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Discrete Mathematics 276 (2004) 111–125
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On i^--ER-critical graphs

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Abstract

We are interested in the behaviour of the independent domination number $i(G)$ of a graph G under edge deletion, and in particular in i^--ER-critical graphs, i.e., graphs for which $i(G)$ decreases whenever an edge e is removed. If $\gamma(G)$ denotes the domination number of G, we determine all the i^--ER-critical graphs G such that $\gamma(G)=2$ and $i(G-e)=2$ for every edge e of G. Different classes of i^--ER-critical graphs such that $i(G-e)>\gamma(G)$ for all or some edges e are described. Finally, for a particular family of circulants, we find the exact value of $i(G-e)$ for every edge e of the graphs of this family and obtain as a corollary the number of automorphism classes of their edge sets.

Keywords: Graph; Domination; Independence; Criticality

1. Introduction

We consider simple graphs $G=(V(G),E(G))$ of order $|V(G)|=n$ and size $|E(G)|=m$. The *neighbourhood* $N(v)$ of a vertex v is $\{u\in V(G) : uv\in E(G)\}$ and its *closed neighbourhood* $N[v]$ is $\{v\}\cup N(v)$. If X and Y are two subsets of $V(G)$, we define $N(X)=\bigcup\{N(x) : x\in X\}$, $N_Y(X)=N(X)\cap Y$ and $E(X,Y)=\{xy\in E(G) : x\in X$ and $y\in Y\}$. We refer the reader to [5] for domination-related concepts not defined here. The minimum cardinality of a set X which is both independent and dominating (or equivalently, maximal independent) is the *independent domination number* and is denoted by $i(G)$. An independent dominating set of order $i(G)$ of G is called an *i-set* of G. Recall that $i(G)\geqslant\gamma(G)$ for all graphs G, where as usual $\gamma(G)$ denotes the domination number of G.

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0012-365X/$ - see front matter
doi:10.1016/S0012-365X(03)00302-9

For many graph parameters, such as those of connectedness, colouring or domination, the study of critical, minimal or maximal graphs under vertex removal, edge removal or edge addition (in this last case, the added edge belongs to the complementary graph $\bar{G}$ of G) is classical. For criticality with respect to domination related parameters, see for instance [1–4,6] and the bibliography in [5]. When we remove an edge from a graph G, the independent domination number $i(G)$ can increase or decrease. For instance, if G is a star $K_{1,p}$ then $i(G)=1$ and $i(G-e)=2$ for all e. If G is a double star formed by two stars $K_{1,p}$ and an edge e joining their centers, $i(G)=p+1$ and $i(G-e)=2$. The graph G is said to be *i^--edge-removal-critical*, *i^--ER-critical* for short, if $i(G-e)<i(G)$ for every edge $e\in E(G)$. Similarly, G is i-ER-critical if $i(G-e)>i(G))$ for every edge. Ao [1] proved that the class of i-ER-critical graphs consists of disjoint union of stars. We have no similar description of i^--ER-critical graphs, which form a much larger and more complicated class. Some of them have been described in [4] and we continue this work here.

In Sections 2 and 3 we construct families of i^--ER-critical graphs with $\gamma(G)=2$ or 3. In Section 4 we completely determine the value of $i(G-e)$ for every edge e of a graph belonging to a particular family of circulants. As $i(G)$ is an invariant, the knowledge of $i(G-e)$ gives interesting information on the classes induced in the edge set $E(G)$ by the automorphism group of G.

Let us begin with some easy observations on i^--ER-critical graphs:

O1. Since $\gamma(G-e)\geqslant\gamma(G)$ for every graph G and every edge $e\in E(G)$, $i(G-e)\geqslant\gamma(G)$ for every edge. Therefore if G is i^--ER-critical, then $i(G)>\gamma(G)$.

O2. The graph G satisfies $\gamma(G)=1$ if and only if G contains a universal vertex, i.e., a vertex v such that $N[v]=V(G)$, and in this case $i(G)=\gamma(G)=1$. Hence if G is i^--ER-critical, then $\gamma(G)\geqslant 2$.

O3. If G is not connected, say $G=G_1\cup G_2\cup\cdots\cup G_p$, then $\gamma(G)=\sum_{j=1}^{p}\gamma(G_j)$, $i(G)=\sum_{j=1}^{p}i(G_j)$, and G is i^--ER-critical if and only if each nontrivial component of G is i^--ER-critical. In particular, every i^--ER-critical graph with $\gamma(G)=2$ (respectively with $\gamma(G)=3$ and without isolated vertices) is connected.

O4. No graph of order $n\leqslant 5$ is i^--ER-critical.

O5. If $i(G-uv)<i(G)$ for some edge uv of G, then $\{u,v\}$ is contained in some i-set of $G-uv$.

More generally, the next result improves Lemma 16 in [4].

Proposition 1. *A graph G is i^--ER-critical if and only if for every $uv\in E(G)$, G has a dominating set D with $|D|<i(G)$ such that u and v are the only non-isolated vertices in $G[D]$.*

Proof. Let uv be any edge of G. If G has a dominating set D with $|D|<i(G)$, where uv is the unique edge of $G[D]$, then D is an independent dominating set of $G-uv$ and thus $i(G-uv)\leqslant|D|<i(G)$. Therefore G is i^--ER-critical.

Conversely, if G is i^--ER-critical, let uv be any edge of G and let I be an i-set of $G-uv$. Then $|I|=i(G-uv)<i(G)$. If $\{u,v\}\nsubseteq I$, then I is an independent dominating

set of G with fewer than $i(G)$ vertices, a contradiction. Hence $\{u,v\} \subseteq I$ and I is a dominating set of G containing the unique edge uv. □

2. i^--ER-critical graphs with domination number two

As seen in Observation O3, every i^--ER-critical graph with $\gamma(G)=2$ is connected. We first characterise the i^--ER-critical graphs such that $i(G-e)=\gamma(G)=2$ for every edge e of G.

Theorem 2. *The graph G is i^--ER-critical with $\gamma(G)=2$ and $i(G-e)=2$ for every $e\in E(G)$ if and only if G is a complete multipartite graph $K_{n_1,n_2,\dots,n_p}$ with $p \geqslant 2$ and $n_j \geqslant 3$ for $1 \leqslant j \leqslant p$.*

Proof. The sufficiency is immediate since $i(K_{n_1,n_2,\dots,n_p}) = \min\{n_1, n_2, \dots, n_p\} \geqslant 3$ and $i(K_{n_1,n_2,\dots,n_p} - e) = 2$ for every edge e.

Conversely let G be a graph satisfying $\gamma(G)=2$, $i(G) \geqslant 3$ and $i(G-e)=2$ for every edge e. By Proposition 1, this implies that for any pair x, y of vertices of G, $\{x, y\}$ is a dominating set of G if and only if xy is an edge of G. By Observation O4, we know that $n \geqslant 6$ and we consider a γ-set $\{x, y\}$ of G (hence $xy\in E(G)$). Let $X = N(x) - N[y]$, $Y = N(y) - N[x]$ and $Z = N(x) \cap N(y)$. Note that $X \neq \phi$, for otherwise y dominates G, a contradiction. Similarly, $Y \neq \phi$. If $G[X]$ contains an edge aa', the set $\{a, a'\}$ dominates G and in particular y, in contradiction to the definition of X. Hence $G[X]$, and similarly $G[Y]$, is independent. Moreover, $G[X, Y]$ is complete bipartite since for every vertex a of X, $\{a, x\}$ dominates Y. We complete the proof by induction on $n \geqslant 6$.

If $n=6$ then, as $i(G) \geqslant 3$, x and y have degree at most 3 and necessarily $Z=\phi$, $|X|=|Y|=2$ and $G \cong K_{3,3}$.

For $n \geqslant 7$, if $Z=\phi$ then G is the complete bipartite graph with bipartition $(X \cup \{y\}, Y \cup \{x\})$. If $Z \neq \phi$ then for every vertex c of Z, $\{x, c\}$ dominates Y and $\{y, c\}$ dominates X. Hence all the edges between Z and X, and between Z and Y, are in G, whence every dominating set of $G[Z]$ is a dominating set of G and thus $\gamma(G[Z]) \geqslant 2$ and $i(G[Z]) \geqslant 3$. If Z is independent, the graph G is complete tripartite with partition $(X \cup \{y\}, Y \cup \{x\}, Z)$. Otherwise, for every edge cc' of $G[Z]$, $\{c, c'\}$ dominates G, and in particular Z, and therefore $\gamma(G[Z]) = i(G[Z] - cc') = 2$. By the induction hypothesis, $G[Z]$ is a complete multipartite graph with partition $(Z_1, \dots, Z_q)$. Hence G is also a complete multipartite graph with partition $(X \cup \{y\}, Y \cup \{x\}, Z_1, \dots, Z_q)$. Since the value of $i(G)$ is the minimum order of the classes in the partition of $V(G)$, each class has at least three elements. □

The next corollary is obvious since if G is i^--ER-critical with $i(G)=3$, then $\gamma(G)=2$ and $i(G-e)=2$ for every edge e of G.

Corollary 3. *The i^--ER-critical graphs with independent domination number 3 are the complete multipartite graphs $K_{n_1,n_2,\dots,n_p}$ with $p \geqslant 2$ and $3 = n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p$.*

We now describe two families of i^--ER-critical graphs with $\gamma(G)=2$ but for which $i(G-e)>2$ for some edges e of G. They can be seen as generalisations of the family of Theorem 2.

The family $\mathscr{F}_1$: Let $H_1, H_2, \ldots, H_p$ be disjoint i^--ER-critical graphs with $i(H_j)=q \geqslant 4$ for $1 \leqslant j \leqslant p$ and $i(H_j-e) \geqslant \gamma(H_j) \geqslant 3$ for every edge of H_j. (The next section contains proofs that such graphs exist for any value of $q \geqslant 4$.) We construct G by adding all possible edges between the H_j. Then $\gamma(G)=2$ and $i(G)=q$ since the only independent dominating sets of G are independent dominating sets of some H_j. If $e \in E(H_j)$ for some j, then $i(G-e)=i(H_j-e)<i(H_j)=i(G)$. If $e \in E(H_j, H_k)$ for some $j \neq k$, then $i(G-e)=2<i(G)$. Therefore G is i^--ER-critical with $\gamma(G)=2$ and $i(G-e)>2$ for each edge e of the H_j's.

The family $\mathscr{F}_2$: The graphs G of Family $\mathscr{F}_2$ are constructed as follows.

- $V(G)=S_1 \cup S_2 \cup \cdots \cup S_p$, where
  - (i) $p \geqslant 3$ and each S_j is independent,
  - (ii) $\min_{1 \leqslant j \leqslant p} |S_j| = q \geqslant 4$,
  - (iii) $|S_j \cap S_k| \leqslant q-3$ for all $1 \leqslant j \neq k \leqslant p$,
  - (iv) if $S_j \cap S_k \neq \phi$ and $S_j \cap S_l \neq \phi$, then $S_k \cap S_l = \phi$,
  - (v) for at least one pair u, v of vertices, $S_u \cap S_v = \phi$ for every set S_u containing u and every set S_v containing v.
- The set $E(G)$ consists of all edges joining two vertices not contained in the same S_j.

By (iv), each vertex belongs to at most two S_j's and has degree at least one. The two vertices u and v of (v) are adjacent and form a dominating set of G. Hence $\gamma(G) \leqslant 2$ and since G has no universal vertex, $\gamma(G)=2$. The maximal independent sets of G are the sets S_j and thus $i(G)=q$.

Let $xy \in E(G)$. If $S_x \cap S_y = \phi$ for every set S_x (respectively S_y) containing x (respectively y), then the set $\{x,y\}$ dominates G and $i(G-xy)=2$. If $S=S_x \cap S_y \neq \phi$ for some S_x and S_y, then neither x nor y dominates S, hence $i(G-xy)>2$ and by (iv), $w \notin S_j$ for every $w \in S$ and every S_j different from S_x and S_y. Therefore $\{x,y\} \cup S$, which contains xy as its only edge, dominates G and thus $i(G-xy) \leqslant |S|+2 \leqslant q-1$ by (iii). Hence $3 \leqslant i(G-xy) < i(G)$.

Therefore G is i^--ER-critical with $\gamma(G)=2$ and if the S_j's are not all disjoint, $i(G-e)$ is not always equal to $\gamma(G)$. When $S_j \cap S_k = \phi$ for all $j \neq k$, the graph G is a complete multipartite graph obtained in Theorem 2.

3. *i*-ER-critical graphs with domination number at least three

In this section we construct two families $\mathscr{F}_3$ and $\mathscr{F}_4$ of i^--ER-critical graphs with $\gamma(G)=3$ and $i(G)$ arbitrarily large. For the graphs G of $\mathscr{F}_3$, $i(G-e)=3$ for every edge of G while for the graphs G of $\mathscr{F}_4$, $i(G-e)>3$ for every edge of G.

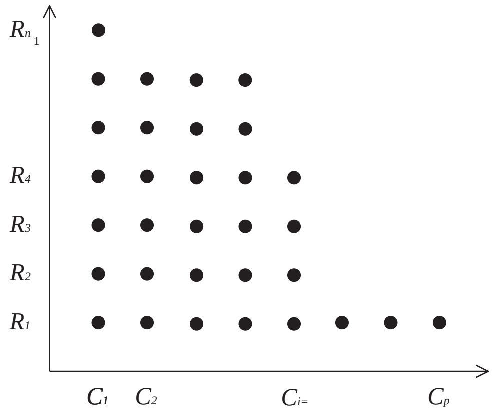

Fig. 1. An array for the graphs in $\mathscr{F}_3$.

The family $\mathscr{F}_3$: Let $\mathbb{Z}^+$ denote the set of positive integers and consider an array $\mathbf{A}$ of points (i,j) in $\mathbb{Z}^+ \times \mathbb{Z}^+$ whose rows R_j and columns C_i are of nonincreasing length as j and i increase. Specifically (see Fig. 1 for an example), for $i \in \{1,\dots,p\},\ p \geqslant 4$, we define the columns C_i by

$$C_i = \{(i,y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : y \in \{1,\dots,n_i\}\},$$

where $n_1 \geqslant \cdots \geqslant n_p$, $n_2 = n_3 = n_4 \geqslant 4$, and if $n_i \neq 1$, then $n_i \geqslant 4$. Let i' be the largest index such that $n_{i'} \neq 1$; note that $i' \in \{4,\dots,i\}$. It follows that the rows R_j satisfy

$$|R_1| = p,$$

$$|R_2| = |R_3| = |R_4| = n_{i'} \geqslant 4$$

(the choice of i' implies that $n_{i'} \geqslant 4$, $n_i = 1$ if $i > i'$, and $n_{i'} - n_{i'+1} \geqslant 3$, whence the equalities) and

$$|R_j| \neq 1 \Rightarrow |R_j| \geqslant 4$$

since $n_2 = n_3 = n_4$. Let j' be the largest index such that $|R_{j'}| \neq 1$.

Now let $H_{\mathbf{A}}$ be the graph with vertex set

$$V(H_{\mathbf{A}}) = \{(x,y) : (x,y) \in \mathbf{A}\}$$

and edge set

$$E(H_{\mathbf{A}}) = \{\{(x,y),(x',y')\} : x \neq x' \text{ and } y \neq y'\}$$

and let G be any graph isomorphic to $H_{\mathbf{A}}$. The family $\mathscr{F}_3$ consists of all graphs G constructed in this way.

No vertex of G is universal. If u and v are two non-adjacent vertices of G, we may suppose without loss of generality that $u,v \in C_i$ for some i, and then $C_i - \{u,v\} \neq \phi$ is not dominated by $\{u,v\}$. If u and v are two adjacent vertices of G, say $u = (i,j)$ and

$v=(k,l)$ with $i<k$ and $j\neq l$, then (i,l) is not dominated by $\{u,v\}$. Hence $\gamma(G)>2$. Since $\{(1,1),(1,2),(2,1)\}$ dominates G, $\gamma(G)=3$.

The maximal independent sets of G are the rows R_j and columns C_i of length bigger than 1. Therefore $i(G)=\min(|R_{j'}|,n_{i'})\geqslant 4$. For any edge $e=\{(i,j),(k,l)\}$ of G with $i<k$ and $j\neq l$, $\{(i,j),(i,l),(k,l)\}$ is an independent dominating set of $G-e$. Therefore $3=\gamma(G)\leqslant i(G-e)\leqslant 3$, that is, $i(G-e)=3$. Hence the graphs G of $\mathscr{F}_3$ are i^--ER-critical with $\gamma(G)=3$ and $i(G-e)=3$ for every edge e.

Remark. Note that if $n_i=p$ for all $i\in\{1,\dots,p\}$, then $G\cong\overline{K_p\times K_p}$. Then $i(G)=p$, $i(G-e)=3$ for every edge of G and $i(G+e)=p-1$ for every edge $e\in E(\bar{G})$. This provides an example of graphs which are both i^--ER-critical and i^--EA-critical (i.e., $i(G)$ decreases under any edge addition).

The family $\mathscr{F}_4$: Consider two complete bipartite graphs with respective bipartitions (X,Y) and (Z,T). Let $|X|=p$, $|Y|=q$, $|Z|=r$, $|T|=s$, and suppose $4\leqslant p<q<r<s$ and $p+r>s+2$. The graph G is obtained by joining an extra vertex w to all the vertices in $X\cup Z$. Let x $(y,z,t$, respectively) be any vertex of X $(Y,Z,T$, respectively). The automorphism group of G induces four orbits on the edge set of G, respectively, formed by the edges of type wx,wz,xy,zt. Hence $i(G-e)$ takes at most four different values. It is easy to check that $\{x,w,z\}$ is a γ-set of G, and that $X\cup Z$ (respectively $\{w,x\}\cup T$, $\{w,z\}\cup Y$, $\{x,y\}\cup Z$, $\{z,t\}\cup X$) is an i-set of G (respectively of $G-wx$, $G-wz$, $G-xy$, $G-zt$). Hence $\gamma(G)=3$, $i(G)=p+r$, $i(G-wx)=s+2$, $i(G-xy)=r+2$, $i(G-wz)=q+2$, $i(g-zt)=p+2$ and thus

$$\gamma(G)<i(G-zt)<i(G-wz)<i(G-xy)<i(G-wx)<i(G).$$

Therefore G is an i^--ER-critical graph with $i(G-e)>\gamma(G)=3$ for every edge e of G.

By starting with more than two complete bipartite graphs, the family $\mathscr{F}_4$ can be generalised to i^--ER-critical graphs with higher values of $\gamma(G)$. For instance, the graph G obtained from $q\geqslant 2$ complete bipartite graphs $K_{p,p}$ with $p\geqslant 3$ by adding an extra vertex w joined to all the vertices of one class of each $K_{p,p}$ satisfies

$$\gamma(G)=q+1,\ i(G-e)=(q-1)p+2>\gamma(G)$$

for every edge e and

$$i(G)=pq>i(G-e).$$

Note that the graphs in the families $\mathscr{F}_3$ and $\mathscr{F}_4$ and its generalisation mentioned above prove that given any two integers $\gamma\geqslant 3$ and $i\geqslant 4$, there exists an i^--ER-critical graph G with $\gamma(G)=\gamma$ and $i(G)=i$.

4. The i^--ER-criticality of a family of circulants

In this section we determine the effect of the removal of an edge on the independent domination number of the graphs of a particular family of circulants. This study was initiated in [4] and we complete it here by finding the value of $i(G-e)$ for all the edges e of G.

The family $\mathscr{F}_5$: The graphs $G=C_n\ \langle 1,3,\ldots,2r-1\rangle$ of $\mathscr{F}_5$ are defined as follows: given the positive integers r,m,q, with $m\geqslant 2$, q odd and $1\leqslant q\leqslant 2r-1$, the order of G is $n=m(2r+1)+q$ and its vertices are labelled $v_1,v_2,\ldots,v_n=v_0$. The edges of G are $v_iv_{i+1},\ v_iv_{i+3},\ldots,v_iv_{i+2r-1}$ with $1\leqslant i\leqslant n$, where the indices are taken modulo n. The labelling of the vertices implicitly defines an orientation on the cycle $C=v_0v_1\cdots v_{n-1}v_n$. The neighbours $v_{i-1},\ v_{i-3},\ldots,v_{i-(2r-1)}$ of the vertex v_i are called its negative neighbours while $v_{i+1},\ v_{i+3},\ldots,v_{i+2r-1}$ are its positive neighbours. For two vertices v_i and v_j of G, $C[v_i,v_j]$ denotes the segment of C between v_i and v_j, respecting its orientation, while $C_G[v_i,v_j]$ denotes the subgraph of G induced by $\{v_i,v_{i+1},\ldots,v_j\}$. If S is a subset of vertices of G, we say that two of its vertices, v_i and v_j, are consecutive on C if one of the two arcs $v_iv_{i+1}\cdots v_j$ and $v_jv_{j+1}\cdots v_i$ of C contains no other vertex of S.

We note that for any graph G in $\mathscr{F}_5$, n and m have different parities, G is vertex-transitive of degree $2r$, and the automorphism group of G induces at most r orbits in the edge set, each of them represented by an edge of the type v_0v_l, $1\leqslant l_{\text{odd}}\leqslant 2r-1$.

Lemma 4. *Let r be a positive integer, C an arbitrarily oriented path or cycle of length at least $2r+3$, and H the graph obtained from C by adding an edge between each pair of vertices at distance $3,5,\ldots,2r-1$ on C. If S is an independent dominating set of H, then any two consecutive vertices of S are at distance 2 or $2r+1$ on C.*

Proof. The vertices of C are labelled $1,2,\ldots,|V(C)|$ in accordance with its orientation. Let $a<b$ be two vertices of S which are consecutive on C. Since S is independent, $b\notin\{a+1,a+3,\ldots,a+2r-1\}$. Suppose $b>a+2$. The vertex $a+2$ must be dominated by S. All the negative neighbours $(a+2)-1,\ (a+2)-3,\ldots,(a+2)-(2r+1)$ of $a+2$ in H, if they exist in the case of a path, are adjacent to a and thus are not in S. Similarly, all the positive neighbours $a+3,\ a+5,\ldots,a+2r+1$ of $a+2$, if they exist, are adjacent to a and thus do not belong to S, except the last one. Hence $a+2r+1\in S$. Moreover the set $\{a,a+2r+1\}$ dominates $C[a,a+2r+1]$ and thus $a+2r+1$ is the first vertex of S after a in C. Therefore $a+2r+1=b$. □

Note that, following the notation of Lemma 4, any pair $\{a,b\}$ of vertices of C with $b-a$ odd and $3\leqslant b-a\leqslant 2r+1$ dominates the whole set $\{a,a+1,a+2,\ldots,b-1,b\}$.

The values of the parameters $\gamma(G)$, $i(G)$ and $\beta(G)$ for the graphs G of a family of circulants more general than $\mathscr{F}_5$ were determined in [4]. Here we just need $\gamma(G)$ and $i(G)$ when $G\in\mathscr{F}_5$. Theorem 5 gives a short proof to determine these values by means of a technique, based on Lemma 4, which will be used repeatedly in the proof of Theorem 6.

Theorem 5. *Let r,m,q be positive integers such that $m\geqslant 2$ and $1\leqslant q_{\text{odd}}\leqslant 2r-1$, and let G be the circulant $C_n\ \langle 1,3,\ldots,2r-1\rangle$ with $n=m(2r+1)+q$. Then $\gamma(G)=m+1$ and $i(G)=m+r+\frac{1}{2}(q-1)$.*

Proof. (1) The graph G is $2r$-regular and thus

$$\gamma(G) \geqslant \left\lceil \frac{n}{2r+1} \right\rceil = m + \left\lceil \frac{q}{2r+1} \right\rceil = m+1.$$

On the other hand, the set

$$D = \{v_0, v_{2r+1}, v_{2(2r+1)}, \dots, v_{m(2r+1)}\}$$

is a dominating set of $m+1$ elements. Indeed, as remarked above,

$$\{v_{k(2r+1)}, v_{(k+1)(2r+1)}\}$$

dominates the set $\{v_i : k(2r+1) \leqslant i \leqslant (k+1)(2r+1)\}$ for $0 \leqslant k \leqslant m-1$, and since $q = n - m(2r+1)$ is odd with $q \leqslant 2r-1$, $\{v_{m(2r+1)}, v_n\}$ dominates the set $\{v_i : m(2r+1) \leqslant i \leqslant n\}$. Hence $\gamma(G) = m+1$.

(2) By Lemma 4, two vertices of an independent dominating set I of G which are consecutive on the cycle $C = v_0 v_1 \cdots v_n$ are at distance 2 or $2r+1$ on this cycle. To construct I as small as possible, we partition C into as many intervals of length $2r+1$ as possible, and intervals of length 2. Since $n = m(2r+1)+q = (m-1)(2r+1)+2r+1+q$ with q odd, and thus $2r+1+q$ even, the greatest number of intervals of length $2r+1$ is $m-1$ and the number of intervals of length 2 is $\frac{1}{2}(2r+q+1)$. Hence

$$i(G) = m - 1 + \tfrac{1}{2}(2r+1+q) = m + r + \tfrac{1}{2}(q-1).$$

For instance, the set

$$I_1 = \{v_{2r}, v_{2(2r+1)-1}, \dots, v_{(m-1)(2r+1)-1}, v_{(m-1)(2r+1)+1}, v_{(m-1)(2r+1)+3}, \dots, v_{n-3}, v_{n-1}\}$$

is an i-set of G. $\square$

The following theorem determines the modification of $i(G)$ under the removal of any edge of G.

Theorem 6. *Let r, m, q be positive integers with $m \geqslant 2$, $r \geqslant 2$, q odd and $1 \leqslant q \leqslant 2r-1$, and let $n = (2r+1)m + q$. Let G be the circulant graph $C_n\langle 1, 3, \dots, 2r-1 \rangle$ and let e_l, with l odd, $1 \leqslant l \leqslant 2r-1$, be any edge of G joining two vertices at distance l on the cycle $C = v_0, v_1, \dots, v_n$. Then*

$$i(G - e_l) = \begin{cases} m + 1 + \dfrac{q-l}{2} & \textit{if } 1 \leqslant l_{\text{odd}} \leqslant q, \\[2ex] m + 1 + \dfrac{q+l}{2} & \textit{if } q < 2r-3 \textit{ and } q < l_{\text{odd}} \leqslant 2r-3, \\[2ex] m + r + \dfrac{q-1}{2} = i(G) & \textit{if } q < 2r-1 \textit{ and } l = 2r-1. \end{cases}$$

Proof. For each value of l, it is sufficient (by symmetry) to prove the result for one particular edge e_l. The proof is illustrated in Figs. 2–6 for $G = C_{38}\langle 1,3,5,7,9 \rangle$, i.e., $m = 3$, $r = 5$ and $q = 5$.

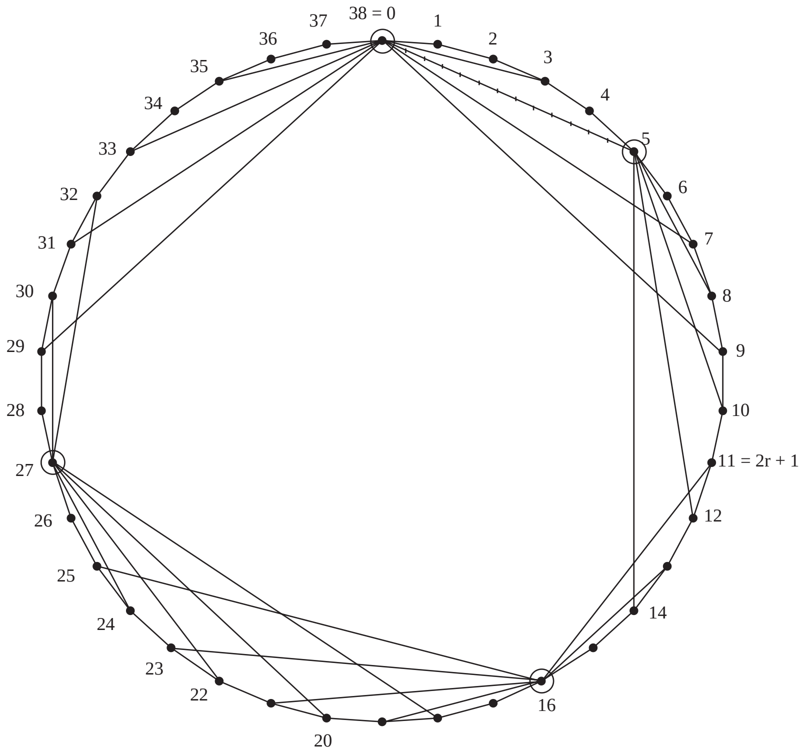

Fig. 2. $C_{38}\langle 1,3,5,7,9\rangle$ with $l=q=5$ and $i(G-e_5)=4=\gamma(G)$.

First consider the case $l=q$ (see Fig. 2). Let $e_q=v_0v_q$. The set

$$D=\{v_0,v_{2r+1},v_{2(2r+1)},\ldots,v_{m(2r+1)}\}$$

defined in the proof of Theorem 5, when rotated clockwise by q vertices, is an independent dominating set of $G-v_0v_q$. Hence

$$i(G-e_q)\leqslant m+1.$$

On the other hand,

$$i(G-e_q)\geqslant \gamma(G-e_q)\geqslant \gamma(G)=m+1$$

by Theorem 5, and thus

$$i(G-e_q)=m+1.$$

Now consider $l_{\mathrm{odd}}\neq q$ and the edge $e_l=v_0v_l$. Let us denote by $\mathscr{T}$ (respectively by $\mathscr{S}$) the set of the independent dominating sets of $G-e_l$ not containing $\{v_0,v_l\}$ (respectively containing $\{v_0,v_l\}$). Then

$$i(G-e_l)=\min\{|T|,|S|:T\in\mathscr{T}\text{ and }S\in\mathscr{S}\}.$$

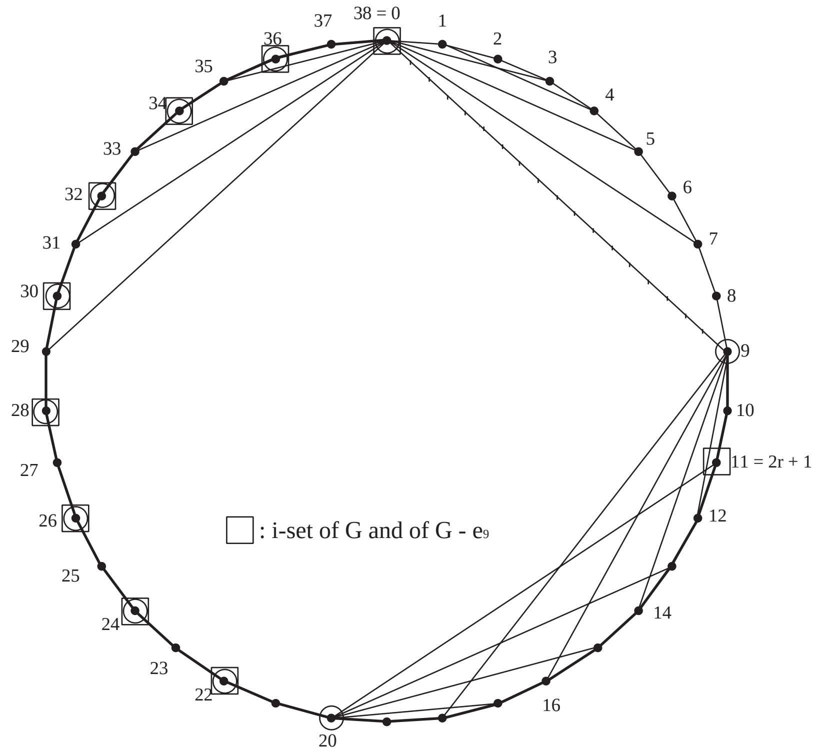

Fig. 3. $C_{38}\langle 1,3,5,7,9\rangle$ with $l = 9 = 2r - 1$, $S_0 = \{9,20,22,24,26,28,30,32,34,36,38\}$, $|S_0| = 11 > i(G) = i(G - e_9) = 10$.

A set T in $\mathscr{T}$ is also an independent dominating set of G and thus $|T| \geqslant i(G)$, so that $\min\{|T| : T \in \mathscr{T}\} \geqslant i(G)$. Since for every $l_{\text{odd}} \leqslant 2r-1$, the i-set I_1 of G given in the proof of Theorem 5 is an independent dominating set of $G - v_0v_l$ not containing $\{v_0, v_l\}$, $\min\{|T| : T \in \mathscr{T}\} = i(G)$. Therefore

$$i(G - e_l) = \min\{i(G), |S| : S \in \mathscr{S}\}.$$

Consider the case $q < 2r-1$, $l = 2r-1$, the edge $e_l = v_0v_l$ and the segment $C[v_l, v_0]$ of the cycle C. (See Fig. 3.) The independent dominating sets S of $G - e_l$ containing $\{v_0, v_l\}$ are disjoint from $\{v_1, v_2, \ldots, v_{l-1}\}$ and are independent dominating sets of the subgraph $C_G[v_l, v_0]$. By Lemma 4, the distance on C between two vertices of such a set S is 2 or $2r+1$. The segment $C[v_l, v_0]$ contains $n - (2r-2)$ vertices determining

$$\begin{aligned} m(2r+1) + q - 2r + 1 &= (m-1)(2r+1) + q + 2 \\ &= (m-2)(2r+1) + q + 2r + 3 \end{aligned}$$

intervals of length 1 on C. Since $q - 2r + 1$ is negative and $q + 2$ is odd, the number of intervals between two consecutive vertices of S is at least $m - 2 + \frac{1}{2}(q + 2r + 3)$.

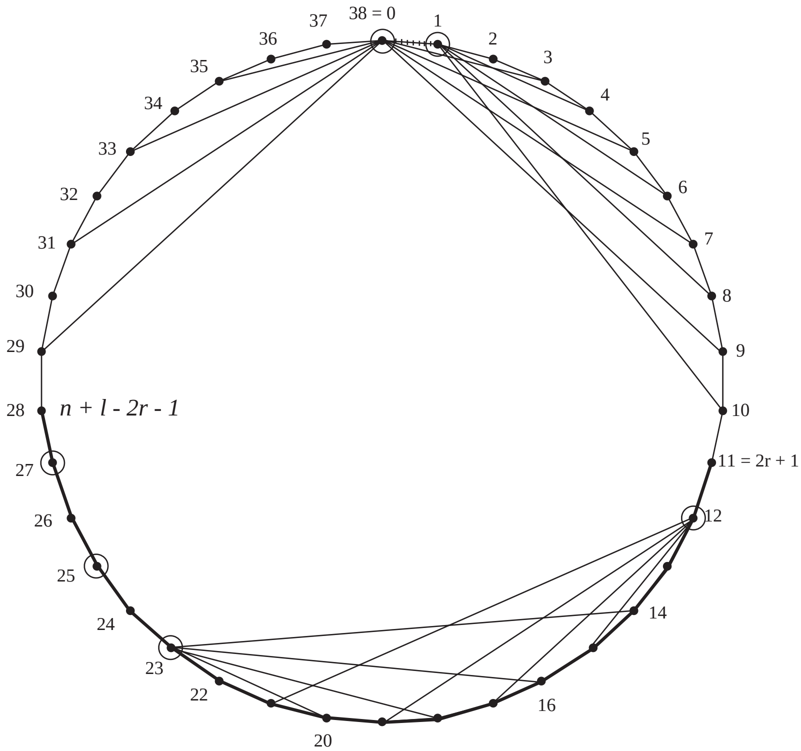

Fig. 4. $C_{38}\langle 1,3,5,7,9\rangle \in \mathscr{F}_5$ with $l=1$, $S''=\{12,23,25,27\}$, $i(G-e_1)=6$.

Hence

$$|S| \geqslant m-2+\tfrac{1}{2}(q+2r+3)+1 = m+r+\tfrac{1}{2}(q+1) > i(G),$$

and thus $i(G-e_l)=i(G)$. Note that the set

$$S_0 = \{v_l, v_{l+(2r+1)}, v_{l+2(2r+1)}, \ldots, v_{l+(m-2)(2r+1)}, v_{l+(m-2)(2r+1)+2},$$

$$v_{l+(m-2)(2r+1)+4}, \ldots, v_{l+(m-2)(2r+1)+q+2r+3}\},$$

where, since $l=2r-1$, $v_{l+(m-2)(2r+1)+q+2r+3}=v_n$, is an independent dominating set of $G-e_l$ containing $\{v_0,v_l\}$ and of order exactly $m+r+\frac{1}{2}(q+1)$.

In the remaining two cases $q>1$ and $1 \leqslant l_{\mathrm{odd}} < q$, or $q<2r-3$ and $q<l_{\mathrm{odd}} \leqslant 2r-3$, we consider as previously the edge $e_l=v_0v_l$ and the set $\mathscr{S}$ of the independent dominating sets of $G-e_l$ containing $\{v_0,v_l\}$. For each set S of $\mathscr{S}$, the set $S'=S-\{v_0,v_l\}$ is equal to $S\cap C[v_{2r+1},v_{n+l-2r-1}]$, where v_{2r+1} is the positive nonneighbour of v_0 at distance $2r+1$ from v_0 on C and $v_{n+l-2r-1}$ is the negative nonneighbour of v_l at distance $2r+1$ from v_l on C. Let S'' be any independent dominating set of $C_G[v_{2r+1},v_{n+l-2r-1}]-N_{C[v_{2r+1},v_{n+l-2r-1}]}(\{v_0,v_l\})$. If $l=1$ (see Fig. 4), then

$$N_{C[v_{2r+1},v_{n+l-2r-1}]}(\{v_0,v_l\})=\phi,$$

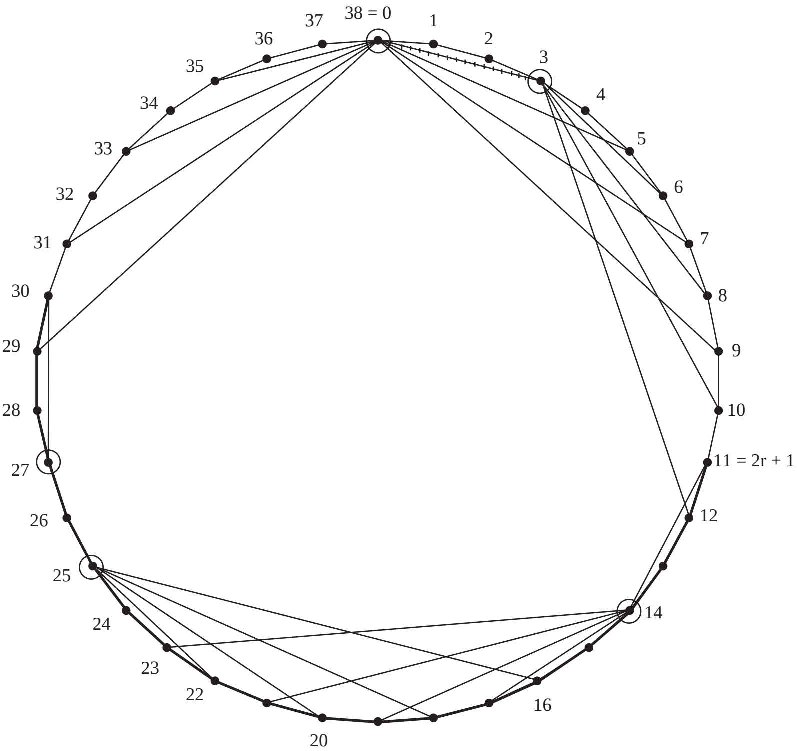

Fig. 5. $C_{38}\langle 1,3,5,7,9\rangle$ with $l=3$, $S''=\{14,25,27\}$ and $i(G-e_3)=5$.

while if $l>1$, then

$$N_{C[v_{2r+1},v_{n+l-2r-1}]}(v_l)=\{v_{2r+2},v_{2r+4},\dots,v_{2r-1+l}\}$$

and

$$N_{C[v_{2r+1},v_{n+l-2r-1}]}(v_0)=\{v_{n-2r+1},v_{n-2r+3},\dots,v_{n-2r+l-2}\}.$$

In the case $q>1$ and $1<l_{\mathrm{odd}}<q$ (Fig. 5), the set

$$S''=\{v_{2r+1+l},v_{2(2r+1)+l},v_{3(2r+1)+l},\dots,v_{(m-1)(2r+1)+l},v_{(m-1)(2r+1)+l+2},$$
$$v_{(m-1)(2r+1)+l+4},\dots,v_{(m-1)(2r+1)+l+(q-l)}\}$$

avoids the vertices of $N_{C[v_{2r+1},v_{n+l-2r-1}]}(\{v_0,v_l\})$ and is a suitable set S'. Its order is $|S''|=m-1+\frac{1}{2}(q-1)$. Adding the two vertices v_0 and v_l to S'' gives

$$i(G-e_l)\leqslant m+1+\tfrac{1}{2}(q-1).$$

It remains to prove that no set S' is smaller than S''. The vertices of S'' were chosen such that $(m-1)(2r+1)+l<n-2r+1$ (where $n-2r+1$ is the last neighbour

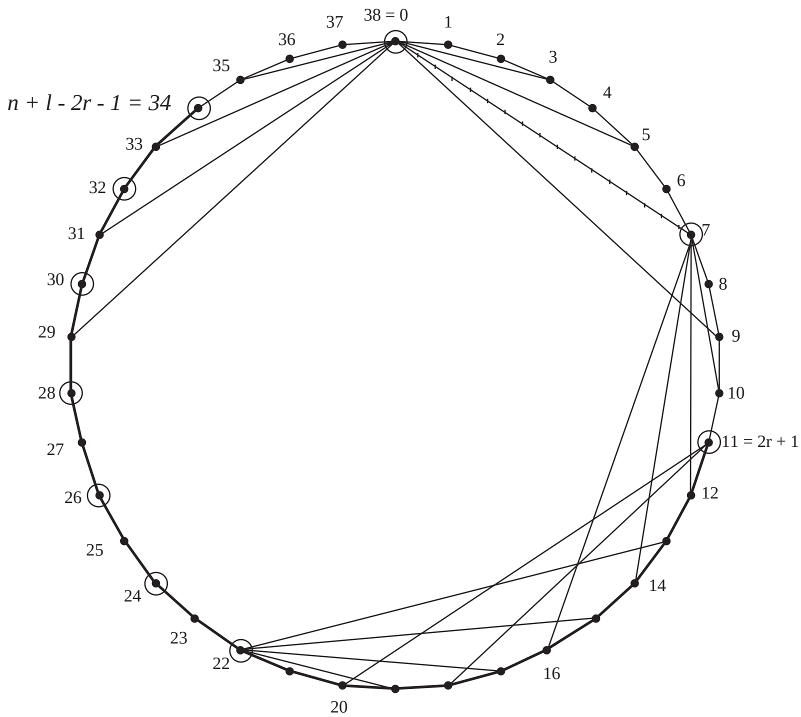

Fig. 6. $C_{38}\langle 1,3,5,7,9\rangle$ with $l=7=2r-3>9$, $S''=\{11,22,24,26,28,30,32,34\}$ and $i(G-e_7)=10=|S''|+2=i(G)$.

of v_0 in $C[v_{2r+1},v_{n+l-2r-1}]$), but $v_{m(2r+1)+l}$ is not in $C[v_{2r+1},v_{n+l-2r-1}]$ since $m(2r+1)+l>n+l-2r-1$. That is, assuming $v_{2r+1+l}\in S'$ and starting from this vertex, we took as many vertices at distance $2r+1$ on C as possible. So we cannot construct a set S' containing v_{2r+1} and smaller than S''. If the set S' does not contain v_{2r+1+l} then, in order to dominate v_{2r+1}, it contains v_{2r+1} itself or one of its neighbours v_{2r+1+j} with $l+2\leqslant j_{\mathrm{odd}}\leqslant 2r-1$, since the other neighbours of v_{2r+1} in $C[v_{2r+1},v_{n+l-2r-1}]$ are in $N_{C[v_{2r+1},v_{n+l-2r-1}]}(v_l)$. If S' contains one vertex v_{2r+1+j} of the second type, then, because of its independence, it cannot contain any neighbour of v_{2r+1+l} and thus cannot dominate v_{2r+1+l}. Hence S' contains v_{2r+1}. Starting from v_{2r+1}, can we construct S' with more intervals of length $2r+1$ between two consecutive vertices than in S'', that is with at least $m-1$ intervals of length $2r+1$? The answer is negative because the vertex $v_{2r+1+(m-1)(2r+1)}=v_{n-q}$, and a fortiori any vertex v_{n-q+2k}, is in $N_{C[v_{2r+1},v_{n+l-2r-1}]}(v_0)$ or is not in $C[v_{2r+1},v_{n+l-2r-1}]$. Hence if S' does not contain v_{2r+1+l}, then $|S'|\geqslant|S''|$. It follows that $\min\{|S|;S\in\mathscr{S}\}=m+1+\frac{1}{2}(q-1)$ and since this value is less than $i(G)=m+r+(q-1)/2$,

$$i(G-e_l)=m+1+\tfrac{1}{2}(q-1).$$

In the case $q < 2r-3$ and $q < l_{\mathrm{odd}} \leqslant 2r-3$ (Fig. 6), the set

$$S'' = \{v_{2r+1}, v_{2(2r+1)}, \dots, v_{(m-1)(2r+1)}, v_{(m-1)(2r+1)+2}, v_{(m-1)(2r+1)+4}, \dots, v_{(m-1)(2r+1)+q+l}\},$$

where $v_{(m-1)(2r+1)+q+l} = v_{n+l-2r-1}$, is a suitable set S' of order $m-1+(q+l)/2$. Does there exist any set S' smaller than S''? Since $m(2r+1)$ is at least $n-2r+1$ and has the same parity as the neighbours of v_0, the sets S' containing v_{2r+1} cannot contain more vertices at distance $2r+1$ on C than S'' and thus are not smaller than S''. If S' does not contain v_{2r+1} then it contains v_{2r+1+l}, for otherwise, in order to dominate both v_{2r+1} and v_{2r+l}, it must contain two adjacent vertices. Starting from v_{2r+1+l}, we cannot include more vertices at distance $2r+1$ on C in S' than in S'', and thus S' is not smaller than S''. Adding $\{v_0, v_l\}$ to S'', we find that the minimum cardinality of an independent dominating set S of $G - e_l$ containing $\{v_0, v_l\}$ is $m+1+(q+l)/2$. Since $l \leqslant 2r-3$, this cardinality is at most $m+r+(q-1)/2 = i(G)$ with equality if and only if $l = 2r-3$. Therefore

$$i(G - e_l) = m + 1 + \tfrac{1}{2}(q+l). \qquad \square$$

Theorem 6 determines which graphs of $\mathscr{F}_5$ are i^--ER-critical.

Corollary 7. *A circulant $C_n\ \langle 1, 3, \dots, 2r-1\rangle$ belonging to the family $\mathscr{F}_5$ is i^--ER-critical if and only if $r \geqslant 2$ and $q = 2r-1$.*

Proof. When $r = 1$, then G is simply the cycle C_n which is not i^--ER-critical since $i(C_n) = i(P_n) = \lceil n/3 \rceil$. When $r \geqslant 2$ we use Theorem 6. If $q < 2r-1$ then $i(G - e_{2r-1}) = i(G)$ and thus G is not i^--ER-critical. If $q = 2r-1$ then for all $1 \leqslant l_{\mathrm{odd}} \leqslant 2r-1$,

$$i(G - e_l) \leqslant m + 1 + \tfrac{1}{2}(q-1) < m + r + \tfrac{1}{2}(q-1) = i(G)$$

and thus G is i^--ER-critical. $\square$

This result provides an example of an i^--ER-critical graph satisfying $\gamma(G) = \gamma$ and $i(G) = i$ for two given integers $\gamma \geqslant 3$ and $i > \gamma$ with $i + \gamma$ even. Indeed, the graph G of $\mathscr{F}_5$ defined by $m = \gamma - 1$, $r = (i - \gamma + 2)/2$ and $q = i - \gamma + 1$ satisfies these conditions.

Another by-product of Theorem 6 is the determination of all the orbits induced by the automorphism group of G in its edge set.

Theorem 8. *Let r, m, q be positive integers with $m \geqslant 2$, $r \geqslant 2$, q odd and $1 \leqslant q \leqslant 2r-1$, and let $n = (2r+1)m + q$ and $G = C_n\ \langle 1, 3, \dots, 2r-1\rangle$. Then the automorphism group of G induces r classes in its edge set E.*

Proof. In the circulant G, any two edges e_l of the same type belong to the same class and thus the number of orbits in E is at most r. Now consider two edges $e_l = v_0 v_l$ and $e_{l'} = v_0 v_{l'}$ with $l \neq l'$. If $i(G - e_l) \neq i(G - e_{l'})$, or if the minimum cardinality of an independent dominating set of $G - e_l$ containing $\{v_0, v_l\}$ is different from the minimum cardinality of an independent dominating set of $G-e_{l'}$ containing $\{v_0, v_{l'}\}$, then e_l and $e_{l'}$ do not belong to the same class.

For each edge e_l the value of $i(G-e_l)$ is given in Theorem 6. If $q=2r-1$ then $i(G-e_l)$ takes r different values, namely $m+1$ for $l=2r-1$, $m+2$ for $l=2r-3,\ldots,m+r$ for $l=1$. If $q=2r-3$ then $i(G-e_l)$ also takes r different values, namely $m+1$ for $l=2r-3$, $m+2$ for $l=2r-5,\ldots,m+r-1$ for $l=1$, $i(G)=m+2r-2$ for $l=2r-1$. If $r\geqslant 3$ and $q<2r-3$ then $i(G-e_l)$ takes the r values $m+1$ for $l=q$, $m+2$ for $l=q-2,\ldots,m+(q+1)/2$ for $l=1$, $m+q+2$ for $l=q+2$, $m+q+3$ for $l=q+4,\ldots,m+r+(q-1)/2$ for $l=2r-3$, $i(G)=m+r+(q-1)/2$ for $l=2r-1$. All these values are distinct except the last two ones. But in this case, as is shown in the proof of Theorem 6, the minimum cardinality of an independent dominating set of $G-e_{2r-1}$ containing $\{v_0, v_{2r-1}\}$ is $m+r+(q+1)/2$ while the minimum cardinality of an independent dominating set of $G-e_{2r-3}$ containing $\{v_0, v_{2r-3}\}$ is $m+r+(q-1)/2$. Hence e_{2r-1} and e_{2r-3} do not belong to the same class, which achieves the proof. □

Acknowledgements

This paper was written while C.M. Mynhardt was employed by the University of South Africa in the Department of Mathematics, Applied Mathematics and Astronomy. Research grants by the University, the South African National Research Foundation and the Natural Sciences and Engineering Research Council of Canada are gratefully acknowledged.

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