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Convergent solutions of linear functional difference equations in phase space

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Abstract

By using weighted summable dichotomies and Schauder's fixed point theorem, we prove the existence of convergent solutions of linear functional difference equations. We apply our result to Volterra difference equations with infinite delay.

Keywords: Functional difference equations; Phase space; Summable dichotomies; Volterra difference equation; Convergent solutions; Schauder's theorem

1. Introduction

In this paper, we are concerned with a system of homogeneous linear functional difference equations

$$x(n+1) = L(n, x_n), \quad n \geqslant n_0 \geqslant 0, \tag{1.1}$$

and its perturbed system

$$x(n+1) = L(n, x_n) + f_1(n, x_n) + f_2(n, x_\bullet), \quad n \geqslant n_0 \geqslant 0, \tag{1.2}$$

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where $L : \mathbb{N}(n_0) \times \mathcal{B} \to \mathbb{C}^r$ is a bounded linear map with respect to the second variable, f_1 (respectively f_2) is $\mathbb{C}^r$-valued function defined on the product space $\mathbb{N}(n_0) \times \mathcal{B}$ (respectively $\mathbb{N}(n_0) \times X_k$) under suitable conditions; $\mathcal{B}$ denotes an abstract phase space that we will explain later, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$ (n_0 is a fixed nonnegative integer) and X_k is an appropriate Banach space endowed with the norm given by (2.7); $x_\bullet$ denotes the $\mathcal{B}$-valued function on $\mathbb{N}(n_0)$ defined by $n \mapsto x_n$ where $x_n(s) = x(n+s)$ for any nonpositive integer s.

The theory of functional difference equations in phase space has drawn the attention of several authors in recent years. We only mention here Murakami [9], Elaydi et al. [7], Cuevas and Pinto [3,4].

The main objective of this work is to establish sufficient conditions on the functions f_1 and f_2 to assure the existence of convergent solutions of Eq. (1.2). As applications and examples we apply our results to concrete model functional difference equations, considering Volterra difference equations.

Some results concerned with convergence problem in ordinary difference equations were established by Cheng et al. [5], Drosdowicz and Popenda [6], Szafranski and Szmanda [11], Lakshmikantham and Trigiante [8], Aulbach [2], Agarwal [1].

In a recent paper Cuevas and Pinto [4] have established results of existence of convergent solutions of nonautonomous Volterra difference systems with infinite delay. In this paper we will extend these results replacing the Lipschitz condition and considering general systems with unbounded delay.

The paper is organized as follows. Section 2 provides the definitions and preliminary results to be used in the theorems stated and proved in this work. In Section 3 we study the existence of convergent solutions of Eq. (1.2). In Section 4 we present applications to Volterra difference equations. Throughout this paper we will always assume that $\mathcal{B}$ is a phase space and L is a bounded linear map with respect to the second variable.

2. Preliminaries and notations

Here we explain some notations and the phase space in this article. As usual, we denote by $\mathbb{Z}$, $\mathbb{Z}^+$ and $\mathbb{Z}^-$ the set of all integers, the set of all nonnegative integers and the set of all nonpositive integers, respectively. Let $\mathbb{C}^r$ be the r-dimensional complex Euclidean space with norm $|\cdot|$. For any function $x : \mathbb{Z} \to \mathbb{C}^r$ and $n \in \mathbb{Z}$, we define the function $x_n : \mathbb{Z}^- \to \mathbb{C}^r$ by $x_n(s) = x(n+s)$ for $s \in \mathbb{Z}^-$. We follow the terminology used in Murakami [9] to define the axioms for space $\mathcal{B}$. Thus the phase space $\mathcal{B} = \mathcal{B}(\mathbb{Z}^-, \mathbb{C}^r)$ is a Banach space (with norm denoted by $\|\cdot\|_{\mathcal{B}}$) which is a subfamily of functions from $\mathbb{Z}^-$ into $\mathbb{C}^r$ and it is assumed to satisfy the following axiom:

(A) There are a positive constant $J > 0$ and nonnegative functions $N(\cdot)$ and $M(\cdot)$ on $\mathbb{Z}^+$ with the property that if $x : \mathbb{Z} \to \mathbb{C}^r$ is a function such that $x_0 \in \mathcal{B}$, then for all $n \in \mathbb{Z}^+$,
  (i) $x_n \in \mathcal{B}$,
  (ii) $J|x(n)| \leqslant \|x_n\|_{\mathcal{B}} \leqslant N(n) \sup_{0 \leqslant s \leqslant n} |x(s)| + M(n)\|x_0\|_{\mathcal{B}}$.

To obtain our results we will need one additional property on $\mathcal{B}$, namely:

(B) The inclusion map $i:(B(\mathbb{Z}^-,\mathbb{C}^r),\|\cdot\|_\infty)\to(\mathcal{B},\|\cdot\|_{\mathcal{B}})$ is continuous, i.e., there is a constant $K\geqslant 0$ such that $\|\varphi\|_{\mathcal{B}}\leqslant K\|\varphi\|_\infty$, for all $\varphi\in B(\mathbb{Z}^-,\mathbb{C}^r)$ (where $B(\mathbb{Z}^-,\mathbb{C}^r)$ represents the bounded functions of $\mathbb{Z}^-$ in $\mathbb{C}^r$).

A typical space which satisfies the conditions (A) and (B) is the Banach space $\mathcal{B}_\alpha$ defined by

$$\mathcal{B}_\alpha=\left\{\Phi:\mathbb{Z}^-\to\mathbb{C}^r\colon \sup_{n\in\mathbb{Z}^+}\frac{|\Phi(-n)|}{\alpha(n)}<+\infty\right\}, \tag{2.1}$$

with norm

$$\|\Phi\|_{\mathcal{B}_\alpha}=\sup_{n\in\mathbb{Z}^+}\frac{|\Phi(-n)|}{\alpha(n)},\quad \Phi\in\mathcal{B}_\alpha, \tag{2.2}$$

where $\alpha:\mathbb{Z}^+\to\mathbb{R}^+:=[0,\infty)$ is an arbitrary positive increasing sequence.

From now on $\mathcal{B}$ will denote a phase space satisfying the axioms (A) and (B). For any $n\geqslant\tau$, we define the operator $T(n,\tau):\mathcal{B}\to\mathcal{B}$ by $T(n,\tau)\varphi=x_n(\tau,\varphi,0)$ for $\varphi\in\mathcal{B}$, where $x(\cdot,\tau,\varphi,0)$ denotes the solution of the homogeneous linear system (1.1) passing through (τ,φ). It is clear that the operator $T(n,\tau)$ is linear and by virtue of Axiom (A) it is bounded on $\mathcal{B}$ and satisfies the following properties:

$$T(n,s)T(s,\tau)=T(n,\tau)\quad\text{for } n\geqslant s\geqslant\tau\quad\text{and}\quad T(n,n)=I\quad\text{for } n\geqslant n_0. \tag{2.3}$$

The operator $T(n,\tau)$ is called the solution operator of the homogeneous linear system (1.1) (see [9] for details).

For convenience we will recall the definition of weighted p-summable dichotomy, because we are interested in establishing our result for this kind of dichotomy. In the following $p\geqslant 1$ and a_1,a_2 are two positive sequences.

Definition 2.1. We say that system (1.1) has an (a_1,a_2) weighted p-summable dichotomy (or simply p-summable) (if $p=1$ we simply say that dichotomy is summable), if there is a positive constant $\tilde{K}$ and a projection operator $P(\tau):\mathcal{B}\to\mathcal{B}$, $(P(\tau)=P^2(\tau))$, $\tau\in\mathbb{Z}$, such that if $Q(\tau)=I-P(\tau)$, then

(i) $T(n,\tau)P(\tau)=P(n)T(n,\tau)$, $n\geqslant\tau$;
(ii) The restriction $T(n,\tau)|R(Q(\tau))$, $n\geqslant\tau$, is an isomorphism of $R(Q(\tau))$ onto $R(Q(n))$ ($R(Q(\cdot))$ denotes the range of $Q(\cdot)$) and we define $T(\tau,n)$ as the inverse mapping;
(iii) $\|\Gamma(n,\cdot)\|_{a_2,p}:=(\sum_{s=n_0}^{\infty}\|\Gamma(n,s)\|^p a_2(s))^{1/p}\leqslant\tilde{K}a_1(n)$, for all $n\geqslant n_0$,

where $\Gamma(n,s)$ denotes the Green function associated with Eq. (1.1), i.e.,

$$\Gamma(n,s)=\begin{cases}T(n,s+1)P(s+1) & \text{if } n-1\geqslant s,\\ -T(n,s+1)Q(s+1) & \text{if } s>n-1.\end{cases}$$

This concept was introduced by Pinto [10]. In general, the dichotomies are decomposed in two important groups. The “uniform” dichotomies and the “weighted summable” dichotomies. The uniform dichotomies are the natural extension of ordinary dichotomy and the weighted summable dichotomies are an extension of exponential dichotomy.

We shall need the following lemma:

Lemma 2.1. *Assume that* (1.1) *has a weighted p-summable dichotomy. Then*

$$\begin{aligned}\bigl\|T(n,m)P(m)\bigr\| &\leqslant \frac{\tilde K a_1(n)}{a_2(m)^{1/p}}\bigl\|T(m+1,m)P(m)\bigr\| \\ &\quad\times \prod_{s=m+1}^{n-1}\frac{\tilde K a_1(s+1)}{[\tilde K^p a_1(s+1)^p + a_2(s)]^{1/p}},\end{aligned}$$

for all $n \geqslant m+1$, where $\tilde K$ is the constant of Definition 2.1.

Proof. Putting $\varphi(n) = \|T(n,m)P(m)\|^{-1}$ and $\psi(n) = \sum_{s=m}^{n-1}\varphi(s+1)^p a_2(s)$, we have

$$\begin{aligned}\frac{\psi(n)}{\varphi(n)^p} &= \sum_{s=m}^{n-1}\varphi(s+1)^p\bigl\|\Gamma(n,s)T(s+1,m)P(m)\bigr\|^p a_2(s) \\ &= \sum_{s=m}^{n-1}\bigl\|\Gamma(n,s)\bigr\|^p a_2(s) \leqslant \tilde K^p a_1(n)^p\end{aligned}$$

and hence $\psi(n) \leqslant \varphi(n)^p\tilde K^p a_1(n)^p$. Moreover, $\psi(n+1)-\psi(n) = \varphi(n+1)^p a_2(n) \geqslant \psi(n+1)a_2(n)/\tilde K^p a_1(n+1)^p$ so that noticing that $\psi(n+1) \geqslant \psi(n)$, we get

$$\psi(n+1) \geqslant \left(1 + \frac{a_2(n)}{\tilde K^p a_1(n+1)^p}\right)\psi(n), \quad n \geqslant m+1.$$

Thus, we deduce that

$$\psi(n) \geqslant \prod_{s=m+1}^{n-1}\left(1+\frac{a_2(s)}{\tilde K^p a_1(s+1)^p}\right)\psi(m+1).$$

Therefore, we complete the proof of lemma. □

In the following paragraphs, we consider the $r\times r$ matrix function, $E^0(t)$, $t\in\mathbb{Z}^-$, defined by

$$E^0(t) = \begin{cases} I & (r\times r \text{ unit matrix}), \quad \text{if } t=0,\\ 0 & (r\times r \text{ zero matrix}), \quad \text{if } t<0.\end{cases}$$

Lemma 2.2 (see [4]). *Assume that a function $z:[\tau,\infty)\to\mathcal{B}$ satisfies the relation*

$$z(n) = T(n,\tau)z(\tau) + \sum_{s=\tau}^{n-1}T(n,s+1)E^0p(s), \quad n\geqslant\tau, \tag{2.4}$$

and define a function $y:\mathbb{Z}\to\mathbb{C}^r$ by

$$y(n) = \begin{cases} [z(n)](0) & \text{if } n \geqslant \tau, \\ [z(\tau)](n-\tau) & \text{if } n < \tau. \end{cases} \tag{2.5}$$

Then $y(n)$ satisfies the equation

$$y(n+1) = L(n, y_n) + p(n), \quad n \geqslant \tau, \tag{2.6}$$

together with the relation $y_n = z(n)$, $n \geqslant \tau$.

Let $\{k(n)\}_{n\in\mathbb{Z}^+}$ be an arbitrary positive sequence. Denoted by X_k the Banach space of all k-bounded functions $\eta : \mathbb{N}(n_0) \to \mathcal{B}$ endowed with the norm

$$\|\eta\|_k = \sup_{n\geqslant n_0} \bigl\|\eta(n)\bigr\|_{\mathcal{B}} k(n)^{-1}. \tag{2.7}$$

Also, we denote by $X_{\infty,k}$ the Banach space of all k-convergent functions $\xi \in X_k$, i.e., for which the limit $Z^k_\infty(\xi) := \lim_{n\to\infty} \xi(n)k(n)^{-1}$ exists, endowed with the norm (2.7). On the other hand, for each $\lambda > 0$ we denote by $X_{\infty,k}[\lambda]$ the ball $\|\xi\|_k \leqslant \lambda$ in $X_{\infty,k}$.

Any concrete nonlinear situation requires a compact operator and hence a compactness criterion. On $l_\infty := l_\infty(\mathbb{N}(n_0), \mathbb{R}^m)$, the Banach space of the uniform bounded sequences there is not a good compactness criterion in l_∞ (here $\mathbb{R}^m$ denotes the m-copies of $\mathbb{R} = (-\infty, \infty)$). We will use the following criterion: A bounded equiconvergent at ∞ subset S of l_∞ is relatively compact. We recall that the set S of sequences $x : \mathbb{N}(n_0) \to \mathbb{R}^m$ in l_∞ is said to be equiconvergent at ∞ if all sequence in S is convergent at the point ∞ and for every $\varepsilon > 0$ there exists a N such that $|x(n) - Z^1_\infty(x)| < \varepsilon$ for $n \geqslant N$, for all $x \in S$, where $Z^1_\infty(x) = \lim_{n\to\infty} x(n)$.

To study general nonlinear perturbations we need a compactness criterion on $X_{\infty,k}$, for using the Schauder's fixed point theorem. This powerful theorem has not been sufficiently used in difference equations. We have the following lemma:

Lemma 2.3 (compactness criterion on $X_{\infty,k}$). *Let S be a subset of $X_{\infty,k}$. Suppose the following conditions are satisfied*:

(C_1) *The set $H^k(n) := \{\xi(n)k(n)^{-1}/\xi \in S\}$ is relatively compact on $\mathcal{B}$, for all $n \in \mathbb{N}(n_0)$;*
(C_2) *S is weighted equiconvergent at ∞, i.e., for every $\varepsilon > 0$ there exists a N such that $\|\xi(n)k(n)^{-1} - Z^k_\infty(\xi)\|_{\mathcal{B}} < \varepsilon$ for $n \geqslant N$, for all $\xi \in S$.*

Then S is relatively compact on $X_{\infty,k}$.

Proof. Let $\{\xi_m\}_m$ be a sequence in S, it follows from (C_1) there is a subsequence $\{\xi_{m_j}\}_j$ of $\{\xi_m\}_m$ such that the limit $a(n) = \lim_{j\to\infty} \xi_{m_j}(n)k(n)^{-1}$ exists. On the other hand, the set $Z^k_\infty(S) = \{Z^k_\infty(\xi)/\xi \in S\}$ is relatively compact in $\mathcal{B}$. Indeed, it follows from (C_1) and (C_2) that $Z^k_\infty(S)$ is the uniform limit of the relatively compact set $H^k(n)$, so it is relatively compact in $\mathcal{B}$. Therefore, we can assume that $\{Z^k_\infty(\xi_{m_j})\}_j$ is a Cauchy sequence on $\mathcal{B}$. A simple computation shows that $\{\xi_{m_j}\}_j$ is a Cauchy sequence on $X_{\infty,k}$. To see this, let N be a number ensured in condition (C_2); then if $n_0 \leqslant n \leqslant N$

$$\begin{aligned}\left\|\xi_{m_j}(n)-\xi_{m_k}(n)\right\|_{\mathcal{B}}k(n)^{-1} \leqslant{} & \left\|\xi_{m_j}(n)k(n)^{-1}-a(n)\right\|_{\mathcal{B}} \\ & + \left\|\xi_{m_k}(n)k(n)^{-1}-a(n)\right\|_{\mathcal{B}},\end{aligned}$$

whereas for $n > N$ we get

$$\begin{aligned}\left\|\xi_{m_j}(n)-\xi_{m_k}(n)\right\|_{\mathcal{B}}k(n)^{-1} \leqslant{} & \left\|\xi_{m_j}(n)k(n)^{-1}-Z_\infty^k(\xi_{m_j})\right\|_{\mathcal{B}} \\ & + \left\|\xi_{m_k}(n)k(n)^{-1}-Z_\infty^k(\xi_{m_k})\right\|_{\mathcal{B}} \\ & + \left\|Z_\infty^k(\xi_{m_j})-Z_\infty^k(\xi_{m_k})\right\|_{\mathcal{B}},\end{aligned}$$

which concludes the proof. □

For $\varphi \in P(n_0)\mathcal{B}$, we introduce the following notation $Z_\varphi(n) = T(n, n_0)P(n_0)\varphi$ and we consider the closed and convex subset $X_{\infty,a_1}[\lambda]$ of X_{∞,a_1} and the operator $\mathcal{M}$ defined by

$$(\mathcal{M}\xi)(n) = Z_\varphi(n) + \sum_{s=n_0}^{\infty} \Gamma(n,s)E^0\big(f_1\big(s,\xi(s)\big) + f_2(s,\xi)\big), \tag{2.8}$$

for any $\xi \in X_{\infty,a_1}[\lambda]$. We need the following lemma:

Lemma 2.4. *Let p and q be conjugated exponents. Suppose the following conditions are satisfied*:

(D_1) *The system* (1.1) *has a weighted p-summable dichotomy*;

(D_2) *There is a constant $\lambda > 0$ and functions $F_i : \mathbb{N}(n_0) \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, with the following properties*:

- (a_1) *$F_i(n, u)$, $i = 1, 2$, is nondecreasing in u for each fixed $n \geqslant n_0$*;
- (a_2) *There is a function $l_1 \in l^q := l^q(\mathbb{N}(n_0), \mathbb{R}^+)$ such that for each $(n, \varphi) \in \mathbb{N}(n_0) \times \mathcal{B}$,*

$$\left|f_1(n,\varphi)\right| \leqslant a_2(n)^{1/p} l_1(n) F_1\big(n, \|\varphi\|_{\mathcal{B}} a_1(n)^{-1}\big);$$

- (a_3) *There is a function $l_2 \in l^q$ such that for each $(n, \xi) \in \mathbb{N}(n_0) \times X_{a_1}$ with $\|\xi\|_{a_1} \leqslant \lambda$,*

$$\left|f_2(n,\xi)\right| \leqslant a_2(n)^{1/p} l_2(n) F_2\big(n, \|\xi\|_{a_1}\big);$$

- (a_4) *$\rho[F_i] := \sup_{n \geqslant n_0} F_i(n, \lambda) < +\infty$, $i = 1, 2$, and*

$$\delta := K\tilde{K}\big(\rho[F_1]\|l_1\|_q + \rho[F_2]\|l_2\|_q\big) < \lambda;$$

(D_3) *The sequences a_1 and a_2 satisfy*

$$\lim_{n\to\infty} \prod_{s=m+1}^{n-1} \frac{\tilde{K} a_1(s+1)}{[\tilde{K}^p a_1(s+1)^p + a_2(s)]^{1/p}} = 0,$$

for all $m \geqslant n_0$;

(D_4) *There are functions $\Theta_i : \mathbb{N}(n_0) \times \mathcal{B} \to \mathbb{C}^r$, $i = 1, 2$, such that for all $\xi \in X_{\infty,a_1}[\lambda]$*:

- (b_1) *$f_1(n, \xi(n)) - \Theta_1(n, Z_\infty^{a_1}(\xi)) = 0(a_2(n)^{1/p} l_1(n))$ as $n \to \infty$*;

(b$_2$) $f_2(n,\xi) - \Theta_2(n, Z_\infty^{a_1}(\xi)) = 0(a_2(n)^{1/p} l_2(n))$ *as* $n \to \infty$;

(D$_5$) *The limits* $R_i(\varphi) := Z_\infty^{a_1}(\Omega_i(\cdot,\varphi))$ $(i = 1, 2)$ *exist for all* $\varphi \in \mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \leqslant \lambda$, *where* $\Omega_i(n,\varphi) := \sum_{s=n_0}^{\infty} \Gamma(n,s) E^0(\Theta_i(s,\varphi))$.

Then there is a constant $M > 0$ *such that for* $\varphi \in P(n_0)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \leqslant (\lambda - \delta)M^{-1}$ *we have that* $\mathcal{M}$ *maps* $X_{\infty,a_1}[\lambda]$ *to itself.*

Proof. We begin defining $M := \sup_{n \geqslant n_0} \|T(n,n_0)P(n_0)\| a_1(n)^{-1}$ and $\varphi \in P(n_0)\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}} \leqslant (\lambda - \delta)M^{-1}$. To establish that $\xi \in X_{\infty,a_1}[\lambda]$ implies $\|\mathcal{M}\xi\|_{a_1} \leqslant \lambda$, using hypothesis (D$_2$), we deduce that

$$\begin{aligned}
&\big\|(\mathcal{M}\xi)(n)\big\|_{\mathcal{B}} a_1(n)^{-1} \\
&\quad \leqslant M\|\varphi\|_{\mathcal{B}} + K a_1(n)^{-1} \big\|\Gamma(n,\cdot)\big\|_{a_2,p} \big(\rho[F_1]\|l_1\|_q + \rho[F_2]\|l_2\|_q\big) \\
&\quad \leqslant M\|\varphi\|_{\mathcal{B}} + \delta \leqslant \lambda.
\end{aligned}$$

The operator $\mathcal{M}$ defined in (2.8) may be rewritten as

$$\begin{aligned}
(\mathcal{M}\xi)(n) = {} & Z_\varphi(n) + (\Omega_1 + \Omega_2)\big(n, Z_\infty^{a_1}(\xi)\big) \\
& + \sum_{s=n_0}^{\infty} \Gamma(n,s) E^0\big(f_1\big(s,\xi(s)\big) - \Theta_1\big(s, Z_\infty^{a_1}(\xi)\big)\big) \\
& + \sum_{s=n_0}^{\infty} \Gamma(n,s) E^0\big(f_2(s,\xi) - \Theta_2\big(s, Z_\infty^{a_1}(\xi)\big)\big).
\end{aligned}$$

From (D$_3$), (D$_4$) and taking into account Lemma 2.1, we can see that the a_1-limit of the last two terms of the right side of above expression are zero. Therefore $Z_\infty^{a_1}(\mathcal{M}\xi) = (R_1 + R_2)(Z_\infty^{a_1}(\xi))$. This completes the proof of Lemma 2.4. □

Remark 2.1. If the system (1.1) has a weighted summable dichotomy the hypothesis (D$_2$)(a$_2$), (D$_2$)(a$_3$), (D$_3$) and (D$_4$) can be considered with $p = 1$ and $l_i \in l_\infty$, $i = 1, 2$. On the other hand, the condition (D$_2$)(a$_4$) is replaced by $\delta := K\tilde{K}(\rho[F_1]\|l_1\|_\infty + \rho[F_2]\|l_2\|_\infty) < \lambda$.

3. Existence of convergent solutions

By using Schauder's fixed point theorem and weighted p-summable dichotomies, we prove the existence of convergent solutions of Eq. (1.2). We have the following result:

Theorem 3.1. *Assume that the hypotheses* (D$_i$), $i = 1, 2, 3, 4$, *of Lemma* 2.4 *hold. In addition suppose also that the following conditions are satisfied*:

(E$_1$) *For every* $n \in \mathbb{N}(n_0)$, *the functions* $a_2(n)^{-1/p} l_i(n)^{-1} f_i(n,\cdot)$, $i = 1, 2$, *are continuous*;

(E$_2$) *The limits* $\pi_1(\xi) = Z_\infty^{\nu_1}(f_1(\cdot, \xi(\cdot)))$ *and* $\pi_2(\xi) = Z_\infty^{\nu_2}(f_2(\cdot, \xi))$ *exist uniformly in* $\xi \in X_{\infty,a_1}[\lambda]$, *where* $\nu_i(n) = a_2(n)^{1/p} l_i(n)$, $i = 1, 2$;

(E$_3$) *There are functions* $G_i : \mathbb{N}(n_0) \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, *with* $\rho[G_i] := \sup_{n \geqslant n_0} G_i(n, \lambda)$, $G_i(n, \cdot)$ *is nondecreasing for each fixed* $n \in \mathbb{N}(n_0)$ *such that for each* $(n, \varphi) \in \mathbb{N}(n_0) \times \mathcal{B}$, $|\Theta_i(n, \varphi)| \leqslant \nu_i(n) G_i(n, \|\varphi\|_{\mathcal{B}})$;

(E$_4$) *The limits* $R_i(\varphi) := Z_\infty^{a_1}(\Omega_i(\cdot, \varphi))$ (*see* (D$_5$)), $i = 1, 2$, *exist uniformly in* $\varphi \in \mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \leqslant \lambda$.

Then there is a constant $M > 0$ *such that for each* $\varphi \in P(n_0)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \leqslant (\lambda - \delta)M^{-1}$ *there is a solution* $y = y(\varphi) = y(n, n_0, \psi)$ *of Eq.* (1.2) *with* $P(n_0)\psi = \varphi$ *such that* $Z_\infty^{a_1}(y_\bullet)$ *exists and* $\|y_\bullet\|_{a_1} \leqslant \lambda$. *The limit* $Z_\infty^{a_1}(y_\bullet)$ *is a fixed point of the operator* $R_1 + R_2$. *Moreover, we have the following asymptotic formula*:

$$y_n(\varphi) = Z_\varphi(n) + (\Omega_1 + \Omega_2)\big(n, Z_\infty^{a_1}(y_\bullet)\big) + o\big(a_1(n)\big), \quad n \to \infty. \tag{3.1}$$

Proof. Putting $M := \sup_{n \geqslant n_0} \|T(n, n_0)P(n_0)\| a_1(n)^{-1}$ and $\varphi \in P(n_0)\mathcal{B}$ with $\|\varphi\|_{\mathcal{B}} \leqslant (\lambda - \delta)M^{-1}$. The Schauder's fixed point theorem will be used to shows that $\mathcal{M}$ (see (2.8)) has a fixed point in $X_{\infty,a_1}[\lambda]$.

We need show that $\mathcal{M}$ is continuous. Let $\{\xi_m\}_{m=1}^\infty$ be a sequence of elements of $X_{\infty,a_1}[\lambda]$ such that $\xi_m \mapsto \xi$ in $X_{\infty,a_1}[\lambda]$. We introduce the following notation: $g_1(n, \varphi) := \nu_1(n)^{-1} f_1(n, \varphi)$, $\varphi \in \mathcal{B}$; $g_2(n, \xi) := \nu_2(n)^{-1} f_2(n, \xi)$, $\xi \in X_{a_1}$. Let $\varepsilon > 0$, and select an $n_1 \in \mathbb{N}(n_0)$ which is large enough to satisfy $\gamma_i := 2K\tilde{K}\rho[F_i](\sum_{s=n_1}^\infty l_i(s)^q)^{1/q} < \varepsilon$, $i = 1, 2$. Then, we obtain the following estimate:

$$\begin{aligned}
&\big\|(\mathcal{M}\xi_m)(n) - (\mathcal{M}\xi)(n)\big\|_{\mathcal{B}} a_1(n)^{-1} \\
&\quad \leqslant K a_1(n)^{-1} \sum_{s=n_0}^{n_1-1} \big\|\Gamma(n, s)\big\| \nu_1(s) \big|g_1\big(s, \xi_m(s)\big) - g_1\big(s, \xi(s)\big)\big| \\
&\qquad + K a_1(n)^{-1} \sum_{s=n_0}^{n_1-1} \big\|\Gamma(n, s)\big\| \nu_2(s) \big|g_2(s, \xi_m) - g_2(s, \xi)\big| + \gamma_1 + \gamma_2 \\
&\quad \leqslant K\tilde{K} \|l_1\|_q \max_{n_0 \leqslant s \leqslant n_1 - 1} \big|g_1\big(s, \xi_m(s)\big) - g_1\big(s, \xi(s)\big)\big| \\
&\qquad + K\tilde{K} \|l_2\|_q \max_{n_0 \leqslant s \leqslant n_1 - 1} \big|g_2(s, \xi_m) - g_2(s, \xi)\big| + 2\varepsilon.
\end{aligned}$$

As an immediate consequence of the continuity of g_i, $i = 1, 2$, it follows that $\mathcal{M}\xi_m$ a_1-converges to $\mathcal{M}\xi$ as $n \to \infty$.

We will show that the image of $\mathcal{M}$ is relatively compact in X_{∞,a_1}. We will prove that $H^{a_1}(n)$ is relatively compact in $\mathcal{B}$ for all $n \in \mathbb{N}(n_0)$. We consider a arbitrary sequence $(\xi_m)_m$ in $X_{\infty,a_1}[\lambda]$ and define the following sequence $\varphi_m(n) := f_1(n, \xi_m(n))/\nu_1(n)$, $n \geqslant n_0$. We note that the set $A_{f_1} = \{\varphi_m / m \in \mathbb{N}\}$ is relatively compact in l_∞. In fact, using (D$_2$), we infer that $\|\varphi_m\|_\infty \leqslant \rho[F_1]$ for every $m \in \mathbb{N}$. In this way A_{f_1} is bounded. On the other hand, by hypothesis (E$_2$) it follows that is equiconvergent. Therefore there is a subsequence $\{\varphi_{m_j}\}_j$ of $\{\varphi_m\}_m$ uniformly convergent in l_∞, i.e., there is a function $\varphi_{f_1} \in l_\infty$ such that $\|\varphi_{m_j} - \varphi_{f_1}\|_\infty \to 0$, as $j \to \infty$.

We next consider a sequence $\psi_{m_j}(n) := f_2(n, \xi_{m_j})/\nu_2(n)$, $n \geqslant n_0$. We can repeat the previous argument to conclude that the set $\{\psi_{m_j}/j \in \mathbb{N}\}$ is relatively compact in l_∞. Hence there is a function $\varphi_{f_2} \in l_\infty$ such that $\|\psi_{m_j} - \varphi_{f_2}\|_\infty \to 0$ as $j \to \infty$. Let $\varphi_i'(n) := \nu_i(n)\varphi_{f_i}(n)$, $i = 1, 2$. Then, we can verify that the sequence $(\mathcal{M}\xi_{m_j})(n)a_1(n)^{-1}$ converges to $Z_\varphi(n)a_1(n)^{-1} + a_1(n)^{-1}\sum_{s=n_0}^{\infty}\Gamma(n,s)E^0(\varphi_1'(s) + \varphi_2'(s))$, as $j \to \infty$.

In fact, we note that

$$a_1(n)^{-1}\left\|\sum_{s=n_0}^{\infty}\Gamma(n,s)E^0\big(\varphi_1'(s)+\varphi_2'(s)\big)\right\|_{\mathcal{B}} \leqslant K\tilde{K}\big(\|\varphi_{f_1}\|_\infty\|l_1\|_q + \|\varphi_{f_2}\|_\infty\|l_2\|_q\big)$$

and

$$\begin{aligned}
&a_1(n)^{-1}\left\|\sum_{s=n_0}^{\infty}\Gamma(n,s)E^0\big(f_1\big(s,\xi_{m_j}(s)\big) - \varphi_1'(s) + f_2(s,\xi_{m_j}) - \varphi_2'(s)\big)\right\|_{\mathcal{B}}\\
&\leqslant Ka_1(n)^{-1}\sum_{s=n_0}^{\infty}\big\|\Gamma(n,s)\big\|\nu_1(s)\big|\varphi_{m_j}(s) - \varphi_{f_1}(s)\big|\\
&\quad + Ka_1(n)^{-1}\sum_{s=n_0}^{\infty}\big\|\Gamma(n,s)\big\|\nu_2(s)\big|\psi_{m_j}(s) - \varphi_{f_2}(s)\big|\\
&\leqslant K\tilde{K}\big(\|\varphi_{m_j} - \varphi_{f_1}\|_\infty\|l_1\|_q + \|\psi_{m_j} - \varphi_{f_2}\|_\infty\|l_2\|_q\big).
\end{aligned}$$

On the other hand, taking into account that $Z_\infty^{a_1}(\|T(\cdot,m)P(m)\|) = 0$, for every $m \in \mathbb{N}$, and condition (E_4), we observe that the weighted equiconvergence at ∞ of the set $\mathcal{M}X_{\infty,a_1}[\lambda]$ is a consequence from the following two inequalities. Let $\xi \in X_{\infty,a_1}[\lambda]$; then

$$\begin{aligned}
&a_1(n)^{-1}\big\|(\mathcal{M}\xi)(n) - (\Omega_1+\Omega_2)\big(n, Z_\infty^{a_1}(\xi)\big)\big\|_{\mathcal{B}}\\
&\leqslant a_1(n)^{-1}\big\|T(n,n_0)P(n_0)\big\|\|\varphi\|_{\mathcal{B}}\\
&\quad + K\big\|\Gamma(n_1,\cdot)\big\|_{a_2,p}a_1(n)^{-1}\big\|T(n,n_1)P(n_1)\big\|\sum_{i=1}^{2}\big(\rho[F_i]+\rho[G_i]\big)\|l_i\|_q\\
&\quad + K\big\|\Gamma(n,\cdot)\big\|_{a_2,p}a_1(n)^{-1}\sum_{i=1}^{2}\big(\rho[F_i]+\rho[G_i]\big)\left(\sum_{s=n_1}^{\infty}l_i(s)^q\right)^{1/q}\\
&\leqslant a_1(n)^{-1}\big\|T(n,n_0)P(n_0)\big\|\|\varphi\|_{\mathcal{B}}\\
&\quad + K\tilde{K}a_1(n_1)a_1(n)^{-1}\big\|T(n,n_1)P(n_1)\big\|\sum_{i=1}^{2}\big(\rho[F_i]+\rho[G_i]\big)\|l_i\|_q\\
&\quad + K\tilde{K}\sum_{i=1}^{2}\big(\rho[F_i]+\rho[G_i]\big)\left(\sum_{s=n_1}^{\infty}l_i(s)^q\right)^{1/q}
\end{aligned}$$

and

$$\begin{aligned}&\left\|a_1(n)^{-1}(\Omega_1+\Omega_2)\big(n, Z_\infty^{a_1}(\xi)\big) - Z_\infty^{a_1}(\mathcal{M}\xi)\right\|_{\mathcal{B}}\\&\quad\leqslant \left\|a_1(n)^{-1}\Omega_1\big(n, Z_\infty^{a_1}(\xi)\big) - R_1\big(Z_\infty^{a_1}(\xi)\big)\right\|_{\mathcal{B}}\\&\qquad+\left\|a_1(n)^{-1}\Omega_2\big(n, Z_\infty^{a_1}(\xi)\big) - R_2\big(Z_\infty^{a_1}(\xi)\big)\right\|_{\mathcal{B}}.\end{aligned}$$

By compactness criterion on X_{∞,a_1}, we conclude that the image of $\mathcal{M}$ is relatively compact in X_{∞,a_1}. Using the Schauder's fixed point theorem $\mathcal{M}$ has a fixed point $\xi \in X_{\infty,a_1}[\lambda]$. Define

$$y(n) = \begin{cases}[\xi(n)](0) & \text{if } n \geqslant n_0,\\ [\xi(n_0)](n-n_0) & \text{if } n < n_0.\end{cases}$$

Lemma 2.2 implies that $y(n)$ is solution of Eq. (1.2) and $y_n = \xi(n)$, $n \geqslant n_0$. This completes the proof of Theorem 3.1. □

We can obtain uniqueness in some sense of solutions of Eq. (1.2) for certain special cases. We have the following corollary:

Corollary 3.2. *Assume that the hypotheses of Theorem* 3.1 *are satisfied, where* $G_i(n,u) = u$ *and* $\Theta_i(n,\varphi)$ *(see* (E$_3$)) $i = 1, 2$, *is a linear operator in* $\varphi \in \mathcal{B}$ *for every* $n \in \mathbb{N}(n_0)$. *Suppose* $K\tilde{K}(\|l_1\|_q + \|l_2\|_q) < 1$; *then there is a constant* $M > 0$ *such that for each* $\varphi \in P(n_0)\mathcal{B}$ *with* $\|\varphi\|_{\mathcal{B}} \leqslant (\lambda-\delta)M^{-1}$, *there is a unique solution* $y = y(\varphi) = y(n, n_0, \psi)$ *with* $P(n_0)\psi = \varphi$ *such that* $Z_\infty^{a_1}(y_\bullet)$ *exists and* $\|y_\bullet\|_{a_1} \leqslant \lambda$ *(uniqueness being understood in the sense that any two such solutions differ by a factor of order* $o(a_1(n))$, *as* $n \to \infty$).

Proof. It is sufficient to observe that $R_1 + R_2$ is contractive on $\mathcal{B}$. For $\xi, \eta \in \mathcal{B}$, we have the following estimate:

$$\begin{aligned}&\left\|\Omega_i(n,\xi) - \Omega_i(n,\eta)\right\|_{\mathcal{B}} a_1(n)^{-1}\\&\quad\leqslant K\sum_{s=n_0}^{\infty}\left\|\Gamma(n,s)\right\|\left|\Theta_i(s,\xi-\eta)\right|a_1(n)^{-1} \leqslant K\tilde{K}\|l_i\|_q\|\xi-\eta\|_{\mathcal{B}}.\end{aligned}$$

Letting $n \to \infty$ in the above estimate, we get

$$\left\|R_i(\xi) - R_i(\eta)\right\|_{\mathcal{B}} \leqslant K\tilde{K}\|l_i\|_q\|\xi-\eta\|_{\mathcal{B}}, \quad i = 1, 2.$$

This implies that

$$\left\|(R_1+R_2)(\xi) - (R_1+R_2)(\eta)\right\|_{\mathcal{B}} \leqslant K\tilde{K}\big(\|l_1\|_q + \|l_2\|_q\big)\|\xi-\eta\|_{\mathcal{B}},$$

and $R_1 + R_2$ is a contraction on $\mathcal{B}$ since $K\tilde{K}(\|l_1\|_q + \|l_2\|_q) < 1$. □

Remark 3.1. If the system (1.1) has a weighted summable dichotomy, taking into account the Remark 2.1, the hypotheses of Theorem 3.1 can be considered with $p = 1$ and $l_i \in l^1$, $i = 1, 2$. On the other hand, the condition $K\tilde{K}(\|l_1\|_q + \|l_2\|_q) < 1$ of Corollary 3.2 is replaced by $K\tilde{K}(\|l_1\|_\infty + \|l_2\|_\infty) < 1$.

Remark 3.2. We emphasize that Theorem 3.1 is interesting in two respects. Firstly, certain conditions guarantee the existence of a weighted convergent solution to Eq. (1.2). Secondly, its usefulness in applications (see Section 4), since the results include a larger class, namely, systems (1.2) where Eq. (1.1) has a weighted p-summable dichotomy.

4. Applications and examples

We complete this work applying our previous results to the Volterra difference equations with infinite delay.

Let $a(n)$, $b(n)$, $k(s)$, $r(t)$ be sequences of complex numbers defined for $n \in \mathbb{N}(n_0)$, $s \in \mathbb{Z}^+$, $t \in \mathbb{Z}$, and let $\alpha : \mathbb{Z}^+ \to \mathbb{R}^+$ be an arbitrary positive increasing sequence such that

$$\sum_{n=0}^{\infty} \bigl|k(n)\bigr|\alpha(n) < +\infty. \tag{4.1}$$

We consider the Volterra difference equations

$$x(n+1) = \sum_{s=-\infty}^{n} a(n)k(n-s)x(s), \tag{4.2}$$

$$y(n+1) = \sum_{s=-\infty}^{n} a(n)k(n-s)y(s) + \nu \sum_{s=-\infty}^{n} b(n)r(s)\bigl(y(s)\bigr)^{\mu}, \tag{4.3}$$

with $\mu \in \mathbb{Z}^+$, $n \geqslant n_0$ and $\nu \in \mathbb{R}$, $|\nu|$ small enough.

Equations (4.2) and (4.3) are viewed as functional difference equations on the phase space $\mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is defined as in the Section 2 with $r = 1$. In fact, let $\xi : \mathbb{N}(n_0) \to \mathcal{B}_\alpha$; we note that

$$L(n,\phi) = \sum_{j=0}^{\infty} a(n)k(j)\phi(-j), \phi \in \mathcal{B}_\alpha, \tag{4.4}$$

$$f(n,\xi) = \sum_{\tau=-\infty}^{n_0-1} \nu b(n)r(\tau)\bigl(\bigl[\xi(n_0)\bigr](\tau-n_0)\bigr)^{\mu} + \sum_{\tau=n_0}^{n} \nu b(n)r(\tau)\bigl(\bigl[\xi(\tau)\bigr](0)\bigr)^{\mu}. \tag{4.5}$$

We have the following result:

Theorem 4.1. *Let p and q be conjugated exponents. Suppose the following conditions are satisfied*:

(H_1) *Equation* (4.2) *possesses a* (a_1,a_2) *weighted p-summable dichotomy with a_1 and a_2 satisfying the condition* (D_3);

(H_2) $\chi_\mu := \sum_{\tau=-\infty}^{\infty} |r(\tau)|(\tilde{a}_1(\tau))^{\mu} < +\infty$, *where*

$$\tilde{a}_1(\tau) = \begin{cases} a_1(\tau)\alpha(0) & \text{if } \tau \geqslant n_0, \\ a_1(n_0)\alpha(n_0-\tau) & \text{if } \tau < n_0; \end{cases}$$

(H₃) $\sum_{s=n_0}^{\infty} |sb(s)|(1 + a_2(s)^{1-q}) < +\infty$;
(H₄) *Let λ be a positive real number and ν a real number such that*

$$\delta := K\tilde{K}\lambda^{\mu}|\nu|\chi_{\mu}\|\cdot b(\cdot)\|_1^{1/p}\|\cdot b(\cdot)a_2^{1-q}\|_1^{1-q} < \lambda,$$

where K is given by axiom (B) *and $\tilde{K}$ is the constant of Definition* 2.1(iii).

Then there is a constant $M > 0$ such that for each $\varphi \in P(n_0)\mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha} \leqslant (\lambda - \delta)M^{-1}$ there is an a_1-convergent solution $y = y(\varphi) = y(n, n_0, \psi)$ of Eq. (4.3) *with $P(n_0)\psi = \varphi$, and $Z_\infty^{a_1}(y_\bullet)$ is a fixed point of R and $\|y_\bullet\|_{a_1} \leqslant \lambda$. Moreover, we have the following asymptotic formula*:

$$y_n(\varphi) = Z_\varphi(n) + o\big(a_1(n)\big), \quad n \to \infty.$$

Proof. We begin defining

$$l(n) = |\nu|a_2(n)^{-1/p}\sum_{\tau=-\infty}^{n}\big|nb(n)\big|\big|r(\tau)\big|\big(a_1(\tau)\big)^{\mu}, \quad n \geqslant n_0.$$

We note that $l \in l^q$. In fact,

$$l(n) \leqslant \chi_{\mu}|\nu|\big\|\cdot b(\cdot)\big\|_1^{1/p}\big|nb(n)\big|^{1/q}a_2(n)^{-1/p},$$

implying that

$$\|l\|_q \leqslant \chi_{\mu}|\nu|\big\|\cdot b(\cdot)\big\|_1^{1/p}\big\|\cdot b(\cdot)a_2^{1-q}\big\|_1^{1/q}.$$

A simple calculation shows that

$$\big|f(n,\xi)\big| \leqslant \left(|\nu|\sum_{\tau=-\infty}^{n}\big|b(n)\big|\big|r(\tau)\big|\big(\tilde{a}_1(\tau)\big)^{\mu}\right)\|\xi\|_{a_1}^{\mu} \leqslant a_2(n)^{1/p}l(n)\|\xi\|_{a_1}^{\mu}.$$

Define a function $\Theta : \mathbb{N}(n_0) \times \mathcal{B}_\alpha \to \mathbb{C}^r$ by

$$\Theta(n,\varphi) = \sum_{\tau=n_0}^{n}\nu b(n)r(\tau)\big(a_1(\tau)\varphi(0)\big)^{\mu} + \sum_{\tau=-\infty}^{n_0-1}\nu b(n)r(\tau)\big(a_1(n_0)\varphi(\tau-n_0)\big)^{\mu}.$$

To prove (D₄)(b₂), let ξ be a function in $X_{\infty,a_1}[\lambda]$, and if we choose n_1 large sufficiently, we have the following estimate:

$$\begin{aligned}
&\big|f(n,\xi) - \Theta\big(n, Z_\infty^{a_1}(\xi)\big)\big| \\
&\quad\leqslant \mu\lambda^{\mu-1}\sum_{\tau=n_0}^{n}|\nu|\big|b(n)\big|\big|r(\tau)\big|\big(\tilde{a}_1(\tau)\big)^{\mu}\left\|\frac{\xi(\tau)}{a_1(\tau)} - Z_\infty^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha} \\
&\qquad+ \mu\lambda^{\mu-1}\sum_{\tau=-\infty}^{n_0-1}|\nu|\big|b(n)\big|\big|r(\tau)\big|\big(\tilde{a}_1(\tau)\big)^{\mu}\left\|\frac{\xi(n_0)}{a_1(n_0)} - Z_\infty^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha} \\
&\quad\leqslant \mu\lambda^{\mu-1}|\nu|\big|b(n)\big|\left(\sum_{\tau=-\infty}^{n_1-1}\big|r(\tau)\big|\big(\tilde{a}_1(\tau)\big)^{\mu}\right)\max_{n_0\leqslant s\leqslant n_1-1}\left\|\frac{\xi(s)}{a_1(s)} - Z_\infty^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha}
\end{aligned}$$

$$+\ \mu\lambda^{\mu-1}|\nu|\left(\sum_{\tau=n_1}^{n}\big|b(n)\big|\big|r(\tau)\big|\big(\tilde{a}_1(\tau)\big)^{\mu}\right)\sup_{s\geqslant n_1}\left\|\frac{\xi(s)}{a_1(s)}-Z_{\infty}^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha}$$

$$\leqslant \mu\lambda^{\mu-1}|\nu|\chi_\mu\big|b(n)\big|\max_{n_0\leqslant s\leqslant n_1-1}\left\|\frac{\xi(s)}{a_1(s)}-Z_{\infty}^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha}$$

$$+\ \mu\lambda^{\mu-1}|\nu|a_2(n)^{1/p}l(n)\sup_{s\geqslant n_1}\left\|\frac{\xi(s)}{a_1(s)}-Z_{\infty}^{a_1}(\xi)\right\|_{\mathcal{B}_\alpha}.$$

We observe that since $Z_{\infty}^{a_1}(\xi)$ exists and $Z_{\infty}^{a_2^{1/p}l}(|b(\cdot)|)=0$, we get from the previous inequality

$$f(n,\xi)-\Theta\big(n,Z_{\infty}^{a_1}(\xi)\big)=o\big(a_2(n)^{1/p}l(n)\big),\quad n\to\infty,$$

for all $\xi\in X_{\infty,a_1}[\lambda]$.

We claim that the function $a_2(n)^{-1/p}l(n)^{-1}f(n,\cdot)$ is continuous for every $n\in\mathbb{N}(n_0)$. Indeed, let $\eta\in X_{a_1}$ fixed, $\varepsilon>0$ and let $\xi\in X_{a_1}$ such that

$$\|\xi-\eta\|_{a_1}<\min\left\{1,\frac{\varepsilon}{\mu(1+\|\eta\|_{a_1})^{\mu-1}}\right\}.$$

We infer that

$$\begin{aligned}&\big|f(n,\xi)-f(n,\eta)\big|a_2(n)^{-1/p}l(n)^{-1}\\&\quad\leqslant\mu\big(1+\|\eta\|_{a_1}\big)^{\mu-1}a_2(n)^{-1/p}l(n)^{-1}|\nu|\left(\sum_{\tau=-\infty}^{n}\big|b(n)\big|\big|r(\tau)\big|a_1(\tau)\right)\|\xi-\eta\|_{a_1}\\&\quad<\varepsilon.\end{aligned}$$

To prove (E_2) with $f_2=f$ and $l_2=l$, we note

$$\sup_{\|\xi\|_{a_1}\leqslant\lambda}\frac{|f(n,\xi)|}{a_2(n)^{1/p}l(n)}\leqslant\frac{\lambda^{\mu}}{n},$$

which implies $Z_{\infty}^{a_2^{1/p}l}(f(\cdot,\xi))=0$ uniformly in $\xi\in X_{\infty,a_1}[\lambda]$.

To prove (E_3) for Θ, we observe

$$\big|\Theta(n,\varphi)\big|\leqslant\frac{1}{n}a_2(n)^{1/p}l(n)\|\varphi\|_{\mathcal{B}_\alpha}^{\mu},$$

for all $\varphi\in\mathcal{B}_\alpha$. Moreover, $Z_{\infty}^{a_2^{1/p}l}(\Theta(\cdot,\varphi))=0$ uniformly in $\varphi\in\mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha}\leqslant\lambda$. To prove ($E_4$), let ε be an arbitrary positive number; we can choose $n_1\geqslant n_0$ and $n_2\geqslant n_1$ such that for all $\varphi\in\mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha}\leqslant\lambda$

$$\big|\Theta(n,\varphi)\big|\leqslant\frac{\varepsilon}{2K\tilde{K}\|l\|_q}a_2(n)^{1/p}l(n),\quad n\geqslant n_1,$$

and

$$\prod_{s=n_1+1}^{n-1}\frac{\tilde{K}a_1(s+1)}{[\tilde{K}^p a_1(s+1)^p+a_2(s)]^{1/p}}<\varepsilon/C,\quad n\geqslant n_2,$$

where

$$C := 2\lambda^{\mu} K \tilde{K}^2 a_1(n_1) a_2(n_1)^{-1/p} \|l\|_q \left\| T(n_1+1, n_1) P(n_1) \right\|.$$

Hence, for $n \geqslant n_2$, taking into account Lemma 2.1, we can infer that

$$\begin{aligned} a_1(n)^{-1} \left\| \Omega(n, \varphi) \right\|_{\mathcal{B}_\alpha} &\leqslant \lambda^{\mu} K \|l\|_q a_1(n)^{-1} \left\| T(n, n_1) P(n_1) \right\| \left\| \Gamma(n_1, \cdot) \right\|_{a_2, p} \\ &\quad + \frac{\varepsilon}{2\tilde{K} \|l\|_q} a_1(n)^{-1} \sum_{s=n_1}^{\infty} \left\| \Gamma(n, s) \right\| a_2(s)^{1/p} l(s) \\ &\leqslant \frac{1}{2} C \prod_{s=n_1+1}^{n-1} \frac{\tilde{K} a_1(s+1)}{[\tilde{K}^p a_1(s+1)^p + a_2(s)]^{1/p}} \\ &\quad + \frac{\varepsilon}{2\tilde{K}} a_1(n)^{-1} \left\| \Gamma(n, \cdot) \right\|_{a_2, p} \\ &< \varepsilon. \end{aligned}$$

Therefore $\Omega(n, \varphi) = o(a_1(n))$ uniformly in $\varphi \in \mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha} \leqslant \lambda$. This completes the proof of Theorem 4.1. □

Remark 4.1. From the proof of Theorem 4.1 we can see that condition (D_3) can be replaced by $\lim_{n\to\infty} a_1(n)^{-1} T(n, m) P(m) = 0$. On the other hand, if the system (4.2) has a weighted summable dichotomy, taking into account the Remark 3.1, the hypotheses of Theorem 4.1 can be considered with $p = 1$ but we need the condition $\| \cdot b(\cdot) a_2^{-1} \|_1 < \infty$ instead conditions (H_3). Moreover, condition (H_4) is replaced by $\delta := K \tilde{K} \lambda^{\mu} |\nu| \chi_\mu \times \| \cdot b(\cdot) a_2^{-1} \|_\infty < \lambda$.

Remark 4.2. Using previous proofs, it is easy to obtain the same type of result (Theorem 4.1) for the following nonautonomous Volterra difference system with infinite delay:

$$x(n+1) = \sum_{s=-\infty}^{n} A(n) K(n-s) x(s), \quad n \geqslant n_0 \geqslant 0, \tag{4.6}$$

and its perturbed system

$$y(n+1) = \sum_{s=-\infty}^{n} \left\{ A(n) K(n-s) + \nu B(n) R(s) \right\} y(s), \quad n \geqslant n_0 \geqslant 0, \tag{4.7}$$

where $A(n)$, $K(m)$, $B(n)$, $R(s)$ are $r \times r$ matrices defined for $n \in \mathbb{N}(n_0)$, $m \in \mathbb{Z}^+$ and $s \in \mathbb{Z}$.

Next, we will to provide an example to illustrate the usefulness of Theorem 4.1. Here $\alpha(n) = 2^n$, $a_1(n) = (1/\sqrt{2})^n$ and $a_2(n) = (1/4)^{n+1}$.

Example 4.1. Let $a > 1$; we consider the following homogeneous linear difference equation:

$$x(n+1) = a^n x(n), \quad n \geqslant n_0 \geqslant 0. \tag{4.8}$$

We begin with a complete analysis to check the dichotomic properties. We recall that solution $x(\cdot, m, \varphi)$ of (4.8) is given by

$$x(n, m, \varphi) = a^{m+(m+1)+\cdots+n-1}\varphi(0) = \sqrt{a}^{(n-m)(n+m-1)}\varphi(0)$$

for $n \geqslant m$. Hence

$$\big[T(n, m)\varphi\big](\theta) = \begin{cases} \varphi(0)\sqrt{a}^{(n+\theta-m)(n+\theta+m-1)}, & m-n \leqslant \theta \leqslant 0, \\ \varphi(n+\theta-m), & n+\theta \leqslant m. \end{cases}$$

A computation shows that

$$T(n, s)T(s, m) = T(n, m), \quad n \geqslant s \geqslant m,$$
$$T(n, n) = I, \quad n \geqslant m.$$

We need to define appropriate projections in this problem. In this case the projections can be taken as $P(n):\mathcal{B}_\alpha \to \mathcal{B}_\alpha$ given by

$$\big[P(n)\varphi\big](\theta) = \begin{cases} \varphi(\theta) - \varphi(0)\sqrt{a}^{(2n\theta+\theta^2-\theta)}, & -n \leqslant \theta \leqslant 0, \\ 0, & \theta < -n. \end{cases}$$

and $Q(n) = I - P(n):\mathcal{B}_\alpha \to \mathcal{B}_\alpha$ given by

$$\big[Q(n)\varphi\big](\theta) = \begin{cases} \varphi(0)\sqrt{a}^{(2n\theta+\theta^2-\theta)}, & -n \leqslant \theta \leqslant 0, \\ \varphi(\theta), & \theta < -n. \end{cases}$$

We can see that for $n \geqslant \tau$, we get

$$T(n, \tau)P(\tau) = P(n)T(n, \tau),$$
$$T(n, \tau)Q(\tau) = Q(n)T(n, \tau).$$

For $n \geqslant \tau$, we observe that $T(n, \tau): Q(\tau)\mathcal{B}_\alpha \to Q(n)\mathcal{B}_\alpha$ is given by

$$\big[T(n, \tau)Q(\tau)\varphi\big](\theta) = \begin{cases} \varphi(0)\sqrt{a}^{(n+\theta-\tau)(n+\theta+\tau-1)}, & -n \leqslant \theta \leqslant 0, \\ \varphi(n+\theta-\tau), & \theta < -n. \end{cases}$$

We can prove that $T(n, \tau)$, $n \geqslant \tau$, is an isomorphism of $Q(\tau)\mathcal{B}_\alpha$ onto $Q(n)\mathcal{B}_\alpha$. We define $T(\tau, n)$ as the inverse mapping which is given by

$$\big[T(\tau, n)Q(n)\varphi\big](\theta) = \begin{cases} \varphi(0)\sqrt{a}^{(\tau+\theta-n)(\tau+\theta+n-1)}, & -\tau \leqslant \theta \leqslant 0, \\ \varphi(\tau-n+\theta), & \theta < -\tau. \end{cases}$$

On the other hand, we can infer that

$$\big\|T(n, m)P(m)\varphi\big\|_{\mathcal{B}_\alpha} \leqslant 2\left(\frac{1}{2}\right)^{n-m} \|\varphi\|_{\mathcal{B}_\alpha}, \quad \varphi \in \mathcal{B}_\alpha,\ n \geqslant s+1, \tag{4.9}$$

which implies that

$$\lim_{n\to\infty} a_1(n)^{-1}\big\|T(n, m)P(m)\big\| = 0. \tag{4.10}$$

A computation shows that

$$\big\|T(n,s+1)P(s+1)\varphi\big\|_{\mathcal{B}_\alpha} \leqslant 2\left(\frac{1}{2}\right)^{n-(s+1)}\|\varphi\|_{\mathcal{B}_\alpha}, \quad \varphi \in \mathcal{B}_\alpha,\ n \geqslant s+1, \tag{4.11}$$

$$\big\|T(n,s+1)Q(s+1)\varphi\big\|_{\mathcal{B}_\alpha} \leqslant 2^{s+1-n}\|\varphi\|_{\mathcal{B}_\alpha}, \quad \varphi \in \mathcal{B}_\alpha,\ s+1 \geqslant n. \tag{4.12}$$

The above estimates imply that $\|\Gamma(n,\cdot)\|_{a_{2,1}} \leqslant 2a_1(n)$, $n \geqslant n_0$. Next we consider the following perturbed equation of (4.8):

$$y(n+1) = a^n y(n) + \nu \sum_{s=-\infty}^{n} b(n)r(s)\big(y(s)\big)^\mu, \quad \mu \in \mathbb{Z}^+, \tag{4.13}$$

where

$$r(s) = l(s)\tilde{r}(s), \quad s \in \mathbb{Z}, \tag{4.14}$$

with $\tilde{r} \in l^1_{a_1^\mu}(\mathbb{N}) \cap l^\infty(\mathbb{Z}^-)$, $l \in l^\infty(\mathbb{N})$ with $l(s) = \alpha(2s)^\mu$, $s \in \mathbb{Z}^-$, and let $b(n)$ be a complex sequence such that $\|\cdot b(\cdot)a_2^{-1}\|_1 < \infty$. We introduce the following notation:

$$\gamma_1 := \big\|\cdot b(\cdot)a_2^{-1}\big\|_\infty,$$
$$\gamma_2 := a_1(n_0\mu)^{-1}\big[\|\tilde{r}\|_\infty + \|l\|_\infty\big]\big[\alpha(\mu)\big/\big(\alpha(\mu)-1\big) + \|\tilde{r}\|_{1,a_1^\mu}\big].$$

Assume that λ is a real positive number and assume ν is a real number such that $2|\nu|\gamma_2\gamma_1 < \lambda^{1-\mu}$. From this construction, we have that hypotheses of Theorem 4.1 are satisfied (see Remark 4.1). In fact, we note that

$$\sum_{\tau=-\infty}^{n}\big|r(\tau)\big|\big(a_1(\tau)\big)^\mu \leqslant a_1(n_0)^\mu \sum_{\tau=-\infty}^{n_0-1}\big|r(\tau)\big|\big(\alpha(n_0-\tau)\big)^\mu + \sum_{\tau=n_0}^{n}\big|r(\tau)\big|\big(a_1(\tau)\big)^\mu$$
$$\leqslant a_1(n_0\mu)^{-1}\left(\|\tilde{r}\|_\infty \sum_{s=0}^{\infty}\left(\frac{1}{\alpha(\mu)}\right)^s + \|l\|_\infty \sum_{\tau=0}^{n}\big|\tilde{r}(\tau)\big|\big(a_1(\tau)\big)^\mu\right) \leqslant \gamma_2.$$

Hence $\chi_\mu < +\infty$. Therefore by Theorem 4.1 there is a constant $M > 0$ such that for each $\varphi \in P(n_0)\mathcal{B}_\alpha$ with $\|\varphi\|_{\mathcal{B}_\alpha} < (\lambda-\delta)M^{-1}$ (δ is the constant of Remark 4.1), there is a solution $y = y(\varphi) = y(n,n_0,\psi)$ of Eq. (4.9) with $P(n_0)\psi = \varphi$ such that $Z_\infty^{a_1}(y_\bullet) = 0$ and $\|y_\bullet\|_{a_1} \leqslant \lambda$. Further, we have the asymptotic formula

$$y_n(\varphi) = T(n,n_0)\varphi + o\big(a_1(n)\big), \quad n \to \infty.$$

Example 4.2. We consider the following homogeneous linear difference equation:

$$x(n+1) = a(n)x(n), \quad n \geqslant n_0 \geqslant 0. \tag{4.15}$$

We note that solution operator is given by

$$\big[T(n,m)\varphi\big](\theta) = \begin{cases} \big(\prod_{\tau=m}^{n+\theta-1} a(\tau)\big)\varphi(0), & m-n \leqslant \theta \leqslant 0,\\ \varphi(n+\theta-m), & \theta < m-n. \end{cases}$$

The projection in this case can be taken as

$$\big[P(n)\varphi\big](\theta) = \begin{cases} \varphi(\theta) - \big(\prod_{\tau=n+\theta}^{n-1} a(\tau)^{-1}\big)\varphi(0), & -n \leqslant \theta \leqslant 0,\\ 0, & \theta < -n, \end{cases}$$

and

$$\big[Q(n)\varphi\big](\theta) = \begin{cases} \big(\prod_{\tau=n+\theta}^{n-1} a(\tau)^{-1}\big)\varphi(0), & -n \leqslant \theta \leqslant 0, \\ \varphi(\theta), & \theta < -n. \end{cases}$$

For $n \geqslant t$, we observe that $T(n,\tau)$ is an isomorphism of $Q(\tau)\mathcal{B}_\alpha$ onto $Q(n)\mathcal{B}_\alpha$ given by

$$\big[T(n,\tau)Q(\tau)\varphi\big](\theta) = \begin{cases} \big(\prod_{s=\tau}^{n-1} a(s)\big)\big(\prod_{s=n+\theta}^{n-1} a(s)^{-1}\big)\varphi(0), & -n \leqslant \theta \leqslant 0, \\ \varphi(n+\theta-\tau), & \theta < -n. \end{cases}$$

We define $T(\tau,n)$ as the inverse mapping of $T(n,\tau)$, which is given by

$$\big[T(\tau,n)Q(n)\varphi\big](\theta) = \begin{cases} \big(\prod_{s=\tau+\theta}^{n-1} a(s)^{-1}\big)\varphi(0), & -\tau \leqslant \theta \leqslant 0, \\ \varphi(\tau-n+\theta), & \theta < -\tau. \end{cases}$$

Let $\alpha(n)$, $a_1(n)$ and $a_2(n)$ be the same functions defined as in Example 4.1. Then, it is easy to see that the solution operator $T(n,m)$ satisfies (4.10), (4.11) and (4.12). Hence, system (4.15) has an (a_1,a_2) weighted summable dichotomy.

Next, we consider the following perturbed equation of (4.15)

$$y(n+1) = a(n)y(n) + \nu \sum_{s=-\infty}^{n} b(n)r(s)\big(y(s)\big)^{\mu}, \quad \mu \in \mathbb{Z}^+,$$

with $b(n)$, $r(s)$, λ and ν as in Example 4.1. Then the hypotheses of Theorem 4.1 are satisfied.

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