NOTE

ON THE HOLONOMY DECOMPOSITION OF TRANSFORMATION SEMIGROUPS

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1. Introduction

In [1] Eilenberg develops the holonomy decomposition of an arbitrary (finite) transformation semigroup. He remarks that this decomposition yields a more efficient method of obtaining a decomposition of a transformation semigroup than the previous method of Krohn–Rhodes, Zeiger, Ginzburg, etc. However he remarks that even his results fail to produce the best known decompositions of some important classes of transformation semigroups. We will discuss here an improvement to the holonomy decomposition theorem and use it to obtain the 'better' decomposition of the cyclic transformation semigroup $C_{(p,r)}$ mentioned by Eilenberg in [1, Chapter II, Example 9.2]. Most of our terminology and notation is the same as [1], with a few minor exceptions.

2. The derived sequence and the holonomy decomposition theorem

Let $X = (Q, S)$ be a transformation semigroup. The family $A$ of subsets of $Q$ consisting of all sets of the form $Qs$ ($s \in S$); the set $Q$, the set $\phi$ and all the singletons of $Q$, will be called the skeleton of $X$.

An admissible subset system of $X$ is a set of non-empty subsets $\{H_i\}_{i \in I}$ of $Q$ which cover $Q$ and satisfy the property that if $i \in I$, $s \in S$ then there exists a $j \in I$ such that $H_s \subseteq H_j$.

If the subsets $\{H_i\}_{i \in I}$ are mutually disjoint, then we call the system an admissible partition.

Let $\pi = \{H_i\}_{i \in I}$ and $\tau = \{K_j\}_{j \in J}$ be admissible partitions of $X$. If $\pi \cap \tau = 1_X$ we call both $\pi$ and $\tau$ orthogonal. (This means that $|H_i \cap K_j| \leq 1$ for all $i \in I$, $j \in J$.) Given an
admissible partition $\pi$ of $X$ we may form the quotient transformation semigroup $X/(\langle \pi \rangle)$ as follows:

$X/(\langle \pi \rangle)$ has state set equal to $\pi$ and the semigroup $S$ of $X$ acts on $\pi$ as follows: Let $H_i \in \pi$, $s \in S$ then $H_i * s$ is defined to be the unique $H_j \in \pi$ satisfying

$$H_i * s \subseteq H_j \text{ in } X$$

unless $H_i s = \emptyset$ in which case $H_i * s$ is undefined.

The pair $(\pi, S)$ can be turned into a transformation semigroup $(\pi, S/\sim)$ where $\sim$ is the congruence defined on $S$ by

$$s \sim s' \iff H_i * s = H_i * s' \text{ for all } i \in I.$$

We write $(\pi, S/\sim)$ as $X/(\langle \pi \rangle)$.

If $\pi$ is an admissible subset system we can define a similar quotient transformation semigroup, but the operation $*$ is no longer uniquely determined and so we have the possibility of a collection of quotient transformation semigroups associated with the admissible subset system $\pi$.

We now quote a standard result.

**Theorem 1.** Let $X = (Q, S)$ be a transformation semigroup and $\pi$, $\tau$ mutually orthogonal partitions, then

$$X < X/(\langle \pi \rangle) \times X/(\langle \tau \rangle).$$

Now let $h : A \rightarrow Z$ be a height function for $X = (Q, S)$. (Note that Eilenberg’s definition of a height function, [1, Chapter II, §6, condition 6.10] should read: ‘$b < a$ with $|a| > 1$ implies $bh < ah$’.)

If $\pi = \{H_i\}_{i \in J}$ is an admissible subset system then we say that $\pi$ is of **rank** $i$ if

(a) $H_j \in \pi \Rightarrow H_i \in A \ (j \in J)$.

(b) $H_i h \leq i \ (j \in J),$

(c) $H_i h = i \text{ for some } j \in J.$

**Theorem 2.** If $\pi$ is an admissible subset system of rank $i$, then there exists an admissible subset system $\pi'$ of rank $i - 1$ with $\pi' \subseteq \pi$.

**Proof.** We briefly sketch the construction of $\pi'$ from $\pi$.

Suppose $\pi = \{H_i\}_{i \in J}$ and $J = K \cup L$ where

$$i \in K \iff H_i h = i \quad \text{and} \quad j \in L \iff H_j h < i.$$

For each $H_i \ (j \in K)$ define $B_{H_i}$ to be the paving of $H_i$, i.e. $B_{H_i} = \{a \in A \mid a \subseteq H_i \text{ such that if } b \in A \text{ and } a \subseteq b \subseteq H_i \text{ then either } b = a \text{ or } b = H_i \}$. Now put $\pi' = \{H_i\}_{i \in L} = (\bigcup_{i \in K} B_{H_i})$. 
Now consider the trivial admissible subset system of rank \( n \) of \( X = (Q, S) \) defined by \( \pi^n = \{ Q \} \) where \( n = Qh \).

Define \( \pi^{n-1} = (\pi^n)' \), \( \pi^{n-2} = (\pi^{n-1})' \), \ldots etc. to obtain a sequence of admissible subset systems

\[
\pi^n \succ \pi^{n-1} \succ \cdots \succ \pi^1
\]

such that \( \pi^r \) is of rank \( r \) (\( 1 \leq r \leq n \)).

We call this the derived sequence of \( X \) (with respect to the given height function \( h : A \to \mathbb{Z} \)).

The following lemma corresponds to Proposition 7.3 of Chapter II of [1].

**Lemma 3.** Let \( X \Delta_{\alpha_i} Y \) be a relational covering of rank \( j \) such that the image of \( \alpha_i \) is \( \pi^j \). There exists a relational covering \( X \Delta_{\alpha_i} , H_{j_i} \circ Y \) such that

(i) the rank of \( \alpha_{i-1} \) is \( j - 1 \),

(ii) the image of \( \alpha_{i-1} \) is \( \pi^{j-1} \).

**Theorem 4.** (The holonomy decomposition theorem). If \( h : A \to \mathbb{Z} \) is a height function for the transformation semigroup \( X = (Q, S) \) and \( Qh = n \), then

\[
X < \overline{H_1} \circ \cdots \circ \overline{H_n}.
\]

**Corollary 5.** \( X < \overline{H_1} \circ \cdots \circ \overline{H_{n-1}} \circ X/(\pi^{n-1}) \) where \( X/(\pi^{n-1}) \) is a suitably chosen quotient transformation semigroup associated with \( \pi^{n-1} \), the first non-trivial element in the derived sequence of \( X \).

(Note that there are other holonomy decompositions involving the generalized transformation groups \( H^+_{i_i} , H^-_{i_i} \) but we will not mention these again here.)

In the next paragraph we will be considering holonomy transformation groups of several transformation semigroups so we will write \( H' (X) \) to denote the holonomy transformation semigroup of \( X \) of height \( i \) (relative to a given height function). Similarly \( A(X) \) will denote the skeleton of the transformation semigroup \( X \).

**Theorem 6.** Let \( X = (Q, S) \) be a transformation semigroup and \( h : A(X) \to \mathbb{Z} \) a height function for \( X \). Consider the derived sequence \( \pi^n \succ \pi^{n-1} \succ \cdots \succ \pi^1 \) of \( X \), where \( Qh = n \). Suppose that \( \pi^p \) and \( \tau \) are mutually orthogonal admissible partitions satisfying the conditions:

(i) \( 1 < p < n \) and

(ii) no admissible partition \( \tau' \) exists such that \( \pi^q \) and \( \tau' \) are mutually orthogonal for \( p < q < n \).
Then
\[ X < X/\langle \tau \rangle \times \overline{H_{p+1}}(X) \circ \cdots \circ \overline{H_n}(X). \]

**Proof.** This follows from Theorem 1, Lemma 3 and from the construction of the derived sequence.

Now we may introduce a height function for the transformation semigroup \( X/\langle \tau \rangle \) and treat the decomposition of this independently from the decomposition of \( X \). The process of theorem 6 can then be continued if the derived sequence of \( X/\langle \tau \rangle \) contains any orthogonal admissible partitions and so on. Each time we use Theorem 6 we will introduce a direct product into the decomposition instead of a wreath product. This will lead to an improved result. Notice, however that Theorem 6 only applies to certain transformation semigroups. If \( X = (Q, S) \) possesses no (non-trivial) orthogonal admissible partitions then this method will not lead to an improved decomposition of \( X \).

We conclude by looking at two examples.

**Example 1.** The cyclic transformation semigroup \( C_{(p,r)} \) is defined by the diagram

\[
\begin{array}{cccccccc}
1 & \xrightarrow{\sigma} & \sigma & \xrightarrow{\sigma} & \sigma^2 & \xrightarrow{\sigma} & \cdots & \sigma^r \\
& & \alpha & & \alpha & & &
\end{array}
\]

where \( r \) and \( p \) are non-negative integers \((p \geq 1)\). The standard holonomy decomposition yields \( C_{(p,r)} < \mathbb{Z}_p \times \mathbb{Z}^r \) but Eilenberg observes that a better decomposition, namely \( C_{(p,r)} < \mathbb{Z}_p \times \langle \mathbb{Z}^r \circ \mathbb{C} \rangle \), exists. We will apply Theorem 6 directly to obtain this decomposition. The element \( \pi^1 \) of the derived sequence of \( C_{(p,r)} \) is an orthogonal admissible partition, namely

\[ \pi^1 = \{[1], \{\sigma\}, \ldots, \{\sigma^{r-1}\}, \{\sigma^r, \ldots, \sigma^{r+p-1}\}\} \]

we define the partition \( \tau \) as follows:

- if \( p < r \) and \( r = qp + t \) with \( 0 \leq t < p \) put

\[
\tau = \{(1, \sigma^p, \ldots, \sigma^{(q+1)p-1}), \{\sigma, \sigma^{p+1}, \ldots, \sigma^{(q+1)p-1}\}, \ldots, \{\sigma^{t-1}, \sigma^{p+t-1}, \ldots, \sigma^{r+p-1}\}, \{\sigma^t, \sigma^{p+t}, \ldots, \sigma^r\}, \{\sigma^{r+1}, \sigma^{p+r+1}, \ldots, \sigma^{r+1}\}, \ldots, \{\sigma^{p-1}, \sigma^{2p-1}, \ldots, \sigma^{(q+1)p-1}\}\};
\]

- if \( p = r \) put

\[
\tau = \{(1, \sigma^r), \{\sigma, \sigma^{r+1}\}, \ldots, \{\sigma^{r-1}, \sigma^{2r-1}\}\};
\]
- if $p > r$ put
  \[ \tau = \{(\sigma'),\{\sigma'^+\}, \ldots,\{\sigma^{p-1}\},\{1, \sigma^n\},\{\sigma,\sigma^{n+1}\}, \ldots,\{\sigma^{-1}, \sigma^{r+p-1}\}\}. \]

In all cases $\pi^1 \cap \tau = 1_{\mathbb{C}(p,r)}$ and

\[ C_{(p,r)} < C_{(p,r)}/(\tau) \times C_{(p,r)}/(\pi^1) \]

Now $C_{(p,r)}/(\tau) \cong \mathbb{Z}_p$ and $C_{(p,r)}/(\pi^1) \cong C_{(1,r)}$.

Thus $C_{(p,r)} < \mathbb{Z}_p \times C_{(1,r)}$.

Now applying Corollary 5 to $C_{(1,r)}$ yields

\[ C_{(1,r)} < \mathbb{Z}^{r-1} \circ C \]

and so

\[ C_{(p,r)} < \mathbb{Z}_p \times (\mathbb{Z}^{r-1} \circ C) \]

as required.

**Example 2.** Let $X = (Q, S)$ be given by the diagram

\[ \begin{array}{c}
| & & | \\
\downarrow & \quad P & \quad \uparrow \\
| & & | \\
\downarrow & \quad q & \quad \uparrow \\
| & & | \\
\downarrow & \quad n & \quad \uparrow \\
| & & | \\
\downarrow & \quad 0 & \quad \uparrow \\
| & & | \\
\downarrow & \quad \tau & \quad \uparrow \\
| & & | \\
\downarrow & \quad 1 & \quad \uparrow \\
| & & | \\
\downarrow & \quad n & \quad \uparrow \\
| & & | \\
\downarrow & \quad 2 & \quad \uparrow \\
| & & | \\
\downarrow & \quad n & \quad \uparrow \\
| & & | \\
\downarrow & \quad 3 & \quad \uparrow \\
| & & | \\
\downarrow & \quad n & \quad \uparrow \\
| & & | \\
\downarrow & \quad 4 & \quad \uparrow \\
| & & | \\
\downarrow & \quad 5 & \quad \uparrow \\
| & & | \\
\downarrow & \quad n & \quad \uparrow \\
| & & | \\
\end{array} \]

This is Example 9.7 of Chapter II of [1]. Eilenberg obtains the decomposition $X < \mathbb{Z}_2 \circ \mathbb{Z}_2 \circ \mathbb{Z}_2 \circ \mathbb{Z}_2$.

Using Theorem 6, however, we obtain

\[ X < \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \]

\[ < (\mathbb{Z}_2 \circ \mathbb{Z}_2) \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2 \]

\[ < [(\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (\mathbb{Z}_2 \times \mathbb{Z}_2)] \times \mathbb{Z}_2 \]

\[ < (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \]

\[ = (\mathbb{Z}_2 \times \mathbb{Z}_2) \circ \mathbb{Z}_2 \circ \mathbb{Z}_2 \circ \mathbb{Z}_2 \]

The existence of an orthogonal admissible partition also allows us to improve our calculation of the complexity of a transformation semigroup. For example if

$C(X)$ denotes the complexity of $X = (Q, S)$, then $C(X/(\pi)) = C(X)$ where $\pi$ is orthogonal since $C(X/(\pi)) \leq C(X)$ and $X \leq X/(\pi) \times X/(\tau)$ for an admissible partition $\tau$ satisfying $\pi \cap \tau = 1_X$, implies that $C(X) \leq \max(C(X/(\pi)), C(X/(\tau))) \leq C(X)$. 


Hence we need only look at the complexity of $C(X/\langle \tau \rangle)$. In Example 2 we can immediately deduce that

$$C(X) = C(X/\langle \pi \rangle) = C(\tilde{Z}) = 1.$$  

This was not immediately apparent from the original holonomy decomposition of $X$.

Reference