\beta$ is a dense open submanifold. Let $A_\alpha$ be an atlas for this structure.

By construction, $\tau_\alpha$ and $A_\alpha$ satisfy (1) and (2). We check (3) from the list of inductive hypotheses. Suppose $\beta + 1 < \alpha$, $\gamma < \beta$, $B_\gamma \subseteq X_\beta$, and $x_\beta \in \overline{B_\gamma^{um}}$. Require $x_\beta \in \overline{B_\gamma^{\tau_\alpha}}$.

Since $B_\gamma \in \mathcal{B}_\beta$, $B_\gamma$ appears infinitely often in the listing $C_{\beta,n}$, so infinitely many of the $y_n$s (used in the definition of the topology at $x_\beta$) are in $B_\gamma$. So every basic neighbourhood, $V_{\beta,k}$, of $x_\beta$ hits (infinitely many) of the $y_n$s in $B_\gamma$, thus $x_\beta \in \overline{B_\gamma^{\tau_\alpha}}$, as required.

*Case*. $\alpha$ is a limit.

Then let $\tau_\alpha$ have basis $\bigcup_{\beta<\alpha}\tau_\beta$. By Lemma 1.2, there is a unique manifold structure on $X_\alpha$ induced by the manifold structures on the open subsets $(X_\beta, \tau_\beta, A_\beta)$. Let $A_\alpha$ be an atlas for this structure. It is easy to check that the induction hypotheses hold.

In the following example we use the fact that $\mathbb{Z}_p$ can be naturally identified with all formal power series of the form $\sum_{n=0}^{\infty} x_n \cdot p^n$, where $x_n \in \{0, 1, \ldots, p-1\}$ $(n \in \omega)$. The valuation on $\mathbb{Z}_p$ is given by $|\sum_{n=0}^{\infty} x_n \cdot p^n| = 1/p^m$, where $m$ is minimal such that $x_m \neq 0$. Addition and multiplication are given by formal addition (respectively, multiplication) of the corresponding formal power series.

**Example 3.5.** There is a space $X$ which is submetrizable, Moore, separable, but not metrizable, which has the structure of a one dimensional $p$-adic analytic manifold.

*Construction*. We give the construction for $p = 2$, but it is clear that the technique can be extended to any prime $p$. The space $X$ has underlying set $\mathbb{Z}_2 \times (\omega + 1)$.

We define a topology $\tau$ on $X$, refining the usual metrizable topology, and check it has the required properties. Then we define charts for $(X, \tau)$, and show they are compatible.

Define, for each $x \in \mathbb{Z}_2$ and $m \in \omega$,

$$U(x,m) = \left(\bigcup_{n=0}^{\infty}\left(\left(\left(\sum_{i=0}^{n-1} x_i \cdot 2^i\right) + 2^n \cdot \mathbb{Z}_2\right) \times \{n\}\right) \cup \big\{(x,\omega)\big\}\right) \cap \big(\mathbb{Z}_2 \times [m,\omega]\big).$$

Topologise $X$ so that $\mathbb{Z}_2 \times \omega$ has the product topology, and each $(x, \omega)$ has local basis $\{U(x,m)\}_{m\in\omega}$. Denote this topology by $\tau$. Thus $(X, \tau)$ is similar to the famous tangent disc space.

Clearly, $(X, \tau)$ is submetrizable, and separable ($\mathbb{Z}_2 \times \omega$ is second countable, and dense in $X$). It is also developable, as we now establish. Let $\{\mathcal{G}_n\}_{n\in\omega}$ be a development for the metrizable space $\mathbb{Z}_2 \times \omega$. Define $\mathcal{H}_n = \{U(x,n)\colon\ x \in \mathbb{Z}_2\} \cup \mathcal{G}_n$. Then the $\mathcal{H}_n$s are open covers of $X$, and $\mathrm{st}((x,\omega),\mathcal{H}_n) = U(x,n)$ (for all $n \geqslant 0$), while $\mathrm{st}((x,m),\mathcal{H}_n) = \mathrm{st}((x,m),\mathcal{G}_n)$ for $n > m$.

It remains to specify compatible charts forming an atlas of a one dimensional $p$-adic analytic manifold structure on $X$. The family $\{U(x,0)\colon x \in \mathbb{Z}_2\}$ is an open cover of $X$. Fix an isomorphism, $\psi$, between $\bigoplus_{n\in\omega}(2^n \cdot \mathbb{Z}_2 \times \{n\})$ and $\mathbb{Z}_2 \setminus \{0\}$. For each $x$ in $\mathbb{Z}_2$, define $\phi_x$ from $U(x,0)$ onto $\mathbb{Z}_2$ by

$$\phi_x\big((x,\omega)\big) = 0, \qquad \phi_x\left(\left(\sum_{i=0}^{n-1} x_i \cdot 2^i + 2^n \cdot y, n\right)\right) = \psi\big((2^n \cdot y, n)\big).$$

The $\phi_x$s are continuous, so $\{(U(x,0),\phi_x)\colon x \in \mathbb{Z}_2\}$ is a family of charts. Further, if $x, y \in \mathbb{Z}_2$, then, setting $W = U(x,0) \cap U(y,0)$, we have

$$\phi_y \circ \big(\phi_x^{-1}|\phi_x(W)\big) = \mathrm{id}_{\phi_x(W)} \quad \text{and} \quad \phi_x \circ \big(\phi_y^{-1}|\phi_y(W)\big) = \mathrm{id}_{\phi_y(W)}.$$

Thus, both transition maps are (trivially) representable as power series.

For our next two examples, observe that if we remove any set of points from $\mathbb{Z}_2 \times \{\omega\}$ in the preceding example, then what remains is an open submanifold of $X$.

**Example 3.6** (MA + ¬CH). There is a space $Y$ which is separable, (perfectly) normal, submetrizable and Moore, but not metrizable, which has the structure of a one dimensional $p$-adic analytic manifold.

*Construction*. Under MA + ¬CH, any subset of $\mathbb{Z}_2$ of size $\aleph_1$ is a $Q$-set (so every subset of that set is a $G_\delta$). Fix such a $Q$-set, $Q$ say, in $\mathbb{Z}_2 \times \{\omega\}$. Remove all points of $\mathbb{Z}_2 \times \{\omega\}$ (considered as a subspace of the preceding example) which are not in $Q$. Denote by $Y$ the resulting one dimensional $p$-adic analytic manifold. Then $Y$ is separable, submetrizable, and Moore. Mimicking the well known proof for the tangent disc space, one can check that $Y$ is normal.

**Example 3.7.** There is a space $Z$ which is submetrizable, quasi-developable, separable, but not perfect, which has the structure of a one dimensional $p$-adic analytic manifold.

*Construction*. Let $B$ be a Bernstein subset of $\mathbb{Z}_2$ and let $\{B_\alpha\}_{\alpha<2^\omega}$ be an enumeration of all countable subsets of $B$ such that $\overline{B_\alpha^{um}}$ is uncountable. For each $\alpha < 2^\omega$, pick

$$x_\alpha \in \overline{B_\alpha^{um}} \setminus \big(B \cup \{x_\beta\}_{\beta<\alpha}\big)$$

and pick points $x_{(\alpha,m)} \in B_\alpha$ such that the sequence $\{x_{(\alpha,m)}\}_{m\in\omega}$ converges to $x_\alpha$ in $\mathbb{Z}_2$ with the ultra-metric topology.

Let $X_0 = (\mathbb{Z}_2 \times \omega) \cup (B \times \{\omega\})$. Consider $(X_0, \tau_0)$ as an open submanifold of the space in Example 3.5. Let $Z = X_0 \cup \{(x_\alpha, \omega)\}_{\alpha<2^\omega}$. We will give $Z$ a topology $\tau$, and manifold structure, so that:

(1) $X_0$ is an open dense submanifold.
(2) $\tau$ refines the *um* topology.
(3) $Z \cap (\mathbb{Z}_2 \times \{\omega\})$ is (naturally) homeomorphic to Gruenhage's example [3, Example 2.17].

First we describe the topology $\tau$ on $Z$. A basis for $\tau$ is $\tau_0 \cup \{V(x_\alpha, n)\colon\ \alpha \in 2^\omega,\ n \in \omega\}$, where $V(x_\alpha, n) = (U(x_\alpha, 0) \cup (\bigcup_{n\in\omega} U(x_{(\alpha,n)}, 0)) \cap [D_{um}(x_\alpha, 1/m) \times [m, \omega]]$, $U(x, m)$ as defined in Example 3.5 and $D_{um}(x_\alpha, 1/m)$ is the open disc of radius $1/m$ about $x$ in the metrizable topology on $\mathbb{Z}_2 \times (\omega + 1)$. Clearly, $(Z, \tau)$ is submetrizable, separable, and not perfect ($(Z, \tau)$ contains a homeomorphic copy of Gruenhage's space, which is not perfect). It is also quasi-developable, as we now establish. Let $\{\mathcal{G}_n\}_{n\in\omega}$ be a development for the metrizable space $\mathbb{Z}_2 \times \omega$. Define $\mathcal{H}_n = \{V(x_\alpha, n)\colon\ \alpha \in 2^\omega\}$ and $\mathcal{J}_n = \{U(x, n)\colon\ x \in B\}$. Then $\{\mathcal{G}_n\}_{n\in\omega} \cup \{\mathcal{H}_n\}_{n\in\omega} \cup \{\mathcal{J}_n\}_{n\in\omega}$ is a countable collection of open families, which form a quasi-development for $X$. To see this, it suffices to note that $\mathrm{st}((x_\alpha, \omega), \mathcal{H}_n) = V(x_\alpha, n)$ and $\mathrm{st}((x, \omega), \mathcal{J}_n) = U(x, n)$ $(x \in B)$.

It remains to give $(Z, \tau)$ a $p$-adic manifold structure. Recall that $(X_0, \tau_0)$ already has a one dimensional $p$-adic manifold structure and is an open subspace of $(Z, \tau)$. For each $(x_\alpha, \omega)$ in $R = Z \setminus X_0$, note that $V(x_\alpha, 0)$ is an open set, meeting $R$ only at $(x_\alpha, \omega)$. By Lemma 1.2, $(Z, \tau)$ has a one dimensional $p$-adic analytic structure, provided we can give $V(x_\alpha, 0)$ a one dimensional $p$-adic analytic structure compatible with that on $X_0$.

Fix $x_\alpha$. Note that $V = V(x_\alpha, 0)$ is the one point compactification of $V^* = V(x_\alpha, 0) \setminus \{(x_\alpha, \omega)\}$. Hence $V^*$ is an open submanifold of $X_0$ isomorphic to $\bigoplus_{n\in\omega} (\mathbb{Z}_p)_n$. Therefore, we may give $V$ the manifold structure of $\mathbb{Z}_p$ via Lemma 1.3, with $V^*$ as an open submanifold.

## 4. Open problems, and dimension

All of the examples constructed above are one dimensional, as $p$-adic manifolds. It is clear that the constructions could be modified to give examples of any desired analytic dimension.

Some results pertaining to real (analytic) manifolds depend on dimension. For example, Rudin and Balogh [1] show that every monotonically normal manifold of dimension at least two is metrizable. It would be interesting to know whether monotonically normal $p$-adic analytic manifolds of *analytic* dimension two (or more) are necessarily metrizable. If true this would give an example of the analytic structure influencing the general topology of a $p$-adic analytic manifold.

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