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# Representation Theory for a Class of Denumerable Markov Chains[1]

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## 1. INTRODUCTION

An interesting and important problem in the theory of denumerable Markov chains is to find a simple, easily computible canonical form for $P^n$, the matrix of the $n$-step transition probabilities. Kemeny has found such a representation for the class of "$k$-speading chains" [1]. A $k$-spreading chain is a denumerable Markov chain with states the natural numbers, with all states communicating, (that is, such that the process, starting in any state, can eventually reach any other state), and with a positive integer $k$ associated with it, such that $P_{i,i+k} > 0$ for all $i$, and $P_{ij} = 0$ if $j > i + k$. Kemeny finds a matrix $R$, depending on $P$, a matrix $Q$, which is a 2-sided inverse of $R$, and a matrix $S$, such that $P = QSR$. All matrices are row-finite, and thus associate. So, $P^n = QS^nR$. Since $S$ is of such simple form that $S^n$ is easy to find, the goal is accomplished.

In a generalization of Kemeny's work, this paper develops such a representation for what we will call $n$-dimensional $k$-spreading chains.[4] These chains are indexed by the $n$-dimensional coordinates (with the natural numbers as entries), and they are basically processes which, when projected on the $x_1$-axis and watched only when the $x_1$-coordinate changes, look like $k$-spreading chains. This class of Markov chains includes (by a trivial renumbering of the states) all $n$-dimensional random walks.

A key tool in this paper will be the establishment of criteria for the existence and uniqueness of inverses for certain types of infinite matrices.

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[4] A further generalization of this representation is presented in Section 11.

DEFINITION. Order the *n-dimensional coordinates* (that is, the $n$-dimensional vectors with the natural numbers, 0, 1, 2,... as entries) as follows:

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \quad \text{iff} \quad x_i = y_i, \quad i = 1, \dots, n$$

$$(x_1, \dots, x_n) < (y_1, \dots, y_n) \quad \text{iff} \quad x_1 < y_1,$$

or

$$x_1 = y_1, \dots, x_s = y_s, \quad x_{s-1} < y_{s+1}, \quad \text{for some} \quad s, \quad 1 \leqslant s < n.$$

Denote each vector by a capital letter, and its coordinates by the corresponding lower-case letter, with subscripts. Also, write the vector $(a, x_2, x_3, \dots, x_n)$ as $(a, \bar{X})$. Thus $X = (x_1, \bar{X})$.

The $n$-dimensional coordinates do not have the order-type of the natural numbers; thus, some elements have an infinite number of predecessors. However, in the generality of such books as [2], we are going to consider matrices indexed by the $n$-dimensional coordinates as states.

Any matrix $M$ which is indexed by the $n$-dimensional coordinates, can be written in the following form:

$$M = \begin{pmatrix} M^{[0,0]} & M^{[0,1]} & M^{[0,2]} & \cdots \\ M^{[1,0]} & M^{[1,1]} & M^{[1,2]} & \cdots \\ M^{[2,0]} & M^{[2,1]} & M^{[2,2]} & \cdots \\ & \vdots & & \end{pmatrix},$$

where $M^{[i,j]}$ is that submatrix of $M$ which is indexed down by all states of form $(i, \bar{X})$, and across by all states of form $(j, \bar{Y})$. Call such an $M^{[i,j]}$ a *basic submatrix* of $M$.

Define a matrix $T$ to be *triangular* if $T_{XX} > 0$ for each $X$ and $T_{XY} = 0$ when $X < Y$. If a triangular matrix $T$ has $T_{XY} = 0$ for $X \neq Y$, call $T$ *diagonal*. All matrices in this paper are assumed to be finite-valued (f.v.), and when necessary are proved to be f.v.

The following useful criteria for associativity and distributivity of infinite matrices, which are proven for example in [2], will be used:

1. Nonnegative matrices associate under multiplication, and distribute.

2. If $A$, $B$, and $C$ are f.v. matrices such that either $A$ is row-finite or $C$ is column-finite, and if $(AB)\,C$ and $A(BC)$ are both well defined, then $(AB)\,C = A(BC)$. Note that if $A$ and $B$ are both row-finite, or if $B$ and $C$ are both column-finite, then $(AB)\,C$ and $A(BC)$ are well defined, since only finite sums are involved.

3. If $AB$, $AC$, and $A(B + C)$ are all well defined, then

$$A(B + C) = AB + AC,$$

and similarly for right distributivity.

4. If $|A|\cdot|B|\cdot|C| < \infty$, then $(AB)C = A(BC)$. If $|A|\cdot|B|$ and $|A|\cdot|C| < \infty$, then $A(B+C) = AB + AC$, and similarly for right distributivity. (By the absolute value of a matrix, $|A|$, we mean a matrix with entries $(|A|)_{XY} = |A_{XY}|$.)

## 2. Row-Finite $N$-Dimensional $K$-Spreading Chains

Definition. An *n-dimensional k-spreading chain (n-k-s chain)* is a Markov chain with states the $n$-dimensional coordinates, with a fixed integer $k > 0$ associated with it, and with transition probabilities as follows:

$P_{XY} = 0$ unless either $y_1 \leqslant x_1 + k - 1$, or $y_1 = x_1 + k$ and $(0, \bar{Y}) \leqslant (0, \bar{X})$. Further,

$$P_{(x_1,\bar{X}),(x_1+k,\bar{X})} > 0.$$

Note that the chain is essentially restricted to being $k$-spreading in only one dimension—there is great freedom of movement in the other dimensions.

Definition. Define the matrix $R$, which we will use in the representation $P^i = QS^iR$, inductively. Write

$$R = \begin{pmatrix} R^{[0,0]} & & \\ R^{[1,0]} & R^{[1,1]} & \\ R^{[2,0]} & R^{[2,1]} & R^{[2,2]} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where each $R^{[i,j]}$ is a basic submatrix, and where omitted basic submatrices are 0. Call $(R^{[i,0]}R^{[i,1]} \cdots R^{[i,i]}\,0\,0 \cdots)$ the "$i$th submatrix row." Then define $R^{[i,j]} = \delta_{ij}I$, if $i < k$, and inductively define the $(i+k)$th submatrix row as the $i$th submatrix row times $P$. Thus

$$R_{XY} = P^{(m)}_{(r,\bar{X}),Y}, \qquad \text{if} \qquad x_1 = mk + r, \qquad 0 \leqslant r < k.$$

The ordering of the states has been defined in precisely such a way that $R$ is triangular: If $x_1 \leqslant k-1$, then the $(x_1, \bar{X})$ row of $R$ is certainly such as to make $R$ triangular. If $x_1 > k-1$, write $x_1 = mk + r$, $0 \leqslant r < k$, $m \geqslant 1$. Then

$$\begin{aligned} R_{XX} &= P^{(m)}_{(r,\bar{X}),(mk+r,\bar{X})}, \\ &= P_{(r,\bar{X}),(k+r,\bar{X})} \cdot P_{(k+r,\bar{X}),(2k+r,\bar{X})} \cdots P_{((m-1)k+r,\bar{X}),(mk+r,\bar{X})} > 0. \end{aligned}$$

Next consider $Y > X$. If $y_1 > x_1 = mk + r$, then

$$R_{XY} = P^{(m)}_{(r,\bar{X}),(y_1,\bar{Y})} = 0,$$

since the first coordinate can increase by at most $k$ on each step. If $y_1 = x_1$, then since $Y > X$, $(0, \bar{Y}) > (0, \bar{X})$, so

$$R_{XY} = P^{(m)}_{(r,\bar{X}),(mk+r,\bar{Y})} = 0,$$

because the first coordinate must increase by $k$ each time, and in so doing, $(0, \bar{X})$ can not increase.

Triangularity by itself is not sufficient to assure that a matrix have a two-sided inverse. In fact, as we shall see in Section 12, there exist $n$-$k$-$s$ chains whose $R$ matrix fails to have such an inverse. However, we shall temporarily assume that $P$ is row-finite, and we will see that no problem then arises.

Define $S$ to be a matrix indexed by the same states as $P$, and with

$$S_{XY} = \delta_{(x_1+k,\bar{X}),Y}.$$

Thus,

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & I & & \\ 0 & 0 & \cdots & 0 & 0 & I & \\ 0 & 0 & \cdots & 0 & 0 & 0 & I \\ & & & \vdots & & & \end{pmatrix}.$$

Note that $S^i$ is simple to find:

$$S^i_{XY} = \delta_{(x_1+ik,\bar{X}),Y}.$$

By definition of $R$, $RP = SR$.

LEMMA 1. *If $M$ and $N$ are matrices, $M$ is triangular and f.v., and $N$ is either a right or a left inverse of $M$ (or both), then $N$ is f.v.*

PROOF. Assume that both $MN = I$, and also some entry $N_{AB}$ of $N$ is $\pm\infty$. Then

$$\delta_{AB} = \sum_Z M_{AZ}N_{ZB} = M_{AA}N_{AB} + \sum_{Z\neq A} M_{AZ}N_{ZB}.$$

Since $M_{AA} > 0$, the right side contains an infinite term, and thus can not equal $\delta_{AB}$. Likewise, any left inverse of $M$ must be f.v.

We will often use this lemma tacitly: for example, we will search for right inverses of a f.v. triangular matrix $T$ by considering only f.v. candidates.

Lemma 2. *Let $T$ be any f.v., row-finite triangular matrix indexed by the n-dimensional coordinates. Then*

1. *$T$ has a unique 2-sided inverse $T'$.*
2. *$T'$ is f.v., triangular, and row-finite.*
3. *$T'$ is the unique right inverse of $T$.*
4. *$T'$ is the unique row-finite left inverse of $T$.*

Note. Even if $n = 1$, $T'$ is not necessarily the only left inverse of $T$, as Example 4 in Section 12 shows.

Proof. Let $T^{\langle i_1, i_2, \ldots, i_r \rangle}$ be the submatrix of $T$ indexed by all states $X$ with $x_1 = i_1$, $x_2 = i_2$,..., $x_r = i_r$. To prove the lemma, we will use "backwards induction" (from $n$ to 0) on the length of the superscript of $T$: that is, we will show that if each $T^{\langle i_1, \ldots, i_r \rangle}$ fulfills the conclusions of the lemma, then so does each $T^{\langle i_1, \ldots, i_{r-1} \rangle}$. When the superscript reaches 0 length, then the lemma is proven for $T$ itself. Since each $T^{\langle i_1, \ldots, i_n \rangle}$ is a positive number, the initial induction step is trivial.

Assume inductively that for each $a = 0, 1, 2,\ldots,$ the matrix $T^{\langle i_1, \ldots, i_{r-1}, a \rangle}$ fulfills the conclusions. Think of $M = T^{\langle i_1, \ldots, i_{r-1} \rangle}$ as being made up of blocks of submatrices as follows (as a short-hand, write $T^{\langle i_1, \ldots, i_{r-1}, a \rangle}$ as $T^{(a)}$):

$$M = \begin{pmatrix} T^{(0)} & & \\ B^{(1,0)} & T^{(1)} & \\ B^{(2,0)} & B^{(2,1)} & T^{(2)} \\ & \vdots & \end{pmatrix};$$

$B^{(i,j)}$ is indexed down by those states $X$ with $x_1 = i_1$,..., $x_{r-1} = i_{r-1}$, $x_r = i$, and across by those states with $x_1 = i_1$,..., $x_{r-1} = i_{r-1}$, $x_r = j$.

Since $T$ is row-finite, so is $T^{\langle i_1, \ldots, i_{r-1} \rangle}$, and hence so is each $B^{(i,j)}$ and each $T^{(a)}$.

Define a triangular matric $C$, indexed by the same states as $M$, as follows: write

$$C = \begin{pmatrix} C^{(0,0)} & & \\ C^{(1,0)} & C^{(1,1)} & \\ C^{(2,0)} & C^{(2,1)} & C^{(2,2)} \\ & \vdots & \end{pmatrix}.$$

Define the submatrix $C^{(i,j)}$, $i \geqslant j$, recursively in $i$, by

1. $C^{(j,j)} = (T^{(j)})'$, the unique two-sided inverse of $T^{(j)}$, which exists, and is row-finite and triangular, by induction hypothesis.

2. $$C^{(i,j)} = -(T^{(i)})' \left( \sum_{t=j}^{i-1} B^{(i,t)} C^{(t,j)} \right), \qquad i > j.$$

Each $C^{(i,j)}$ is well-defined and row-finite, since : $C^{(j,j)}$ certainly is. Given inductively that $C^{(j,j)}$, $C^{(j+1,j)}, \ldots, C^{(i-1,j)}$ are, then so is $C^{(i,j)}$, since the products and sums of row-finite matrices are row-finite.

We know that

1′. $T^{(j)}C^{(j,j)} = I$

2′. $$-(T^{(i)})\, C^{(i,j)} = \sum_{t=j}^{i-1} B^{(i,t)}C^{(t,j)}, \qquad i > j$$

by multiplying Eq. 1 on the left by $T^{(j)}$, and Eq. 2 on the left by $- T^{(i)}$, and associating by row-finiteness. But these are precisely the conditions for $C$ to be a right inverse of $M$.

$C$ is a matrix just like $M$, so we can identically construct a matrix $C_1$ which is row-finite, and which is a right inverse of $C$.

Now $M = M(CC_1) = (MC)\, C_1 = C_1$; the second equality follows from row-finiteness of $M$ and $C$. So, $C$ is a two-sided inverse of $M$. Assume $M$ has another right inverse $N$. Then $N = (CM)\, N = C(MN) = C$.

Finally, assume $M$ has another row-finite left inverse $L$. Then

$$L = L(MC) = (LM)\, C = C;$$

the second equality follows from row-finiteness of $L$ and $M$.

The induction is complete, and the lemma is proven.

THEOREM 1. *Let $P$ be an n-k-s chain, which is row-finite. Then*

1. *The matrix $R$ associated with $P$ has a unique two-sided inverse $Q$.*
2. *$R$ and $Q$ are f.v., triangular, and row-finite.*
3. *$Q$ is the unique right inverse of $R$, and $R$ is the unique right inverse of $Q$.*
4. *$Q$ is the unique row-finite left inverse of $R$, and $R$ is the unique row-finite left inverse of $Q$.*
5. *$P^i = QS^iR$, $i = 0, 1, 2, \ldots$ .*

PROOF. Each row of $R$ is a row of some $P^i$. Thus $R$ is row-finite, and we can apply Lemma 2. We can then apply Lemma 2 to $Q$.

Conclusion 5 follows from multiplying both sides of $RP = SR$ on the left by $Q$, giving $P = QSR$, which implies $P^i = QS^iR$. Since $P$, $Q$, $S$, and $R$ are all row-finite, there is free associativity.

## 3. BASIC QUANTITIES OBTAINABLE FROM THE REPRESENTATION

Several fundamental quantities associated with the Markov chain $P$ can be found from the matrices $Q$ and $R$. In this section, we shall generalize some of Kemeny's formulas in [2] to cover row-finite *n-k-s* chains.

If $P$ has rows-sums unity, then so does $R$, and hence so does $Q$, since $Q1 = Q(R1) = (QR)1 = 1$, where $1$ is a column vector of all ones. If $P1 \neq 1$, then the row sum of the $(x_1, \bar{X})$ row of $R$, where $x_1 = mk + r$, $0 \leqslant r < k$, equals

$$\sum_Y P^{(m)}_{(r,\bar{X}),Y} = \text{probability that the process, started in state } (r, \bar{X}),$$
$$\text{has not stopped with } m \text{ steps.}$$

As for the columns of $R$: the interesting quantity is not $1^T R$, which gives column sums, but $V^A R$, where $V^A$ is a row vector defined for each $A = (r, \bar{A})$, $0 \leqslant r < k$, $a_i$ arbitrary, $i = 2, 3, 4, \ldots$, by

$$(V^A)_Y = \begin{cases} 1, & Y = (r + sk, \bar{A}) \qquad s = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

For,

$$(V^A R)_Y = \sum_{s=0}^{\infty} R_{(r+sk,\bar{A}),Y} = \sum_{s=0}^{\infty} P^{(s)}_{AY} = N_{AY},$$

the mean number of times $Y$ is eventually reached, starting in $A$. If all states are transient, we can find all of $N$. Define

$$T^{(n)} = \sum_{t=0}^{n} S^t, \qquad \text{and} \qquad T = \lim_n T^{(n)}.$$

Then

$$\begin{aligned} N &= \lim_n QT^{(n)}R \\ &= Q \lim(T^{(n)}R) \text{ since } Q \text{ is row-finite} \\ &= Q(TR) \text{ by monotonicity.} \end{aligned}$$

Further, it is worth mentioning that $(QT)R = Q(TR) = N$. For, if $x_1 = mk + r$, $0 \leqslant r < k$, then

$$(TR)_{XY} \leqslant (V^{(r,\bar{X})}R)_Y = N_{(r,\bar{X}),Y} \leqslant N_{YY}.$$

So, by row-finiteness of $Q$, $|Q| \, TR$ is f.v. Let us note for later that each column of $TR$ is uniformly bounded.

## 4. The Regular Functions of $P$

The set of all functions (column vectors indexed by the $n$-dimensional coordinates) forms a vector space over the reals, where we even allow infinite sums of functions whenever the sum is well-defined and f.v. in each entry. A countable set of functions $\{f_i\}$ is then linearly independent whenever

$$\sum_{i=1}^{\infty} a_i f_i = 0 \qquad \text{implies that each} \qquad a_i = 0.$$

The set of all f.v. regular functions of a row-finite Markov chain forms a vector subspace. For, assume

$$g = \sum_{i=1}^{\infty} a_i f_i \text{ is f.v., where each } f_i \text{ is regular.}$$

Then

$$Pg = P \sum a_i f_i = \sum a_i P f_i = \sum a_i f_i = g,$$

with the second equality following since $P$ is row-finite.

For each state $A = (r, \bar{A})$, $0 \leqslant r < k$, of an $n$-$k$-$s$ chain $P$, define a column vector $E^A$ by

$$(E^A)_X = \begin{cases} 1, & X = (r + sk, \bar{A}), \qquad s = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Thus $E^A = (V^A)^T$ (see Section 3).

THEOREM 2. *Let $P$ be any row-finite $n$-$k$-$s$ chain $P$. Then the vectors $QE^A$ form a linearly independent set which spans (allowing infinite sums) the subspace $V$ of f.v. regular functions of $P$. In particular, if $n > 1$, then $P$ has a countably infinite number of linearly independent regular functions, and if $n = 1$, then the subspace of f.v. regular functions is $k$-dimensional.*

PROOF. Each $QE^A$ is well-defined, since $Q$ is row-finite. Each $QE^A$ is regular, since from $PQ = QS$, we have $PQE^A = QSE^A = QE^A$. Thus, the space spanned by the $\{QE^A\}$ is contained in $V$.

The $\{QE^A\}$ are linearly independent: assume

$$\sum a_A Q E^A = 0.$$

Then

$$\begin{aligned} 0 &= R0 \\ &= R \sum a_A Q E^A \\ &= \sum a_A R Q E^A \qquad \text{by row-finiteness} \\ &= \sum a_A E^A, \end{aligned}$$

and by the obvious linear independence of the $\{E^A\}$, each $a_A = 0$.

Now assume $f$ is f.v. and regular. Then $Rf = RPf = SRf$. Since $g = Rf$ fulfills $g = Sg$, we know $(Rf)_X = (Rf)_{(x_1+mk, \bar{X})}$, $m \geqslant 0$. So,

$$Rf = \sum a_A E^A, \qquad \text{where} \qquad a_A = (Rf)_A .$$

Then

$$f = QRf = Q \sum a_A E^A = \sum a_A (QE^A).$$

The last statement of the theorem is now obvious by counting the number of $E^A$'s.

We now have a representation for f.v. regular functions of a row-finite $n$-$k$-$s$ chain. First, write each function

$$g = \begin{pmatrix} g^{[0]} \\ g^{[1]} \\ g^{[2]} \\ \vdots \end{pmatrix},$$

where $g^{[i]}$ is a basic subcolumn, indexed by all states $X$ with $x_1 = i$. Then a function $f$ is regular iff it is of form $Qg$, where $g^{[0]}, g^{[1]}, \ldots, g^{[k-1]}$ are completely arbitrary, and $g^{[i]} = g^{[j]}$ whenever $i \equiv j \bmod k$.

## 5. A Shortcut for Obtaining $Q$

At this stage, the only method we have for finding the matrix $Q$ associated with a given $n$-$k$-$s$ chain $P$ is to first find $R$, and then to find $Q$ as the unique two-sided inverse of $R$. However, it would be desirable to have a more direct way of obtaining $Q$. The conclusion of the following theorem is completely analogous to the fact that $R$ is characterized by $R_{XY} = \delta_{XY}$ for $x_1 < k$, and $RP = SR$.

Theorem 3. *The $Q$ matrix for a row-finite $n$-$k$-$s$ chain $P$ is characterized by $Q_{XY} = \delta_{XY}$ for $x_1 < k$, and $PQ = QS$.*

Proof. First, the $Q$ matrix for $P$ obviously fulfills this. Assume some other matrix $Q_1$ fulfills these conditions. Then $PQ_1 = Q_1 S$ tells us that $Q_1$ is indexed by the same states as $Q$ (and $P$ and $S$). Let $P^{[i,j]}$, $Q^{[i,j]}$, $S^{[i,j]}$, and $Q_1^{[i,j]}$ be basic submatrices. Then for $i < k$ and arbitrary $j$, $Q_1^{[i,j]} = Q^{[i,j]}$. Assume inductively that $Q_1^{[i,j]} = Q^{[i,j]}$, $i \leqslant r - 1$ and all $j$, where $r \geqslant k$. Then from $PQ_1 = Q_1 S$,

$$\sum_{x=0}^{r} P^{[r-k,x]} Q_1^{[x,j]} = Q_1^{[r-k,j-k]} = Q^{[r-k,j-k]},$$

if we define the "basic" submatrix $Q_1^{[s,t]} = Q^{[s,t]} = 0$ when $t < 0$. So,

$$P^{[r-k,r]}Q_1^{[r,j]} = Q^{[r-k,j-k]} - \sum_{x\ 0}^{r-1} P^{[r-k,x]}Q^{[x,j]} \text{ by induction hypothesis}$$

$$= P^{[r-k,r]}Q^{[r,j]}, \qquad \text{since} \qquad PQ = QS. \tag{1}$$

Now $P^{[r-k,r]}$ can be considered as being indexed by the $(n-1)$-dimensional coordinates, and it then fulfills the hypothesis of Lemma 2. Multiply both sides of (1) by $(P^{[r-k,r]})^{-1}$ on the left, and then $Q_1^{[r,j]} = Q^{[r,j]}$, completing the induction.

Let us calculate $R$ and $Q$ for a given example, to demonstrate the usefulness of Theorem 3. Assume $P$ is a row-finite $n$-1-$s$ chain (that is, an $n$-$k$-$s$ chain with $k = 1$) of form

$$\begin{pmatrix} P^{[0,0]} & P^{[0,1]} & & & \\ P^{[1,0]} & 0 & P^{[1,2]} & & \\ P^{[2,0]} & 0 & 0 & P^{[2,3]} & \\ P^{[3,0]} & 0 & 0 & 0 & P^{[3,4]} \\ & & \vdots & & \end{pmatrix}$$

$R$ can be found from its recusive definition :

$$R^{[i,j]} = \begin{cases} I, & i = j = 0 \\ \sum_{r=0}^{i-1} R^{[i-1,r]}P^{[r,0]}, & i > j = 0 \\ R^{[i-1,j-1]}P^{[j-1,j]}, & i \geqslant j > 0 \\ 0, & i < j. \end{cases}$$

Obviously it would be difficult to try to find $Q$ as a two-sided inverse of $R$. However, starting from $Q^{[0,i]} = \delta_{0,i}I$ and $PQ = QS$, we easily find that for $i \geqslant 1$,

$$Q^{[i,j]} = \begin{cases} -(P^{[i-j-1,i-j]}P^{[i-j,i-j+1]} \cdots P^{[i-1,i]})^{-1}P^{[i-1-j,0]}, & i > j \\ (P^{[0,1]}P^{[1,2]} \cdots P^{[i-1,i]})^{-1}, & i = j \\ 0 \quad \text{otherwise.} \end{cases}$$

To verify that these equations correctly define $Q$, we need only verify now, according to Theorem 3, that $PQ = QS$, which is easy to check.

## 6. Infinite-Dimensional $K$-Spreading Chains

It is certainly natural to try to consider infinite-dimensional coordinates (that is, vectors containing a countable number of natural numbers as entries) as a set of states. However, this set of vectors forms an uncountable collection, and thus can not be considered as the states of a denumerable Markov chain. One solution is to consider only those vectors which "terminate," that is, which have only zeroes as entries from some point on, since these form a countable collection.

DEFINITION. Call these vectors the *infinite-dimensional terminating coordinates*, and order them as we did the finite-dimensional coordinates:

$$(x_1, x_2, \dots) = (y_1, y_2, \dots) \quad \text{iff} \quad x_i = y_i \quad \text{for all} \quad i$$

$$(x_1, x_2, \dots) < (y_1, y_2, \dots) \quad \text{iff} \quad x_1 < y_1$$

or

$$x_1 = y_1, \dots, x_s = y_s, x_{s+1} < y_{s+1} \qquad \text{for some} \qquad s \geqslant 1.$$

Adopt the same conventions as before, e.g., $X = (x_1, \bar{X})$. Define now an infinite-dimensional $k$-spreading chain ($\omega$-$k$-$s$ chain) by carrying over exactly the definition of an $n$-$k$-$s$ chain (with the exception, of course, of the set of states).

Assume now that each $\omega$-$k$-$s$ chain $P$ fulfills the following additional restriction: there is some integer $c \geqslant 1$ associated with $P$ such that

$$P_{(x_1, \bar{X}), (x_1+k, \bar{Y})} = 0$$

unless not only $(0, \bar{Y}) \leqslant (0, \bar{X})$, but also

$$(0, 0, \dots, 0, x_{c+1}, x_{c+2}, \dots) = (0, 0, \dots, 0, y_{c+1}, y_{c+2}, \dots).$$

Thus, when the process takes its maximum jump in the $x_1$-direction, at most $c - 1$ other components can change. Our last restriction can be weakened, although we will not consider that here: for example, $c$ can be nonconstant, but instead a function of $x_1$. Let us show that if such a chain $P$ is row-finite, it is representable as $P^i = QS^iR$.

LEMMA 3. *Let $T$ be any f.v., row-finite matrix indexed by the infinite-dimensional terminating coordinates. Suppose that $T$ has associated with it a constant $d \geqslant 1$, such that each submatrix $T^{\langle i_1, i_2, \dots, i_d \rangle}$ (defined in the proof of Lemma 2) is diagonal. Then $T$ fulfills the conclusions of Lemma 2.*

PROOF. The proof of Lemma 2 holds, if we only change the initial induction step: the backwards induction runs from $d$ to 0, with each $T^{\langle i_1, \dots, i_d \rangle}$, being diagonal, fulfilling the conclusions of the lemma.

THEOREM 4. *Let $P$ be a row-finite $\omega$-$k$-$s$ chain, with the following additional* *restriction: There exists* $c \geqslant 1$ *such that* $P_{(x_1, \bar{X}), (x_1+k, \bar{Y})} = 0$ *unless not only* $(0, \bar{Y}) \leqslant (0, \bar{X})$ *but also*

$$(0, 0, \dots, 0, x_{c+1}, x_{c+2}, \dots) = (0, 0, \dots, 0, y_{c+1}, y_{c+2}, \dots).$$

*Then P fulfills the conclusions of Theorems 1, 2, and 3.*

PROOF. Each $R^{\langle i_1, \dots, i_c \rangle}$ is diagonal: it is obviously triangular, since it is "on the diagonal" of $R$. Let $i_1 = mk + r$, $0 \leqslant r < k$. If

$$(0, \dots, 0, x_{c+1}, x_{c+2}, \dots) \neq (0, \dots, 0, y_{c+1}, y_{c+2}, \dots),$$

then

$$R_{(i_1, \dots, i_c, x_{c+1}, x_{c+2}, \dots), (i_1, \dots, i_c, y_{c+1}, y_{c+2}, \dots)}$$

$$= P^{(m)}_{(r, i_2, \dots, i_c, x_{c+1}, x_{c+2}, \dots), (mk+r, i_2, \dots, i_c, y_{c+1}, y_{c+2}, \dots)} = 0.$$

Thus Lemma 3 applies, and the conclusion to Theorem 1 holds.

The proof of Theorem 2 carries over word for word. The dimension of the subspace of regular functions is then, of course, countably infinite, just as in the case $n > 1$.

The only change necessary in the proof of Theorem 3 lies in showing that $P^{[r-k, r]}$ has a row-finite left inverse. There are 2 cases. If $c = 1$, then $P^{[r-k, r]}$ is diagonal, and the result follows; if $c > 1$, then $P^{[r-k, r]}$ can be considered as being indexed by the infinite-dimensional terminating coordinates, and it then fulfills the hypotheses of Lemma 3, with $d = c - 1$.

We have shown that all $\omega$-$k$-$s$ chains with a certain natural restriction are representable ($P^i = QS^iR$). Let us show (mainly as an interesting exercise in the renumbering of states) that if we slightly modify the definition of $\omega$-$k$-$s$ chains, we can obtain the desirable result that *all* row-infinite $\omega$-$k$-$s$ chains (of the modified variety) are representable.

Define a new ordering on the infinite-dimensional terminating coordinates, as follows:

$$(x_1, x_2, \dots) =' (y_1, y_2, \dots) \quad \text{iff} \quad x_i = y_i \quad \text{for all} \quad i$$

$$(x_1, x_2, \dots) <' (y_1, y_2, \dots) \quad \text{iff} \quad p_1^{x_1} p_2^{x_2} \cdots < p_1^{y_1} p_2^{y_2} \cdots,$$

where $p_i$ is the $i$th prime ($p_1 = 2$, $p_2 = 3$, etc.).

Since the entries are all 0 from some point on, this is well-defined.

DEFINITION. A *modified* $\omega$-$k$-$s$ *chain* is a Markov chain with states the infinite-dimensional terminating coordinates, and with

$P_{XY} = 0$ unless either $y_1 \leqslant x_1 + k - 1$, or $y_1 = x_1 + k$ and $(0, \bar{Y}) \leqslant' (0, \bar{X})$. Further, $P_{(x_1, \bar{X}), (x_1+k, \bar{X})} > 0$.

Note that this definition is identical to the definition of an $\omega$-$k$-$s$ chain with $\leqslant'$ substituted for $\leqslant$. A modified $\omega$-$k$-$s$ chain is not merely a weakened $\omega$-$k$-$s$ chain, since for example in a modified $\omega$-$k$-$s$ chain, the process can move directly from $(1, 1, 1, 0, 0, 0, 0,\ldots)$ to $(1 + k, 2, 0, 0, 0,\ldots)$, which is impossible in regular $\omega$-$k$-$s$ chain.

THEOREM 5. *A modified $\omega$-$k$-$s$ chain which is row-finite is representable.*

PROOF. We will show that by renumbering the states of a modified $\omega$-$k$-$s$ chain, we get nothing other than an ordinary 2-$k$-$s$ chain. Then the result will follow immediately, since row-finite $n$-$k$-$s$ chains are representable.

Define a 1-1 correspondence between the infinite-dimensional terminating coordinates and the positive integers by

$$f : (a_1, a_2, \ldots) \to p_1^{a_1} p_2^{a_2} \cdots .$$

Define a 1-1 correspondence between the positive integers and the 2-dimensional coordinates by

$$g : 2^a(2b + 1) \to (a, b).$$

Then the 1-1 correspondence $gf$ maps $(a_1, a_2, \ldots)$ onto $(a_1, (p_2^{a_2} p_3^{a_3} \cdots - 1)/2)$. Relabel states $X$ of $P$ as $g(f(X))$, and let

$$x_1' = \frac{p_2^{x_2} p_3^{x_3} \cdots - 1}{2}, \qquad y_1' = \frac{p_2^{y_2} p_3^{y_3} \cdots - 1}{2}.$$

Then it is easy to check that the process is now simply a 2-$k$-$s$ chain.

## 7. BLOCK-COLUMN-FINITE $n$-$k$-$s$ CHAINS

We have proved that each row-finite $n$-$k$-$s$ chain $P$ has the property that its $R$ matrix has a two-sided inverse. We might naturally hope that this would be true also of column-finite $n$-$k$-$s$ chains. However, there is an immediate stumbling block: even if an $n$-$k$-$s$ chain $P$ is column-finite, its $R$ matrix is not necessarily column-finite. In fact, it is easy to show that if any two states $A$ and $B$ of $P$ communicate (that is, $P_{AB}^{(r)} > 0$ and $P_{BA}^{(s)} > 0$ for some $r$, $s$), then $R$ is not column-finite. However, as we will see later, the $R$ matrix for a column-finite $n$-$k$-$s$ chain does have another property which is similar to column-finiteness, a property we will call "block-column-finiteness."

DEFINITION. A matrix $M$ which is indexed by the $n$-dimensional or infinite-dimensional terminating coordinates is *block-column-finite* if each of its basic submatrices $M^{[i,j]}$ is column-finite. In particular, every column-finite matrix indexed by multi-dimensional coordinates is block-column-finite.

Unfortunately, not even all block-column-finite, triangular matrices have a two-sided inverse. In fact, in Section 12 we see a counterexample, of a column-finite, 2-1-$s$ chain $P$ whose $R$ matrix does not have a right inverse. So some further restriction is necessary to guarantee that the $R$ matrix of a column-finite $n$-$k$-$s$ chain $P$ be invertible. A very natural restriction is that each $P^{[i,i+k]}$ be diagonal—that is, that when the process takes its maximum jump of $k$ units in the first coordinate, then no other coordinate can change. If we adopt this assumption, then we can prove even more: that any such *block*-column-finite $n$-$k$-$s$ (or, in fact, $\omega$-$k$-$s$) chain $P$ has a (unique) two-sided inverse for its $R$ matrix. This is quite significant, since then with a little more work we can have a representation for a class of Markov chains which need not be either row-finite or column-finite.

We begin with 4 lemmas.

LEMMA 4. *Assume that a f.v. triangular matrix $T$ is indexed by either the n-dimensional or the infinite-dimensional terminating coordinates, and that each $T^{[i,i]}$ is diagonal. Then if $T$ has a right inverse $C$, $C$ is triangular.*

PROOF. The proof is exactly like that for a finite triangular matrix, except that matrix blocks are used instead of numbers. A diagonal submatrix corresponds to a nonzero number, which always has a unique inverse.

LEMMA 5. *Let $T$ be as in Lemma 4. Then $T$ has at most one triangular left inverse.*

PROOF. Assume $LT = I$. Then

$$L^{[i,i]}T^{[i,i]} = I$$

$$L^{[i,j]}T^{[j,j]} = -\sum_{r=j+1}^{i} L^{[i,r]}T^{[r,j]}, \qquad j < i.$$

Denote the (unique), diagonal, two-sided inverse of the diagonal matrix $T^{[i,i]}$ by $(T^{[i,i]})^{-1}$. Then the above two equations give us

$$L^{[i,i]} = (T^{[i,i]})^{-1}$$

$$L^{[i,j]} = -\left(\sum_{r=j+1}^{i} L^{[i,r]}T^{[r,j]}\right)(T^{[j,j]})^{-1}, \qquad j < i.$$

These are recursion equations in $j$, which determine first $L^{[i,i]}$, and then $L^{[i,j]}$ $(j < i)$ in terms of $L^{[i,i]}, L^{[i,i-1]},\ldots, L^{[i,j+1]}$. Hence there is at most one solution for $L$, which can only exist if all matrix products and sums above are well defined.

LEMMA 6. *Let $A$ and $B$ be two nonnegative f.v. block-column-finite matrices, both indexed by the n-dimensional, or the infinite-dimensional terminating coordinates. Assume there exists $r_A$ such that $A_{XY} = 0$ whenever $y_1 > x_1 + r_A$, and likewise for $B$. Then $C = AB$ has the same properties: it is nonnegative, f.v., block-column-finite, and there exists $r_C$ such that $C_{XY} = 0$ whenever $y_1 > x_1 + r_C$. Thus any product of a finite number of such matrices is again such a matrix, and in particular is f.v.*

PROOF. Let $C = AB$. Then

$$C^{[i,j]} = \sum_{m=0}^{i+r_A} A^{[i,m]}B^{[m,j]}.$$

Since each $A^{[i,m]}$ and each $B^{[m,j]}$ is column-finite, so is $C^{[i,j]}$. Thus, $C$ is f.v. and block-column-finite.

Finally, if $j > i + r_A + r_B$, then for $m = 0, 1,\ldots, i + r_A$, we have $B^{[m,j]} = 0$, so

$$C^{[i,j]} = \sum_{m=0}^{i+r_A} A^{[i,m]}B^{[m,j]} = 0.$$

Thus $r_A + r_B$ can serve as $r_C$.

We are now ready to prove:

LEMMA 7. *Let $T$ be a f.v., block-column-finite triangular matrix, indexed by either the n-dimensional or the infinite-dimensional terminating coordinates. If each $T^{[i,i]}$ is diagonal, then*

1. *$T$ has a unique two-sided inverse $C$.*
2. *$C$ is f.v., triangular, and block-column-finite and each $C^{[i,i]}$ is diagonal.*
3. *$C$ is the unique right inverse of $T$.*
4. *$C$ is the only left inverse of $T$ which is triangular.*

NOTE. Even if $n = 1$, $C$ is not necessarily the unique left inverse of $T$, as we see from counterexample 4 in Section 12. The given matrix is block-column-finite, where a number serves as a block.

PROOF. By Lemma 4, if $T$ is to have a right inverse $C$, $C$ must be triangular. Hence a necessary and sufficient condition for a matrix $C$ to be a right inverse of $T$ is

$$\begin{aligned} C^{[i,j]} &= 0, & j > i \\ T^{[j,j]}C^{[j,j]} &= I \\ -\, T^{[i,i]}C^{[i,j]} &= \sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]}, & i > j. \end{aligned}$$

Since each $T^{[i,i]}$ is diagonal, an equivalent set of conditions is

$$\begin{aligned} C^{[i,j]} &= 0, & j > i \\ C^{[j,j]} &= (T^{[j,j]})^{-1} \\ C^{[i,j]} &= -\,(T^{[i,i]})^{-1} \sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]}, & i > j. \end{aligned} \tag{2}$$

Since condition (2) is a set of recursion equations in $i$, we can get at most one solution for $C$. Let us prove that we do indeed get a solution, that is, that each $C^{[i,j]}$ is well-defined; simultaneously let us show that each $C^{[i,j]}$ is column-finite. Certainly each $C^{[i,j]}$, $i \leqslant j$, is well-defined and column-finite. Assume inductively that $C^{[j,j]}$, $C^{[j+1,j]}, \ldots, C^{[i-1,j]}$ are well-defined and column-finite. Then since products and finite sums of column-finite matrices are column-finite, so is $C^{[i,j]}$.

Since the matrix $C$ we have constructed fulfills the hypothesis of the lemma, $C$, by an identical argument, has a unique right inverse $C'$, which is f.v., triangular, and block-column-finite. Now, by the final conclusion of Lemma 6, $|T| \cdot |C| \cdot |C'|$ is f.v. So, $T = T(CC') = (TC)\,C' = C'$.

The final conclusion follows from Lemma 5.

THEOREM 6. *Let $P$ be any block-column-finite n-k-s or $\omega$-k-s chain, and assume that each $P^{[i,i+k]}$ is diagonal. Then:*

1. *The $R$ matrix associated with $P$ has a unique two-sided inverse $Q$.*
2. *$R$ and $Q$ are f.v., triangular, and block-column-finite.*
3. *$Q$ is the unique right inverse of $R$, and $R$ is the unique right inverse of $Q$.*
4. *$Q$ is the only left inverse of $R$ which is triangular, and $R$ is the only left inverse of $Q$ which is triangular.*
5. $P^i = QS^iR$, $i = 0, 1, 2, \ldots$ .

PROOF. Let us show that $R$ fulfills the hypothesis of Lemma 7. $R$ is f.v. and triangular. $R$ is also block-column-finite: since $P$ is block-column-finite, so is each $P^m$, $m = 0, 1, 2, \ldots$, by the final conclusion of Lemma 6. And, it is easy to show that for $i \geqslant j$, $R^{[i,j]} = (P^m)^{[r,j]}$, $i = mk + r$, $0 \leqslant r < k$.

Each $R^{[i,i]}$ is diagonal: automatically (by triangularity), $R_{XX} > 0$. And, if $(0, \bar{X}) \neq (0, \bar{Y})$, and $i = mk + r$, $0 \leqslant r < k$, then

$$R_{(i,\bar{X}),(i,\bar{Y})} = P^{(m)}_{(r,\bar{X}),(mk+r,\bar{Y})} = 0.$$

To show conclusion 5, we need only show that $P$, $Q$, $R$, and $S^i$ $(i \geqslant 0)$ all freely associate. This is satisfied if all finite products among themselves of $P$, $|Q|$, $R$, and $S$ are again f.v. But this holds by Lemma 6, where $r_P = r_S = k$, and $r_R = r_{|Q|} = 0$.

Note that Theorem 3 holds for this class of chains also: we can carry the proof over completely, with only one change—$P^{[r-k,r]}$ has an inverse since it is diagonal.

Surprisingly, unlike the row-finite case it is not necessarily true that the matrix $N$ of mean number of visits is given by $N = QTR$. A counterexample is given in Section 12.

## 8. Another Class of Representable $n$-$k$-$s$ Chains

In the previous section, we saw that $n$-$k$-$s$ chains $P$ are representable when

1. Each $P^{[i,i+k]}$ is diagonal
2. $P$ is block-column-finite.

Neither condition alone is sufficient, as the counterexamples in Section 12 show. Since condition 1 is so natural—it says that when the process takes its maximum jump of $k$ steps in the first coordinate, no other coordinate changes—we seek another class of $n$-$k$-$s$ chains which are representable because of this condition along with some other conditions. One such additional condition is

2. There exists $d > 0$ such that $P_{(i,\bar{X}),(i+k,\bar{X})} \geqslant d$. We can weaken this condition even further to:

2. There exists a set of positive scalars $\{d_i\}$ such that $P_{(i,\bar{X}),(i+k,\bar{X})} \geqslant d_i$.

All of these condition 2's together are not sufficient unless we include condition 1, as counterexample 3 in Section 12 shows.

We begin with two lemmas.

Lemma 8. *Let $A$ and $B$ be two nonnegative, f.v. matrices, both indexed by the n-dimensional or the infinite-dimensional terminating coordinates. Assume:*

1. *There exists a set of nonnegative scalars $\{a_i\}$ such that the row sum of the $(i, \bar{X})$ row of $A$ is less than or equal to $a_i$, uniformly in $\bar{X}$.*

2. *There exists a constant $r_A$ such that $A_{XY} = 0$ whenever $y_1 > x_1 + r_A$.*

Assume $B$ also has a set of scalars $\{b_i\}$ fulfilling hypothesis 1, and a constant $r_B$ fulfilling hypothesis 2.

Then $C = AB$ has the same properties: $C$ is nonnegative, f.v., and has a set of scalars $\{c_i\}$ fulfilling 1, and a constant $r_C$ fulfilling 2. Thus, the product of any finite number of such matrices is again such a matrix, and in particular is f.v.

PROOF. Let $C = AB$. Then

$$C^{[i,j]} = \sum_{m=0}^{i+r_A} A^{[i,m]}B^{[m,j]}.$$

As in Lemma 6, $r_A + r_B$ can serve as $r_C$. So 2 holds for $C$.

To show 1 holds for $C$, we need only show that there exists a set of scalars $\{c_{ij}\}$ such that $C^{[i,j]}1 \leqslant c_{ij}1$, since then we can set

$$c_i = \sum_{j=0}^{i+r_A} c_{ij}.$$

This will also show, of course, that $C$ is f.v. Now

$$\begin{aligned} C^{[i,j]}1 &= \sum_{m=0}^{i+r_A} A^{[i,m]}B^{[m,j]}1 \text{ by nonnegativity} \\ &\leqslant \sum_{m=0}^{i+r_A} A^{[i,m]}\, b_m 1 \\ &\leqslant \max\{b_m\} \cdot \sum_{m=0}^{i+r_A} A^{[i,m]}1 \\ &\leqslant a_i \max\{b_m\}1, \end{aligned}$$

where the maximum is taken over $0 \leqslant m \leqslant i + r_A$. We can let

$$c_{ij} = a_i \max\{b_m\}.$$

LEMMA 9. *Assume $T$ is any f.v., triangular matrix indexed by either the n-dimensional or the infinite-dimensional terminating coordinates, and assume also that:*

A. *There exists a set $\{a_i\}$ of scalars, such that the row sum of the $(i, \bar{X})$ row of $|T|$ is less than or equal to $a_i$, uniformly in $\bar{X}$.*

B. *Each $T^{[i,i]}$ is diagonal.*

C. *There exists a set of positive scalars $\{b_i\}$, such that $b_i \leqslant |T^{[i,i]}|_{XX}$ for all $X$.*

*Then:*

1. *$T$ has a unique two-sided inverse $C$.*
2. *$C$ is f.v. and triangular, and fulfills each of $A$, $B$, and $C$.*
3. *$C$ is the unique right inverse of $T$.*
4. *$C$ is the only left inverse of $T$ which is triangular.*

NOTE. Again, Section 12 shows that $C$ is not necessarily the unique left inverse of $T$.

PROOF. As in Lemma 7, necessary and sufficient conditions for a matrix $C$ to be a right inverse of $T$ are

$$\begin{aligned} C^{[i,j]} &= 0, && i<j \\ C^{[i,j]} &= (T^{[j,j]})^{-1} \\ C^{[i,j]} &= -\,(T^{[i,i]})^{-1}\sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]}, && i>j. \end{aligned} \tag{3}$$

Since these equations are recursion equations in $i$, we can get at most one solution. The matrix $C$, if it exists, certainly fulfills hypothesis $B$ and $C$: $C^{[i,i]} = (T^{[i,i]})^{-1}$, which is diagonal; and,

$$\frac{1}{a_i} \leqslant \frac{1}{|\,T^{[i,i]}\,|_{XX}} = |\,C^{[i,i]}\,|_{XX}\,.$$

Let us prove that equations (3) give us a well-defined solution $C$ (i.e., that each $C^{[i,j]}$ is well defined), and simultaneously, let us show that the matrix $C$ fulfills hypothesis $A$. Fulfilling hypothesis $A$ is equivalent to there being $\{a'_{ij}\}$ such that $|\,C^{[i,j]}\,| \leqslant a'_{ij}1$. Now $C^{[j,j]} = (T^{[j,j]})^{-1}$ is well defined, and satisfies this condition, with $a'_{jj} = 1/b_j$. Assume inductively that $C^{[j,j]}$, $C^{[j+1,j]},\ldots, C^{[i-1,j]}$ are all well defined and satisfy $|\,C^{[r,j]}\,|\,1 \leqslant a'_{rj}1$ for some $a'_{rj} < \infty$. Then, first, each $T^{[i,m]}C^{[m,j]}$, $m < i$, is well defined, since we need only show each entry of $|\,T^{[i,m]}\,|\,|\,C^{[m,j]}\,|$ is finite, for $m < i$.

$$\begin{aligned} (|\,T^{[i,m]}\,|\,|\,C^{[m,j]}\,|)_{XY} &= \sum_Z |\,T^{[i,m]}\,|_{XZ}\,|\,C^{[m,j]}\,|_{ZY} \\ &\leqslant \sum_Z |\,T^{[i,m]}\,|_{XZ}\, a'_{mj} \\ &= a'_{mj}\sum_Z |\,T^{[i,m]}\,|_{XZ} \\ &\leqslant a'_{mj}a_i\,. \end{aligned}$$

Thus the finite sum

$$\sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]}$$

is well defined, and therefore so is

$$-(T^{[i,i]})^{-1}\sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]},$$

since $(T^{[i,i]})^{-1}$ is diagonal. Thus each $C^{[i,j]}$ is indeed well defined, and so $C$ is well defined. And,

$$|C^{[i,j]}| = \left|(T^{[i,i]})^{-1}\sum_{t=j}^{i-1} T^{[i,t]}C^{[t,j]}\right| 1$$

$$\leqslant \left(\sum_{t=j}^{i-1} |(T^{[i,i]})^{-1}|\,|T^{[i,t]}|\,|C^{[t,j]}|\right) 1.$$

So, to finish off this induction, we need only show that there exists some constant $c = c(i, t, j)$ such that

$$|(T^{[i,i]})^{-1}|\,|T^{[i,t]}|\,|C^{[t,j]}|\,1 \leqslant c1, \qquad j \leqslant t < i.$$

Now

$$\begin{aligned}|(T^{[i,i]})^{-1}|\,|T^{[i,t]}|\,|C^{[t,j]}|\,1 &= |(T^{[i,i]})^{-1}|\,|T^{[i,t]}|\,(|C^{[t,j]}|\,1)\\ &\leqslant a'_{tj}\,|(T^{[i,i]})^{-1}|\,|T^{[i,t]}|\,1\\ &\leqslant a'_{tj}a_i\,|(T^{[i,i]})^{-1}|\,1\\ &\leqslant a'_{tj}a_i\,\frac{1}{b_i}.\end{aligned}$$

The induction is complete, and we have proven that $T$ has a unique right inverse $C$. This matrix $C$ we have constructed fulfills all the hypotheses of the lemma, as we proved, so $C$ has a unique right inverse $C'$, which also fulfills the hypotheses. By the final conclusion of Lemma 8, $|T| \cdot |C| \cdot |C'|$ is f.v. Thus $C' = (TC)C' = T(CC') = T$.

Lastly, conclusion 4 follows from Lemma 5.

THEOREM 7. *Let $P$ be any $n$-$k$-$s$ or $\omega$-$k$-$s$ chain $P$ with the following two properties:*

1. *Each $P^{[i,i+k]}$ is diagonal.*

2. *There exists a set of positive scalars $\{d_i\}$ such that $P_{(i,\mathcal{X}),(i+k,\mathcal{X})} \geqslant d_i$, uniformly.*

*Then:*

1. *The matrix $R$ associated with $P$ has a unique two-sided inverse $\underset{\sim}{Q}$.*

2. *$\underset{\sim}{Q}$ is f.v., triangular, and there is a set $\{f_i\}$ of positive scalars, such that the row sum of the $(i, \bar{X})$ row of $\underset{\sim}{Q}$ is less than or equal to $f_i$, uniformly.*

3. *$\underset{\sim}{Q}$ is the unique right inverse of $R$, and $R$ is the unique right inverse of $\underset{\sim}{Q}$.*

4. *$\underset{\sim}{Q}$ is the only left inverse of $R$ which is triangular, and $R$ is the only left inverse of $\underset{\sim}{Q}$ which is triangular.*

5. $P^i = \underset{\sim}{Q}S^iR,\ i = 0, 1, 2, \ldots$ .

PROOF. To prove conclusions 1-4, we need only show that $R$ fulfills hypotheses $A$, $B$, and $C$ of Lemma 9. Then we can apply the results of Lemma 9 to $\underset{\sim}{Q}$ also.

A. Set $a_i = 1$.

B. This follows, as in the proof of Theorem 6.

C. If $i = mk + r$, then it is easy to show that we can set

$$b_i = d_r\, d_{k+r} \cdots d_{(m-1)k+r}\,.$$

To prove conclusion 5, we need only show, as in Theorem 6, that $P$, $|\underset{\sim}{Q}|$, $R$, and $S$ fulfill the hypotheses of Lemma 8. The $\{a_i\}$ of part 1 exist for $|\underset{\sim}{Q}|$ by conclusion 2 of this theorem, and $a_i = 1$ for $P$, $R$, and $S$. As for part 2: $r_{|Q|} - r_R = 0$, and $r_P - r_S = k$.

Theorem 3 applies to this class of chains, with the same modification in the proof as in the previous section.

Finally, as in the row-finite case (but unlike the block-column-finite case), the matrix $N$ is given by $N = \underset{\sim}{Q}TR$. To show this, first note that each column of $TR$ is uniformly bounded, as the last paragraph of Section 3 shows. Then, since $|\underset{\sim}{Q}|$ has finite row sums, we have

$$\begin{aligned} N &= \lim_n \underset{\sim}{Q}T^{(n)}R \\ &= \underset{\sim}{Q} \lim_n T^{(n)}R \text{ by dominated convergence and the above remarks} \\ &= \underset{\sim}{Q}(TR) \text{ by monotonicity} \\ &= \underset{\sim}{Q}TR \text{ since } |\underset{\sim}{Q}|\, TR \text{ is f.v., by the above remarks.} \end{aligned}$$

## 9. SUMS OF VECTOR-VALUED INDEPENDENT RANDOM VARIABLES

By renumbering the states we can turn many Markov chains into representable $n$-$k$-$s$ chains. Various classes of sums of independent random variables are of this type. For example, we can represent sums of $n$-dimensional or infinite-dimensional vector-valued (with integral entries) independent random variables, with the following restrictions: in $S_i = X_1 + \cdots + X_i$

(where each $X_r$ is identically distributed), $X_r$ can only take on a finite number of values in the first coordinate; when $X_r$ takes on its maximum $a$ or its minimum $b$ in the first coordinate, all the other coordinates must be 0. The values $a$ and $b$ must not, of course, both be 0. And, in the infinite-dimensional case, we must also add that each value of $X_r$ has 0's in all but a finite number of entries. Then after renumbering the states, we will have an $n$-$k$-$s$ or $\omega$-$k$-$s$ chain of the type in Section 8. The method of renumbering, which is done in [2], is as follows. If $a > 0$ and $b < 0$, then let $c = -b$, and renumber the first coordinate of the states as follows:

$$0, 1, \ldots, a - 1, -1, -2, \ldots, -c, a, a + 1, \ldots,$$
$$2a - 1, -c - 1, -c - 2, \ldots, -2c, 2a, 2a + 1, \ldots .$$

Then $k = a + c$. If $b \geqslant 0$, do not renumber the first coordinate of the states; then $k = a$. If $a \leqslant 0$, renumber the first coordinate of the states $0, -1, -2, \ldots$; then $k = |\, b \,|$. And in all cases, renumber the other coordinates $0, 1, -1, 2, -2, 3, -3, \ldots$ .

The most famous such chains are the $n$-dimensional random walks. By other renumbering schemes, we can represent reflecting random walks.

## 10. A Semi-Representation for the Most General Denumerably Infinite Markov Chain

Because of the great freedom which an $n$-$k$-$s$ chain has in all but one dimension, the reader may have already anticipated a theorem of the type we are about to prove.

By $P^E$ we mean the process which is obtained by watching a Markov chain only when it enters a given set of states $E$. $P^E$ is easily proven (in [1]) to be a Markov chain in its own right.

THEOREM 8. *Any Markov chain $A$ with a countably infinite number of states is of form $P^E$, where $E = \{(0, 0), (0, 1), (0, 2), \ldots\}$, with $P$ a representable* 2-1-$s$ *chain of the type in Section* 8.

PROOF. Without loss of generality, let $A$ be indexed by the natural numbers.

Define

$$P = \begin{pmatrix} \frac{1}{2}A & \frac{1}{2}I & & & \\ \frac{1}{2}A & 0 & \frac{1}{2}I & & \\ \frac{1}{2}A & 0 & 0 & \frac{1}{2}I & \\ \frac{1}{2}A & 0 & 0 & 0 & \frac{1}{2}I \\ & & \vdots & & \end{pmatrix};$$

$\frac{1}{2}A$, $\frac{1}{2}I$, and 0 are being used as basic submatrices.

$P$ can be considered as being of form

$$\begin{array}{cc} & \begin{array}{cc} E & \tilde{E} \end{array} \\ \begin{array}{c} E \\ \tilde{E} \end{array} & \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}; \end{array}$$

by $\tilde{E}$ we mean the set of all states excluding those in $E$. Thus $M_2$ is indexed down by all states in $E$, and across by all states not in $E$, etc. It is proven in [1] that $P^E$ then equals $M_1 + M_2\,(\sum_{m=0}^{\infty} (M_4)^m)\, M_3$. So, in this case,

$$\begin{aligned} P^E &= \tfrac{1}{2} A + (\tfrac{1}{2} I \quad 0 \quad 0 \quad \cdots) \left( \sum_{m=0}^{\infty} \begin{pmatrix} 0 & \tfrac{1}{2} I & & \\ 0 & 0 & \tfrac{1}{2} I & \\ 0 & 0 & 0 & \tfrac{1}{2} I \\ & & \vdots & \end{pmatrix}^m \right) \begin{pmatrix} \tfrac{1}{2} A \\ \tfrac{1}{2} A \\ \tfrac{1}{2} A \\ \vdots \end{pmatrix} \\ &= \tfrac{1}{2} A + (\tfrac{1}{2} I \quad \tfrac{1}{4} I \quad \tfrac{1}{8} I \quad \tfrac{1}{16} I \quad \cdots) \begin{pmatrix} \tfrac{1}{2} A \\ \tfrac{1}{2} A \\ \tfrac{1}{2} A \\ \vdots \end{pmatrix} \\ &= \tfrac{1}{2} A + \tfrac{1}{4} A + \tfrac{1}{8} A + \tfrac{1}{16} A + \cdots \\ &= A. \end{aligned}$$

Note that $A$ is the process obtained by projecting $P$ on the $x_2$-axis and watching the process only when it changes $x_2$-values. After projection, the process changes $x_2$-values with probability one, since the probability that it does not is equal to $(\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{1}{2}) \cdots = 0$.

Certain properties which have a simple formula for a representable $P$ can be used to give information about $A = P^E$. For example, if $i, j \in E$ then $N_{ij}$ is the same whether computed for $A$ or for $P$. The same is true for other quantities, such as "hitting probabilities." So, if such a quantity is obtainable from the representation in some way, then it can thus be obtained for the most general Markov chain $A$. It is, however, not clear whether this is a useful technique.

## 11. Advancing Chains

We can generalize row-finite $n$-$k$-$s$ chans to get an even larger class of representable chains. An *$n$-dimensional advancing chain* is a row-finite Markov chain with states the $n$-dimensional coordinates, and which has associated with it a function $f$ defined on the states, with the following properties:

1. $P_{X,f(X)} > 0$ and $P_{XY} = 0$ for $Y > f(X)$. That is, $f(X)$ tells the largest state that the process can move to in one step from $X$.

2. $f$ is strictly monotone increasing. That is, $X < Y$ implies $f(X) < f(Y)$.
3. $f(X) > X$ for every $X$.

Note that even when $n = 1$, this is still a generalization of 1-$k$-$s$ chains.

Define the matrix $S$, which is indexed by the $n$-dimensional coordinates, by $S_{XY} = \delta_{Y,f(X)}$. Note that $S^i$ is still extremely simple to find:

$$S^i_{XY} = \delta_{X,f^{(i)}(X)},$$

where $f^{(2)}(X) = f(f(X))$, etc.

Now let us define the matrix $R$, which is again indexed by the $n$-dimensional coordinates. Denote the set of states by $C$. Set

$$R_{XY} = \begin{cases} \delta_{XY}, & X \notin f(C) \\ \sum_W R_{ZW} P_{WY}, & X = f(Z). \end{cases} \tag{4}$$

Since $0 \notin f(C)$, the 0th row of $R$ is certainly well defined. If $X = f(Z)$, then $R_{XY}$ is well defined, since $X > Z$, and so the $X$th row of $R$ is defined only in terms of an earlier row.

By construction, $RP = SR$, since

$$(RP)_{ZY} = R_{f(Z),Y} = \sum_W S_{ZW} R_{WY} = (SR)_{ZY}.$$

Let us show that $R$ is f.v., row-finite, and triangular. The 0th row of $R$ is all right (i.e., such as to make $R$ f.v., row-finite, and triangular). Assume inductively that for all $U < X$, the $U$th row of $R$ is all right. Then let us show that the $X$th row of $R$ is all right. This is certainly the case if $X \notin f(C)$. So assume $X \in f(C)$. Write $X = f(Z)$. For each $Y$, $R_{XY}$ is finite, since $R_{XY}$ is defined in (4) as a finite sum of finite numbers by induction hypothesis, since $Z < X$.

Let

$$S_1 = \{W \mid R_{ZW} > 0\}$$

$$S_2 = \{Y \mid P_{WY} > 0 \text{ for some } W \in S_1\}.$$

By induction hypothesis, $S_1$ is a finite set. Since $P$ is row-finite, $S_2$ is also a finite set. Then $R_{XY} = 0$ unless $Y \in S_2$, so the $X$th row of $R$ has a finite number of nonzero entries.

We need now only show triangularity to complete the induction.

$$R_{XX} = \sum_{W} R_{ZW} P_{W,f(Z)}$$

$$= \sum_{W \geq Z} R_{ZW} P_{W,f(Z)} \text{ by induction hypothesis,} \qquad \text{since} \qquad Z < X$$

$$= R_{ZZ} P_{Z,f(Z)} \qquad \text{since} \qquad P_{W,f(Z)} = 0 \qquad \text{for} \qquad W > Z$$

$$\neq 0 \text{ by induction hypothesis.}$$

Lastly, if $Y > X = f(Z)$, then

$$R_{XY} = \sum_{W \geq Z} R_{ZW} P_{WY} = 0,$$

since

$$P_{WY} = 0 \qquad \text{for all} \qquad Y > f(Z) \geq f(W).$$

Lemma 2 now holds for $R$, and so $P$ fulfills the conclusions of Theorem 1 and 3. In particular, for every $i$, $P^i = QS^iR$.

## 12. Five Counterexamples

Before presenting the counterexamples, let us first prove a lemma.

Lemma 10. *If a f.v. triangular matrix $T$ which is indexed by the $n$-dimensional coordinates has a right inverse $T_1$, $T_1$ is triangular. Further, the only solution to $TA = 0$ is $A = 0$.*

Proof. By induction on $n$. The lemma is true for $n = 1$: by Lemma 2, $T$ has a unique right inverse $T_1$, which is triangular, and which is also a left inverse of $T$. And, $TA = 0$ implies

$$0 = T_1(TA) = (T_1T)A = A.$$

Assume inductively that the lemma is true for dimension $n - 1$, where $n > 1$. Then in terms of basic submatrices, $TT_1 = I$ becomes

$$\sum_{s=0}^{i} T^{[i,s]} T_1^{[s,j]} = \delta_{ij} I.$$

Each $T^{[i,j]}$ can be considered as being indexed by the $(n - 1)$-dimensional coordinates. From $T^{[0,0]} T_1^{[0,j]} = \delta_{ij} I$, we have by induction hypothesis that $T_1^{[0,0]}$ is triangular, and that $T_1^{[0,j]} = 0$ when $j > 0$. Thus all submatrices $T_1^{[0,j]}$ are all right (i.e., such as to make $T_1$ triangular). Assume inductively that all submatrices $T_1^{[r,i]}$, $r = 0, 1, \dots, m-1$ and $i = 0, 1, 2, \dots$ are all right. Then if $y \geqslant m$,

$$\delta_{my} I = \sum_{s \leqslant m} T^{[m,s]} T_1^{[s,y]} = T^{[m,m]} T_1^{[m,y]},$$

since by induction hypothesis, $T_1^{[s,y]} = 0$ for $s < m \leqslant y$. Thus by induction hypothesis, each $T_1^{[m,y]}$ is all right.

Lastly, if $TA = 0$, then $T^{[0,0]} A^{[0,i]} = 0$, so $A^{[0,i]} = 0$ for all $i$. And, if $A^{[r,i]} = 0$ for all $r < m$ and for all $i$, then $T^{[m,m]} A^{[m,i]} = 0$, and so $A^{[m,i]} = 0$ for all $i$.

COUNTEREXAMPLE 1. A simple example of a triangular matrix indexed by the 2-dimensional coordinates which has no right inverse.

Let

$$A = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & 1 \\ & \vdots & & \end{pmatrix} \qquad B = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \\ & \vdots & & \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ & & \vdots & & \end{pmatrix}.$$

Then $AB = I$, and by Lemma 2, $B$ is the unique right inverse of $A$. Set

$$T = \begin{pmatrix} A & & & \\ E & A & & \\ E & E & A & \\ E & E & E & A \\ & \vdots & & \end{pmatrix}.$$

By Lemma 10, if $T$ has a right inverse $T_1$, $T_1$ is triangular. So, the basic submatrix in the upper left corner of $T_1$ would be $B$. But then $TT_1$ involves the product $EB$, which has all infinite entries, a contradiction.

COUNTEREXAMPLE 2. A 2-1-$s$ chain $P$ with each $P^{[i,i+1]}$ diagonal, but whose $R$ matrix does not have a right inverse.

Define:

$$P_{(0,0)\,Y} = \delta_{Y,(1,0)}$$

$$\begin{array}{c} P_{(0,a),Y} = \\ (a = 1, 2, 3, \ldots) \end{array} \begin{cases} \frac{1}{2}, & Y = (0, 0) \\ \frac{1}{2} - (\frac{1}{2})^a, & Y = (0, 1) \\ (\frac{1}{2})^a, & Y = (1, a) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(1,0),Y} = \begin{cases} \frac{1}{2}, & Y = (2, 0) \\ (\frac{1}{2})^{a+2}, & Y = (1, a), \qquad a = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(a,b),Y} = \delta_{Y,(a+1,b)}$$

$$(a = 1 \text{ and } b \geqslant 1, \text{ or } a \geqslant 2).$$

From our recursive definition of $R$, we easily find:

$$R^{[0,0]} = I$$

$$R^{[1,1]} \text{ is diagonal with } (R^{[1,1]})_{ii} = (\tfrac{1}{2})^i$$

$$(R^{[1,0]})_{i0} = \tfrac{1}{2}(1 - \delta_{i0})$$

$$(R^{[2,1]})_{0i} = (\tfrac{1}{2})^{i+2}. \tag{5}$$

Assume now $R$ has a right inverse $Q$. $Q$ is triangular, by Lemma 10. From $RQ = I$, we get:

$$Q^{[0,0]} = I, \qquad R^{[1,0]}Q^{[0,0]} + R^{[1,1]}Q^{[1,0]} = 0$$

$$R^{[2,0]}Q^{[0,0]} + R^{[2,1]}Q^{[1,0]} + R^{[2,2]}Q^{[2,0]} = 0 \tag{6}$$

From the first two of Eqs. (6), $-R^{[1,0]} = R^{[1,1]}Q^{[1,0]}$. Since $R^{[1,1]}$ is triangular and row-finite, $Q^{[1,0]} = -(R^{[1,1]})^{-1} R^{[1,0]}$. So the third equation becomes

$$R^{[2,0]} - R^{[2,1]}((R^{[1,1]})^{-1} R^{[1,0]}) + R^{[2,2]}Q^{[2,0]} = 0. \tag{7}$$

When we use Eqs. (5) to obtain the entry in the 0th row and 0th column of the matrix Eq. (7), an infinite entry appears, which is a contradiction.

COUNTEREXAMPLE 3. A 2-1-$s$ chain $P$ which is column-finite and which fulfills $P_{(x_1,x_2),(x_1+1,x_2)} \geqslant \frac{1}{6}$, but whose $R$ matrix does not have a right inverse.

Define

$$P_{(0,0),Y} = \begin{cases} \frac{1}{2}, & Y = (0,0) \\ \frac{1}{2}, & Y = (1,0) \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{c} P_{(0,a),Y} = \\ (a = 1, 2, 3, \ldots) \end{array} \begin{cases} \frac{1}{2}, & Y = (0,a) \\ \frac{1}{3}, & Y = (1, a-1) \\ \frac{1}{6}, & Y = (1,a) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(1,0),Y} = \begin{cases} \frac{1}{2}, & Y = (2,0) \\ (\frac{1}{2})^{a+1}, & Y = (1,a), \qquad a = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(a,b),Y} = \delta_{Y,(a+1,b)} \qquad (a = 1 \text{ and } b \geqslant 1, \text{ or } a \geqslant 2).$$

We easily find that:

$$R^{[1,0]} = \tfrac{1}{2} I$$

$$(R^{[2,1]})_{0,i} = (\tfrac{1}{2})^{i+2}$$

$$(R^{[1,1]})_{ij} = \begin{cases} \frac{1}{2}, & i = j = 0 \\ \frac{1}{3}, & j + 1 = i \geqslant 1 \\ \frac{1}{6}, & j = i \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2, $R^{[1,1]}$ has a unique two-sided inverse, which is triangular. $(R^{[1,1]})^{-1}_{i,0} = (-1)^i 2^{i+1}$, as we can easily show by induction on $i$.

Now assume that $R$ has a right inverse $Q$. By Lemma 10, $Q$ is triangular. Equation (7) of the previous counterexample holds, just as before. But

$$R^{[2,1]}((R^{[1,1]})^{-1} R^{[1,0]}) = \tfrac{1}{2} R^{[2,1]}(R^{[1,1]})^{-1},$$

which is undefined in the upper left entry.

COUNTEREXAMPLE 4. A nonnegative triangular matrix indexed by the natural numbers with more than one left inverse.

This example will also have row sums less than one.

Define

$$T_{ab} = \begin{cases} (\frac{1}{2})^{a+1}, & a \text{ odd and } b \leqslant a, \text{ or } a \text{ even and } b \text{ even and } b \leqslant a \\ (\frac{1}{2})^{a}, & a \text{ even and } b \text{ odd and } b \leqslant a \\ 0, & b > a \end{cases}$$

$$V_{ab} = \begin{cases} -2^{(b-2)/2}, & b \text{ even} \\ \phantom{-}2^{(b-1)/2}, & b \text{ odd,} \end{cases}$$

Then $T$ is triangular and of course row-finite, so it has a left inverse $T_1$ by Lemma 2. It is easy to check that $VT = 0$. Hence $T_1 - V$ is a second left inverse of $T$.

COUNTEREXAMPLE 5. A representable transient 2-1-$s$ chain for which $N \neq QTR$.

We will show that $N \neq Q(TR)$ and also $N \neq (QT)R$ for this chain.

Define $P$ as follows:

$$\underset{(a = 0, 1, 2,\ldots)}{P_{(0,a),Y}} = \begin{cases} 1 - (\frac{1}{2})^{a+1}, & Y = (0, a) \\ (\frac{1}{2})^{a+1}, & Y = (1, a) \\ 0 & \text{otherwise} \end{cases}$$

$$P_{(1,0),Y} = \begin{cases} (\frac{1}{2})^{a+1}, & Y = (1, a), \quad a = 1, 2, 3,\ldots \\ \frac{1}{2}, & Y = (2, 0) \\ 0 & \text{otherwise} \end{cases}$$

$$\underset{(a = 1, 2, 3,\ldots)}{P_{(1,a),Y}} = \delta_{(2,a),Y}$$

$$\underset{(a = 2, 3, 4,\ldots)}{P_{(a,0),Y}} = \begin{cases} = \dfrac{a(a+2)}{(a+1)^2}, & Y = (a + 1, 0) \\ > 0, & Y < (a + 1, 0) \\ = 0, & Y > (a + 1, 0) \end{cases} \quad \text{(This is obviously possible to bring about, and with } \textstyle\sum_Y P_{(a,0),Y} = 1\text{)}$$

$$\underset{\substack{(a = 2, 3, 4,\ldots)\\(b = 1, 2, 3,\ldots)}}{P_{(a,b),Y}} = \begin{cases} \delta_{a+1,b}, & b > a \\ \frac{1}{2}, & Y = (a + 1, b) \quad \text{and} \quad b \leqslant a \\ \frac{1}{2}, & Y = (0, 0) \quad \text{and} \quad b \leqslant a \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that $P$ is a 2-1-$s$ chain of the type in Section 7, and with all row sums unity. One can also check that (0, 0) communicates with every state, and so all states communicate. To show that all states of $P$ are transient, we need only to show that (0, 0) is transient, since all states communicate. Now the chain, starting in (0, 0), can "drift off to infinity" along the $x_1$-axis: that is, from (0, 0) the process can move to (1, 0), and from there to (2, 0), and from there to (3, 0), etc., with probability

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\prod_{a=2}^{\infty}\frac{a(a+2)}{(a+1)^2}.$$

The infinite product is easily shown to converge to $\frac{2}{3} > 0$.

Now let us show that $N \neq Q(TR)$ and $N \neq (QT)R$. The basic submatrix (assuming it exists) $(Q(TR))^{[2,0]}$ must equal

$$Q^{[2,0]}(TR)^{[0,0]} + Q^{[2,1]}(TR)^{[1,0]} + Q^{[2,2]}(TR)^{[2,0]}.$$

So to show that the f.v. matrix $N$ does not equal $Q(TR)$, we need only show that there is an infinite entry in

$$Q^{[2,1]}(TR)^{[1,0]} = Q^{[2,1]}\left(\sum_{k=1}^{\infty} R^{[k,0]}\right).$$

Let us now show that this condition is also sufficient to prove that $N \neq (QT)R$.

Assuming it exists, the basic submatrix $((QT)R)^{[2,0]}$ is easily shown to equal

$$Q^{[2,0]} + (Q^{[2,0]} + Q^{[2,1]})R^{[1,0]} + \sum_{k=2}^{\infty}((Q^{[2,0]} + Q^{[2,1]} + Q^{[2,2]})R^{[k,0]}).$$

Since each $Q^{[i,j]}$ and $R^{[i,j]}$ is column-finite (by Theorem 6, conclusion 2), we can distribute in the previous expression. Hence, we can reach our contradiction if we can show that

$$\sum_{k=1}^{\infty} Q^{[2,1]}R^{[k,0]} \tag{8}$$

has an infinite entry. By "solving" $RQ = I$, and freely associating and distributing by column-finiteness of the basic submatrices, we get

$$-Q^{[2,1]} = (R^{[2,2]})^{-1}R^{[2,2]}(R^{[1,1]})^{-1} \geqslant 0.$$

So we can distribute in (8) by nonnegativity (the minus sign can be pulled outside), and to show that $N \neq (QT)R$, we now need only show that

$$Q^{[2,1]}\sum_{k=1}^{\infty} R^{[k,0]} \tag{9}$$

contains an infinite entry. As promised, this is precisely the condition we found as sufficient to prove that $N \neq Q(TR)$.

Index $Q^{[2,1]}$ and each $R^{[i,j]}$ by the natural numbers. Then

$$\begin{aligned}
(Q^{[2,1]})_{0,a} &= -((R^{[2,2]})^{-1}R^{[2,1]}(R^{[1,1]})^{-1})_{0,a}\\
&= -((R^{[2,2]})^{-1})_{0,0}(R^{[2,1]})_{0,a}((R^{[1,1]})^{-1})_{a,a}\\
&= -(4)((\tfrac{1}{2})^{a+2})(2^{a+1}), \text{ as simple calculations show}\\
&= -2,
\end{aligned}$$

where the second equality follows since $(R^{[2,2]})^{-1}$ and $(R^{[1,1]})^{-1}$ are diagonal.

Now

$$\left(\sum_{k=1}^{\infty} R^{[k,0]}\right)_{a,0} = \sum_{k=1}^{\infty} P^{(k)}_{(0,a),(0,0)} = N_{(0,a),(0,0)} - \delta_{a,0}\,.$$

So, the entry in the 0th row and 0th column of (9) is

$$-\,2(N_{(0,0),(0,0)} - 1) - 2\sum_{a=1}^{\infty} N_{(0,a),(0,0)}$$

$$= -\,2(N_{(0,0),(0,0)} - 1) - 2N_{(0,0),(0,0)}\sum_{a=1}^{\infty} H_{(0,a),(0,0)}\,,$$

where $H_{XY}$ is the probability that the process, starting in state $X$, eventually reaches state $Y$. We are clearly through if we now show that

$$H_{(0,a),(0,0)} \geqslant \tfrac{1}{2}\,, \qquad a \geqslant 3.$$

With probability one, the process moves from $(0, a)$ to $(1, a)$ in a finite number of steps, since the probability that it does not is the probability that it remains at $(0, a)$ at every stage, which is

$$\left(1 - \frac{1}{2^{a+1}}\right)\left(1 - \frac{1}{2^{a+1}}\right)\left(1 - \frac{1}{2^{a+1}}\right)\cdots = 0.$$

From $(1, a)$ the process moves deterministically to $(2, a)$, from $(2, a)$ deterministically to $(3, a)$,..., and then deterministically to $(a, a)$, and then to $(0, 0)$ on the next step with probability $\frac{1}{2}$. So $H_{(0,a),(0,0)} \geqslant \frac{1}{2}$, and we are through.

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