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A multiplier theorem using the Schechter's method of interpolation ☆

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Abstract

Let m be a measurable bounded function and let us assume that there exists a bounded functions S so that $m(\xi)S(\xi)^{it-1}$ is a Fourier multiplier on L^p uniformly in $t \in \mathbb{R}$. Then, using the analytic interpolation theorem of Stein, one can show that necessarily m is a L^p multiplier. The purpose of this work is to show that under the above conditions, it holds that, for every $k \in \mathbb{N}$, $m(\log S)^k \in M_p$. The technique is based on the Schechter's interpolation method.

Keywords: Multiplier; Interpolation; Analytic family of operators; Schechter's method; Endpoint estimates

1. Introduction

Let $1 \leqslant p \leqslant \infty$ and let M_p be the class of measurable bounded functions such that the operator given by

$$Tf(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m(\xi) e^{2\pi i \xi x} \, d\xi,$$

where $\hat{f}$ is the Fourier transform, is bounded on $L^p(\mathbb{R}^n)$. M_p is the so-called class of Fourier multipliers on L^p, and $||m||_{M_p}$ denotes the norm of the

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corresponding bounded operator. Well-known properties about M_p are for example:

(1) M_p is a Banach algebra contained in $M_2 = L^\infty$ and $M_p = M_{p'}$, where $1/p + 1/p' = 1$.

(2) If $m \in M_p$ and $q \in [p, p']$, then $m \in M_q$, and $\|m\|_{M_q} \leqslant \|m\|_{M_p}$.

The theory of multipliers has been widely studied for a long time up to our days (see, for example, Refs. [9,11,13,14,16] just to mention a few of them and the works [8,10]). One of the technique which is an extremely useful tool to deal with multiplier questions is the theory of interpolation (see [1,2]). In particular, property 2, above mentioned, is proved by using the classical Riesz–Thorin interpolation theorem.

Let now m be a measurable function and let us assume that there exists a bounded function S so that

$$m(\xi)S(\xi)^{it-1} \in M_p$$

uniformly in $t \in \mathbb{R}$. Then, using appropriately the analytic interpolation theorem of Stein (see [15]), one can show that necessarily $m \in M_p$.

The purpose of this work is to show that under the above conditions, it holds that, for every $k \in \mathbb{N}$, $m(\log S)^k \in M_p$. The technique is based on the Schechter's interpolation method (see [12]).

The paper is organized as follows: in Section 2, we present the main tool we need to show our main result (Theorems 3.2 and 4.5); that is we need some features about Schechter's interpolation method. For simplicity in the explanation of our method, we shall present, in Section 3, all our results for the case of the first derivative $(n = 1)$, but we want to emphasize that, up to some computations, all the results can be extended to a general $n \in \mathbb{N}$. We shall state the main results for the case $n > 1$ in the last section.

Given two Banach spaces A and B, we write $A \approx B$ to indicate that they have equivalent norms and $A = B$ means that $A \approx B$ and that the constants in the equivalence are independent of θ.

We shall write a universal constant C if $C = C(\theta)$ remains bounded when $\theta \to 0$ and, such universal constants C may change from one occurrence to the next. As usual, the symbol $f \approx g$ will indicate the existence of a positive universal constant C so that $(1/C)f \leqslant g \leqslant Cf$ and, by $f \lesssim g$ we mean that $f \leqslant Cg$. $\lambda_f(y) = \mu\{x; |f(x)| > y\}$ is the usual distribution function and $L^{p,\infty}$ is defined as the set of measurable functions so that $\|f\|_{p,\infty} = \sup_{y>0} y^{1/p} \lambda_f(y) < \infty$.

2. Schechter method of interpolation

Let us start by giving a short description of the main objects of our method.

Let $\Omega = \{z \in \mathbb{C}; 0 < \operatorname{Re} z < 1\}$ be the unit strip and $A(\Omega)$ is the algebra of Ω; that is the set of analytic functions on Ω and continuous on $\bar{\Omega}$. Let $\bar{A} = (A_0, A_1)$ be a compatible couple of Banach spaces and let $\mathscr{F}(\bar{A}) = \mathscr{F}(A_0, A_1)$ be the space of analytic vector functions of the Calderón complex interpolation method (see [3]); that is, the set of all functions $f:\bar{\Omega}\to A_0+A_1$ such that

(i) for every $l\in(A_0+A_1)^*$, $l(f(\cdot))\in A(\Omega)$,
(ii) $f(z)\in A_j$ for $\operatorname{Re} z=j$ and $j=0,1$,
(iii) $f(j+i\cdot)$ is A_j-continuous,
(iv) $\|f\|_{\mathscr{F}(\bar{A})}=\max_{j=0,1}\sup_{t>0}\{\|f(j+it)\|_{A_j}\}<\infty$.

Let

$$\mathscr{G}(\bar{A})=\mathscr{G}(A_0,A_1)=\left\{\sum_{\text{finite}}\varphi_j x_j;\ x_j\in A_0\cap A_1,\ \varphi_j\in A(\Omega)\right\},$$

and let us recall that $\mathscr{G}(\bar{A})$ is dense in $\mathscr{F}(\bar{A})$ (see [3]).

The classical complex interpolation space (Calderón space) is defined, for $0<\theta<1$, by

$$\bar{A}_\theta=\{a=f(\theta); f\in\mathscr{F}(\bar{A})\}$$

with norm $\|a\|_{\bar{A}_\theta}=\inf\{\|f\|_{\mathscr{F}(\bar{A})}; a=f(\theta)\}$.

In [12], the following interpolation spaces were introduced:

$$\bar{A}_{\delta^{(n)}(\theta)}=[A_0,A_1]_{\delta^{(n)}(\theta)}=\{x\in A_0+A_1;\ \exists f\in\mathscr{F}(\bar{A}),\ f^{(n)}(\theta)=x\}$$

with the norm

$$\|x\|_{\bar{A}_{\delta^{(n)}(\theta)}}=\inf\{\|f\|_{\mathscr{F}(\bar{A})};\ f^{(n)}(\theta)=x\},$$

and $\{A_0,A_1\}_{\delta^{(n)}(\theta)}$ is the completion of the intersection $A_0\cap A_1$ with respect to the norm

$$\|x\|_{\{A_0,A_1\}_{\delta^{(n)}(\theta)}}=\inf\{\|g\|_{\mathscr{F}(\bar{A})};\ g\in\mathscr{G}(\bar{A}),\ g^{(n)}(\theta)=x\}.$$

Also,

$$\begin{aligned}\bar{A}^{\delta^{(n)}(\theta)}&=[A_0,A_1]^{\delta^{(n)}(\theta)}\\&=\{x\in A_0+A_1;\ \exists f\in\mathscr{F}(\bar{A}),\ f(\theta)=x,\ f^{(m)}(\theta)=0,\ m=1,\dots,n\}\end{aligned}$$

with the corresponding norm of the infimum and $\{A_0,A_1\}^{\delta^{(n)}(\theta)}$ defined in analog way with $\mathscr{G}(\bar{A})$ instead of $\mathscr{F}(\bar{A})$. We shall call them Schechter spaces. If $\bar{A}=(L^{p_0},L^{p_1})$, then $\bar{A}_{\delta^{(n)}(\theta)}=\{A_0,A_1\}_{\delta^{(n)}(\theta)}$ and $\bar{A}^{\delta^{(n)}(\theta)}=\{A_0,A_1\}^{\delta^{(n)}(\theta)}$ and we shall refer to them as the first and second Schechter method, respectively.

Remark 2.1. Some general facts concerning these spaces which will be useful in the sequel are the following:

(i) If $a\in\bar{A}_\theta$ and $\varphi_\theta:\Omega\to D$ is a conformal map from Ω onto the unit disc D so that $\varphi_\theta(\theta)=0$, then,

$$|\varphi_\theta'(\theta)|=\frac{\pi}{2\sin\pi\theta},$$

and thus $|\varphi'_\theta(\theta)| = O(1/\theta)$ when $\theta \to 0$. Therefore, if $F \in \mathscr{F}$ is such that $F(\theta) = a$, the function $G = \varphi_\theta F \in \mathscr{F}$ satisfies that $||G||_{\mathscr{F}} = ||F||_{\mathscr{F}}$ and $G'(\theta) = \varphi'_\theta(\theta)a$. Hence, $\bar{A}_\theta \subset \bar{A}_{\delta'(\theta)}$ with norm less than or equal to $\frac{2}{\pi}\sin \pi\theta$.

(ii) If $F \in \mathscr{F}$ satisfies that $F(\theta) = 0$, then $G = F/\varphi_\theta \in \mathscr{F}$ and $F'(\theta) = \varphi'_\theta(\theta)G(\theta)$. Therefore, $F'(\theta) \in \bar{A}_\theta$ and $||F'(\theta)||_\theta \leqslant \frac{\pi}{2 \sin \pi\theta} ||F||_{\mathscr{F}}$.

From now on, φ_θ will be the conformal map from Ω onto D we have mentioned above.

Let us now recall the definition of Calderón analytic families of operators (see [4,7]):

Definition 2.2. Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces. Let $\bar{L} = \{L_\xi\}_{\xi \in \bar{\Omega}}$ be such that

$$L_\xi : A_0 \cap A_1 \to B_0 + B_1.$$

We say that $\bar{L}$ is a Calderón analytic family of operators and we write $\bar{L} \in C$, if the following conditions hold:

(i) for every $l \in (B_0 + B_1)^*$ and every $a \in A_0 \cap A_1$, the function $\langle l, L_\xi a \rangle \in A(\Omega)$,

(ii) for every $t \in \mathbb{R}$,

$$L_{j+it} : (A_0 \cap A_1, || \; ||_{A_j}) \to (B_j, || \; ||_{B_j})$$

is bounded and there exist two continuous functions $M_j : \mathbb{R} \to \mathbb{R}^+$ so that $\log M_j(\cdot) \in L^1(\mu_j(\xi, \cdot))$, where μ_j is the Poisson kernel for Ω $(j = 0, 1)$ and $||L_{j+it}|| \leqslant M_j(t)$,

(iii) for every $a \in A_0 \cap A_1$, the function $M^{-1}L.a \in \mathscr{F}(\bar{B})$, where $M = \exp(\Psi)$ and Ψ is an analytic function in Ω whose real part is

$$\sum_{j=0}^{1} \int_{-\infty}^{\infty} \log M_j(t)\mu_j(\xi, t)\, dt.$$

Moreover, if $||M||_\infty \leqslant 1$, we say that $\bar{L}$ is a uniformly bounded Calderón analytic family and we write $\bar{L} \in BC$.

Then, the following extension of the theorem of Stein (see [15]) and Theorem 1 in [7] was proved in [4]:

Theorem 2.3. *Let* (A_0, A_1) *and* (B_0, B_1) *be two compatible couples of Banach spaces and let* $\bar{L} \in BC$. *Then,*

$$(L_\xi)^{(n)}(\theta) : \{A_0, A_1\}^{\delta^{(n)}(\theta)} \to \bar{B}_{\delta^{(n)}(\theta)}$$

is bounded with norm less than or equal to 1; *that is*

$$||(L_\xi)^{(n)}(\theta)a||_{\bar{B}_{\delta^{(n)}(\theta)}} \leqslant ||a||_{\{A_0,A_1\}^{\delta^{(n)}(\theta)}}.$$

Now, let us assume that, for each $0<\theta<1$, we consider a family $\overline{L^\theta}\in BC$. Then, by Theorem 2.3, we can conclude that

$$\overline{\lim_{\theta\to 0}}\ ||(L_\xi^\theta)^{(n)}(\theta)a||_{\bar{B}_{\delta^{(n)}(\theta)}} \leqslant \overline{\lim_{\theta\to 0}}\ ||a||_{\{A_0,A_1\}^{\delta^{(n)}(\theta)}}. \tag{1}$$

Our purpose now is to show that inequality (1) gives us, when applied to the couples $\bar{A}=(L^{p_0}(\mu),L^{p_1}(\mu))$ and $\bar{B}=(L^{q_0}(v),L^{q_1}(v))$, a sufficient condition for the (p_0,q_0) boundedness of the linear operator $Tf=\lim_{\theta\to 0}\theta^n(L_\xi^\theta)^{(n)}(\theta)$, whenever this limit exists. As a consequence, we shall get the announced multiplier result.

From (1), we notice that one of the things we have to do is to study the behavior of the Schechter interpolation spaces $(L^{p_0}(\mu),L^{p_1}(\mu))_{\delta^{(n)}(\theta)}$ and $\{L^{p_0}(\mu),L^{p_1}(\mu)\}^{\delta^{(n)}(\theta)}$ when θ goes to zero.

For the classical Calderón interpolation spaces, it is known that $(L^{p_0}(\mu),L^{p_1}(\mu))_\theta$ is isometric to $L^{p(\theta)}(\mu)$ where, as usual,

$$\frac{1}{p(\theta)}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \tag{2}$$

and hence, for every $f\in L^{p_0}(\mu)\cap L^{p_1}(\mu)$,

$$\overline{\lim_{\theta\to 0}}\ ||f||_{(L^{p_0}(\mu),L^{p_1}(\mu))_\theta}=||f||_{p_0}.$$

In what follows, $p(\theta)$ will be defined by (2).

Also, it is known (see [5,6]) that

$$\{L^{p_0}(\mu),L^{p_1}(\mu)\}_{\delta'(\theta)}=(L^{p_0}(\mu),L^{p_1}(\mu))_{\delta'(\theta)}\approx L_\phi$$

and

$$\{L^{p_0}(\mu),L^{p_1}(\mu)\}^{\delta'(\theta)}=(L^{p_0}(\mu),L^{p_1}(\mu))^{\delta'(\theta)}\approx L_\varphi,$$

where L_ϕ and L_φ are Orlicz spaces with

$$\phi(t)=\left(\frac{t}{1+|\log t|}\right)^{p(\theta)} \quad\text{and}\quad \varphi(t)=(t(1+|\log t|))^{p(\theta)}.$$

However, the constants in the above equivalence depend on θ and hence, our first goal consists in finding a better equivalence than the above one.

In this section, we shall prove the equivalence of the spaces $(L^{p_0},L^{p_1})_{\delta^{(n)}(\theta)}$ and $(L^{p_0},L^{p_1})^{\delta^{(n)}(\theta)}$ with some Orlicz spaces so that the constants in such equivalence do not depend on θ.

Let us start by analyzing the right-hand side of (1). In Ref. [6] it was shown that

$$\{L^{p_0},L^{p_1}\}^{\delta_\theta^{(n)}}=(L^{p_0},L^{p_1})^{\delta_\theta^{(n)}}=\left\{f \text{ measurable};\ f\left(1+\left|\log\frac{|f|}{||f||_{p(\theta)}}\right|\right)^n\in L^{p(\theta)}\right\}$$

and

$$||f||_{(L^{p_0},L^{p_1})^{\delta^{(n)}_\theta}} \lesssim \left|\left| f\left(1+\left|\log\frac{|f|}{||f||_{p(\theta)}}\right|\right)^n \right|\right|_{p(\theta)} \lesssim \theta^{-n}||f||_{(L^{p_0},L^{p_1})^{\delta^{(n)}_\theta}}.$$

Modifying slightly the proof of the above theorem (we include here the new proof for the sake of completeness) we get the following result:

Theorem 2.4. *$f \in (L^{p_0}, L^{p_1})^{\delta'(\theta)}$ if and only if $f(1+|\log|f||) \in L^{p(\theta)}$ and*

$$||f||_{(L^{p_0},L^{p_1})^{\delta'(\theta)}} \approx ||f||_{p(\theta)} + \frac{\theta(p_0-p_1)}{p_1+\theta|p_0-p_1|}\left|\left| f\log\frac{f}{||f||_{p(\theta)}}\right|\right|_{p(\theta)}.$$

Proof. Let $f \in (L^{p_0}, L^{p_1})^{\delta'(\theta)}$ and let $\varepsilon > 0$. Then, there exists $F \in \mathscr{F}$ such that $F(\theta) = f$, $F'(\theta) = 0$ and $||F||_{\mathscr{F}} \leqslant ||f||_{(L^{p_0},L^{p_1})^{\delta'(\theta)}} + \varepsilon$. The first condition implies that $f \in L^{p(\theta)}$ and $||f||_{p(\theta)} \leqslant ||F||_{\mathscr{F}}$.

Let $H = F - H_f$, where

$$H_f(\xi) = \frac{f}{|f|}\left(\frac{|f|}{||f||_{p(\theta)}}\right)^{((1-\xi)p_1+\xi p_0)\frac{p(\theta)}{p_0p_1}} ||f||_{p(\theta)}.$$

Then $H(\theta) = 0$ and

$$H'(\theta) = \frac{(p_0-p_1)}{p_1+\theta(p_0-p_1)} f\log\frac{|f|}{||f||_{p(\theta)}}.$$

Hence, by Remark 2.1(ii), $f\log\frac{|f|}{||f||_{p(\theta)}} \in L^{p(\theta)}$, and

$$\left|\left| f\log\frac{|f|}{||f||_{p(\theta)}}\right|\right|_{p(\theta)} \lesssim \frac{p_1+\theta(p_0-p_1)}{\theta|p_0-p_1|}||H||_{\mathscr{F}}$$
$$\lesssim \frac{p_1+\theta(p_0-p_1)}{\theta|p_0-p_1|}(||F||_{\mathscr{F}} + ||f||_{p(\theta)}) \lesssim \frac{p_1+\theta(p_0-p_1)}{\theta|p_0-p_1|}||F||_{\mathscr{F}}.$$

Therefore,

$$||f||_{p(\theta)} + \frac{\theta(p_0-p_1)}{p_1+\theta|p_0-p_1|}\left|\left| f\log\frac{f}{||f||_{p(\theta)}}\right|\right|_{p(\theta)} \lesssim ||F||_{\mathscr{F}} \leqslant ||f||_{(L^{p_0},L^{p_1})^{\delta'(\theta)}} + \varepsilon.$$

Letting ε tend to zero, we are done.

Conversely, if $f(1+\log||f|) \in L^{p(\theta)}$ and we call $g = f\log\frac{|f|}{||f||_{p(\theta)}}$, we only have to consider the function

$$G = \varphi'_\theta(\theta)\frac{p_1+\theta(p_0-p_1)}{(p_0-p_1)}H_f - \varphi_\theta H_g$$

to see that $f \in (L^{p_0}, L^{p_1})^{\delta'(\theta)}$ and that the corresponding inequality for $f \in (L^{p_0}, L^{p_1})^{\delta'(\theta)}$ holds. $\square$

Theorem 2.5. *For every $f \in L^{p_0} \cap L^{p_1}$, we have that*

$$\overline{\lim_{\theta \to 0}} \; ||f||_{(L^{p_0}, L^{p_1})^{\delta'(\theta)}} \approx ||f||_{p_0}.$$

Corollary 2.6. *Let $\bar{L} : (L^{p_0}, L^{p_1}) \to (L^{p_0}, L^{p_1})$ be a uniformly bounded Calderón analytic family of operators. Then, for every $f \in L^{p_0} \cap L^{p_1}$,*

$$\lim_{\theta \to 0} \; ||L'_\theta f||_{(L^{p_0}, L^{p_1})_{\delta'(\theta)}} \lesssim ||f||_{L^{p_0}}.$$

To analyze the left-hand side of (1) and also of Corollary 2.6, we have to study the behavior when θ tends to zero of the first method of Schechter. Carro and Cerdà [5] proved that

$$[L^{p_0}, L^{p_1}]_{\delta_\theta^{(n)}} \approx M_\Psi,$$

where

$$M_\Psi = \{f;\; f = f_0 + f_1 \log |f_1|;\; f_0, f_1 \in L^{p(\theta)}\}$$

endowed with the norm

$$||f||_{M_\Psi} \approx ||f_0 + f_1 \log ||f_1||_{p(\theta)}||_{p(\theta)} + ||f_1||_{p(\theta)},$$

and the constants in the above equivalence are as follows:

$$||f||_{\bar{A}_{\delta^{(n)}(\theta)}} \lesssim ||f||_{M_{\Psi_n}} \lesssim \theta^{-n} ||f||_{\bar{A}_{\delta^{(n)}(\theta)}}.$$

To avoid the dependence in θ of the above constants, we have to modify the proof of Theorem 3.1 in [5] to obtain the following one:

Theorem 2.7. *Let*

$$M_{\Psi_\theta} = \left\{ f;\; f = \frac{f_0}{\theta} + f_1 \log \frac{|f_1|}{||f_1||_{p(\theta)}};\; f_0, f_1 \in L^{p(\theta)} \right\}$$

with

$$||f||_{M_{\Psi_\theta}} = \inf\{||f_0||_{p(\theta)} + ||f_1||_{p(\theta)}\},$$

where the infimum extends to the collection of all functions f_0 and f_1 satisfying that $f = \frac{f_0}{\theta} + f_1 \log |f_1| / ||f_1||_{p(\theta)}$. Then,

$$(L^{p_0}, L^{p_1})_{\delta'(\theta)} = M_{\Psi_\theta}.$$

To study the space M_{Ψ_θ}, set

$$\gamma_\theta(x) = \frac{x}{\theta}(1 + \theta|\log x|).$$

Obviously, γ_θ is an increasing and one-to-one function from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and hence we can define the increasing and one-to-one function

$$\Psi_\theta = (\gamma_\theta^{-1})^{p(\theta)}.$$

Then, one can easily see that

$$\Psi_\theta^{-1}(x) = \frac{x^{1/p(\theta)}}{\theta}\left(1 + \frac{\theta}{p(\theta)}|\log x|\right).$$

From this expression we get that $\Psi_\theta(x) \approx (\theta x(1 + \frac{\theta}{p(\theta)}|\log x|)^{-1})^{p(\theta)}$ and hence there exists a constant C such that $\Psi(2t) \leqslant C\Psi(t)$. Therefore,

$$L_{\Psi_\theta} = \left\{ f; \int \Psi_\theta(|f(x)|)\, dx < +\infty \right\}$$

is a linear space. Set

$$||f||_{\Psi_\theta} = \inf\left\{ k > 0; \int \Psi_\theta\left(\frac{|f(x)|}{k}\right) dx \leqslant 1 \right\}.$$

Since, in general Ψ_θ is not a convex function, we do not have that the above expression is a norm but we have the following properties:

(i) $||\lambda f||_{\Psi_\theta} = |\lambda| ||f||_{\Psi_\theta}$, for every $\lambda \in \mathbb{R}$,
(ii) if $||f||_{\Psi_\theta} = 1$ then $\int \Psi_\theta(|f(x)|)\, dx = 1$,
(iii) $||f||_{\Psi_\theta} \leqslant \theta ||f||_{p(\theta)}$,
(iv) if A is such that

$$\sup_t \frac{2\Psi_\theta(t/A)}{\Psi_\theta(t)} \leqslant 1$$

then

$$||f + g||_{\Psi_\theta} \leqslant 2A(||f||_{\Psi_\theta} + ||g||_{\Psi_\theta}).$$

Theorem 2.8.

$$L_{\Psi_\theta} = M_{\Psi_\theta}.$$

Proof. Let $p = p(\theta)$ and let us assume without loss of generality that $f \geqslant 0$. If $||f||_{\Psi_\theta} = 1$, we have that $\int \Psi_\theta(f) = 1$, and hence, if $g = \Psi_\theta(f)^{1/p}$, we obtain that $g \in L^p$ and $||g||_p = 1$. Now,

$$f = \Psi_\theta^{-1}(\Psi_\theta(f)) = \gamma_\theta(\Psi_\theta(f)^{1/p}) = \gamma_\theta(g) = \frac{g}{\theta}(1 + \theta|\log g|),$$

and thus $||f||_{M_{\Psi_\theta}} \lesssim 1$.

Let $f = f_0/\theta + f_1 \log \frac{|f_1|}{\|f_1\|_p}$. Then, for k to be chosen later on

$$\int \Psi_\theta\left(\frac{|f|}{2k\|f_1\|_p}\right) \leqslant \int \Psi_\theta\left(\frac{\max(|\frac{f_0}{\theta} + f_1 \log k|, |f_1|(|\log \frac{|f_1|}{k\|f_1\|_p}|))}{k\|f_1\|_p}\right)$$

$$\leqslant \int \Psi_\theta\left(\frac{|\frac{f_0}{\theta} + f_1 \log k|}{k\|f_1\|_p}\right) + \int \Psi_\theta\left(\frac{|f_1|(|\log \frac{|f_1|}{k\|f_1\|_p}|)}{k\|f_1\|_p}\right)$$

$$\leqslant \theta^p \int \left(\frac{|\frac{f_0}{\theta} + f_1 \log k|}{k\|f_1\|_p}\right)^p + \frac{1}{k^p}$$

$$\leqslant \frac{2^p}{k^p}\left(\frac{\|f_0\|_p^p}{\|f_1\|_p^p} + |\log k|^p + 1\right),$$

and using that $\log k \leqslant k/e$, we get that if

$$k = \frac{(\|f_0\|_p^p + \|f_1\|_p^p)^{1/p}}{\|f_1\|_p}(2^{-p} - e^{-p})^{-1/p}$$

then

$$\int \Psi_\theta\left(\frac{|f|}{2k\|f_1\|_p}\right) \leqslant 1.$$

Therefore, $\|f\|_{\Psi_\theta} \leqslant 2(2^{-p} - e^{-p})^{-1/p}(\|f_0\|_p + \|f_1\|_p)$. □

Theorem 2.9. *For every* $0 < \theta < 1$, *it holds that*

$$(L^{p_0}, L^{p_1})_{\delta'(\theta)} = L_{\Psi_\theta}.$$

Lemma 2.10.

(i) *For every* $f \in L^{p_0} \cap L^{p_1}$,

$$\overline{\lim_{\theta \to 0}} \frac{1}{\theta}\|f\|_{\Psi_\theta} \lesssim \|f\|_{p_0}.$$

(ii) *For every* f,

$$\|f\|_{p_0,\infty} \lesssim \underline{\lim_{\theta \to 0}} \frac{1}{\theta}\|f\|_{\Psi_\theta}.$$

Proof. (i) First we observe that if $f \in L^{p_0} \cap L^{p_1}$, then

$$\|f\|_{p(\theta)} \leqslant \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta}.$$

And using Remark 2.1(i), we have that

$$\overline{\lim_{\theta\to 0}} \frac{1}{\theta} ||f||_{\Psi_\theta} \approx \overline{\lim_{\theta\to 0}} \frac{1}{\theta} ||f||_{\bar{A}_{\delta'(\theta)}} \lesssim \overline{\lim_{\theta\to 0}} ||f||_{\bar{A}_\theta} = \overline{\lim_{\theta\to 0}} ||f||_{p(\theta)} \leqslant \overline{\lim_{\theta\to 0}} ||f||_{p_0}^{1-\theta} ||f||_{p_1}^{\theta} = ||f||_{p_0}.$$

(ii) To prove the second part, we observe that

$$\Psi_\theta\left(\frac{t}{\lambda}\right)\lambda_f(t) \leqslant \int_0^\infty \Psi_\theta\left(\frac{|f(x)|}{\lambda}\right) dx$$

and therefore,

$$||f||_{\Psi_\theta} \geqslant \inf\left\{\lambda>0; \sup_{z>0} \Psi_\theta(z/\lambda)\lambda_f(z) \leqslant 1\right\}.$$

Now, $\sup_{z>0} \Psi_\theta(z/\lambda)\lambda_f(z) \leqslant 1$ if and only if $\lambda \geqslant \theta \sup_z \frac{z\lambda_f(z)^{1/p(\theta)}}{1+\frac{\theta}{p(\theta)}|\log \lambda_f(z)|}$, and, hence,

$$||f||_{\Psi_\theta} \geqslant \theta \sup_z \frac{z\lambda_f(z)^{1/p(\theta)}}{1+\frac{\theta}{p(\theta)}|\log \lambda_f(z)|}.$$

From this, we deduce that, for every $z>0$,

$$\underline{\lim_{\theta\to 0}} \frac{z\lambda_f(z)^{1/p(\theta)}}{1+\frac{\theta}{p(\theta)}|\log \lambda_f(z)|} \lesssim \underline{\lim_{\theta\to 0}} \frac{1}{\theta} ||f||_{\Psi_\theta},$$

and hence

$$||f||_{p_0,\infty} \lesssim \underline{\lim_{\theta\to 0}} \frac{1}{\theta} ||f||_{\Psi_\theta}. \qquad \square$$

Theorem 2.11. *For every* $f \in L^{p_0} \cap L^{p_1}$, *it holds that*

$$\overline{\lim_{\theta\to 0}} \frac{1}{\theta} ||f||_{\Psi_\theta} \approx ||f||_{p_0}.$$

Proof. Let $\lambda_\theta = \frac{1}{\theta}||f||_{\Psi_\theta}$. Then, by the previous lemma, we have that, if $f \neq 0$,

$$0 < \underline{\lim_{\theta\to 0}} \lambda_\theta \leqslant \overline{\lim_{\theta\to 0}} \lambda_\theta < \infty,$$

and hence $\lim_{\theta\to 0} \theta|\log(y/(\theta\lambda_\theta))| = 0$. Therefore,

$$\underline{\lim_{\theta\to 0}} \Psi_\theta\left(\frac{y}{\theta\lambda_\theta}\right) = \left(\frac{y}{\overline{\lim}_{\theta\to 0} \lambda_\theta}\right)^{p_0}.$$

From this, using Fatou's lemma we get that

$$\left(\frac{\|f\|_{p_0}}{\overline{\lim}_{\theta\to 0}\ \lambda_\theta}\right)^{p_0} = \int \left(\frac{|f(x)|}{\overline{\lim}_{\theta\to 0}\ \lambda_\theta}\right)^{p_0} dx \leqslant \int \underline{\lim}_{\theta\to 0}\ \Psi_\theta\left(\frac{|f(x)|}{\theta\lambda_\theta}\right) dx$$
$$\leqslant \underline{\lim}_{\theta\to 0} \int \Psi_\theta\left(\frac{|f(x)|}{\theta\lambda_\theta}\right) dx \leqslant 1,$$

from which the result follows. □

In fact, modifying slightly the above proof, one can easily see that something stronger holds:

Theorem 2.12. *If f_θ converges to f almost everywhere, then*

$$\|f\|_{p_0} \lesssim \overline{\lim}_{\theta\to 0} \frac{1}{\theta} \|f_\theta\|_{\Psi_\theta}.$$

3. A multiplier result

Let us formulate our main result.

Theorem 3.1. *Let T be a linear operator such that, for every θ, there exists a uniformly bounded Calderón analytic family of operators $\overline{L^\theta} : (L^{p_0}, L^{p_1}) \to (L^{q_0}, L^{q_1})$ so that, for every $f \in L^{p_0} \cap L^{p_1}$, $\theta(L^\theta)'_\theta f$ converges to Tf almost everywhere. Then*

$$T : L^{p_0} \to L^{q_0}$$

is bounded.

Proof. By Theorem 2.3, we have that

$$\frac{1}{\theta} \|\theta(L^\theta)'_\theta f\|_{\Psi_\theta} \leqslant \|f\|_{(L^{p_0}, L^{p_1})^{\delta'(\theta)}}.$$

Then, letting $\theta \to 0$ and applying Theorems 2.5 and 2.12, we get that, for every $f \in L^{p_0} \cap L^{p_1}$:

$$\|Tf\|_{q_0} \lesssim \|f\|_{p_0}. \qquad \square$$

In the following example, we have that $(L^\theta)'_\theta = \frac{1}{\theta} T$, and hence the conclusion of our theorem holds.

Theorem 3.2. *Let $m \in L^\infty$ and let us assume that there exists a bounded function S such that*

$$\frac{m(\xi)}{S(\xi)} S(\xi)^{it} \in M_p,$$

uniformly in $t \in \mathbb{R}$. Then, $m(\log S) \in M_p$.

Proof. Let f be in the Schwartz class of rapidly decreasing C^∞ functions and let K be a compact set in $\mathbb{R}^n$. Let us consider the operators

$$T_z^\theta f(x) = \chi_K(x) \int_{\mathbb{R}^n} \hat{f}(\xi) \frac{m(\xi)}{S(\xi)} S(\xi)^{z/\theta} e^{2\pi i x\xi}\, d\xi.$$

Then, if $z = it$, we have, by hypothesis, that $T_{it}^\theta f \in L^p$ and $\sup_{t\in\mathbb{R}} ||T_{it}^\theta f||_{p,p} = C_1 < \infty$. Also, using the dominated convergence theorem, one can show that $T_{it}^\theta f$ is an L^p-continuous function in the variable t. On the other hand, using the boundedness of S, one can immediately see that $T_{1+it}^\theta f \in L^2$ uniformly in $t \in \mathbb{R}$ with constant $||m/S||_\infty ||S||_\infty^{1/\theta}$, and this function defines an L^2-continuous function in the variable t. Hence, if we consider the analytic families of operators

$$\overline{T^\theta} : (L^p, L^2) \to (L^p, L^2),$$

and $M_\theta(\xi) = C^{\frac{(1-\theta)}{\theta}\xi}$ with $C = \max(C_1, ||S||_\infty)$, we obtain that the family $\overline{L^\theta} = \overline{T^\theta}/M_\theta \in BC$ and

$$(L^\theta)'(\theta) = \frac{C^{\theta-1}}{\theta}(T - L),$$

where $L = L_\theta^\theta$ and T is the operator associated to the multiplier $m \log S$. Now, we know (see [7]) that $L = L_\theta^\theta : L^{p(\theta)} \to L^{q(\theta)}$ uniformly in θ and hence $L : L^p \to L^p$ is bounded.

From this and Theorem 3.1, it follows that the operator T is bounded on L^p. Expanding K up to $\mathbb{R}^n$, we get the conclusion of the theorem. □

4. The case $n > 1$

As was said in the introduction all the results of this paper can be extended to the general case $n \in \mathbb{N}$. In particular, the following results hold:

Theorem 4.1. *For every* $f \in L^{p_0} \cap L^{p_1}$,

$$\overline{\lim_{\theta\to 0}} ||f||_{(L^{p_0}, L^{p_1})^{\delta^{(n)}(\theta)}} \approx ||f||_{p_0}.$$

Theorem 4.2.

$$(L^{p_0}, L^{p_1})_{\delta^{(n)}(\theta)} = L_{\Psi_{n,\theta}},$$

where $\Psi_{n,\theta} = (\gamma_{n,\theta}^{-1})^{p(\theta)}$ *and* $\gamma_{n,\theta}(x) = \frac{x}{\theta}(n + \theta \log x)^n$.

Theorem 4.3.

(i) *For every* $f \in L^{p_0} \cap L^{p_1}$,

$$\overline{\lim_{\theta \to 0}} \frac{1}{\theta^n} ||f||_{\Psi_{n,\theta}} \approx ||f||_{p_0}.$$

From these results, one can easily deduce the following generalization of Theorem 3.1 and the corresponding extension of Theorem 3.2.

Theorem 4.4. *Let T be a linear operator such that, for every* θ, *there exists a uniformly bounded Calderón analytic family of operators* $\overline{L^\theta} : (L^{p_0}, L^{p_1}) \to (L^{q_0}, L^{q_1})$ *so that* $\theta^n (L^\theta)_\theta^{(n)} f$ *converges to Tf almost everywhere, for every* $f \in L^{p_0} \cap L^{p_1}$. *Then*

$$T : L^{p_0} \to L^{q_0}$$

is bounded.

Theorem 4.5. *Under the hypotheses of Theorem* 3.2, *we have that, for every* $k \in \mathbb{N}$, $m(\log S)^k \in M_p$.

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