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Coloring some classes of mixed graphs

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Abstract

We consider the coloring problem for mixed graphs, that is, for graphs containing edges and arcs. A mixed coloring c is a coloring such that for every edge $[x_i, x_j]$, $c(x_i) \neq c(x_j)$ and for every arc (x_p, x_q) , $c(x_p) < c(x_q)$. We will analyse the complexity status of this problem for some special classes of graphs.

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1. Introduction

Scheduling problems containing incompatibility constraints are very often modelled by undirected graphs: every vertex corresponds to a job and two vertices are joined by an edge if the corresponding jobs cannot be processed at the same period. A vertex coloring of the graph then gives a possible schedule respecting the constraints. In general scheduling problems, there are often more requirements than just incompatibility constraints. Hence the ordinary coloring model is too limited to be useful in many scheduling applications. We will consider here scheduling problems containing incompatibility and precedence constraints: several pairs of jobs have to be processed in a given order. To handle these problems, we have to introduce a more general model, able to take into account these requirements: mixed graphs. These graphs have been introduced for the first time in [11].

A *mixed graph* $G_M = (X, U, E)$ is a graph containing edges (set E) and arcs (set U). An edge joining vertices x_i and x_j will be denoted by $[x_i, x_j]$ and an arc with tail x_p and head x_q by (x_p, x_q) . Thus a precedence constraint, saying that job p must be processed before job q , will be represented by an arc (x_p, x_q) . A k -coloring of a mixed graph $G_M = (X, U, E)$ is a function $c: X \rightarrow \{0, 1, \dots, k-1\}$ such that for $[x_i, x_j] \in E$, $c(x_i) \neq c(x_j)$ and for $(x_p, x_q) \in U$, $c(x_p) < c(x_q)$. Notice that the mixed graph G_M must be acyclic, i.e. must not contain any directed circuit, otherwise no proper k -coloring would exist. Also notice that there is a one-to-one correspondence between a feasible schedule in k time units and a k -coloring of the mixed graph G_M . The smallest k such that there exists a k -coloring of G_M is called the *mixed chromatic number* and will be denoted by $\gamma(G_M)$. Let $G_M^0 = (V, U, \emptyset)$ be the directed partial graph of G_M . If $\gamma(G_M^0)$ denotes the chromatic number of G_M^0 , that is, the length of a longest directed path in G_M^0 plus one, then we

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conclude that $\gamma(G_M) \geq \gamma(G_M^0)$. In this paper we only consider finite mixed graphs G_M containing no directed circuits, no multiple edges or multiple arcs and no loops.

Obviously, coloring the vertices of a mixed graph is more general than the ordinary vertex coloring problem and thus it is *NP*-complete. There is not much literature about mixed graph coloring. In [3], an $O(n^2)$ -algorithm to color optimally mixed trees and bounds on the mixed chromatic number for general mixed graphs are given. For mixed bipartite graphs, the mixed chromatic number is bounded above by $\gamma(G_M^0) + 1$ and hence can only take two values. In [3] an open question is the complexity to decide whether it is $\gamma(G_M^0)$ or $\gamma(G_M^0) + 1$ for mixed bipartite graphs. Rote has shown with an elementary construction that this problem is *NP*-complete [8]. Here, we will strengthen this result by proving that it is *NP*-complete even for planar bipartite graphs and for bipartite graphs with maximum degree 3. In [9,10] the unit-time job-shop problem is considered via mixed graph coloring. In this case, G_M^0 is the union of disjoint paths and (V, \emptyset, E) is the union of disjoint cliques. In [10] three branch-and-bound algorithms are developed and tested on randomly generated mixed graphs of order at most 200 for the exact solution and of order at most 900 for the approximate solution. In [12] mixed graph colorings ϕ for which an arc (x_p, x_q) implies that $\phi(x_p) \leq \phi(x_q)$ are considered.

In this paper, we will consider some special classes of graphs and analyse the complexity status of the mixed graph coloring problem for these classes.

2. Some complexity results

First, we will give some definitions taken from [3] which we will use throughout this paper.

Definitions: Let $G_M = (X, U, E)$ be a mixed graph. The *inrank* of a vertex x_i , denoted by $\text{in}(x_i)$, is the length of a longest directed path ending at x_i and the *outrank* of x_i , denoted by $\text{out}(x_i)$, is the length of a longest directed path starting at x_i .

We denote by n the number of vertices in a mixed graph $G_M = (X, U, E)$, i.e. $n = |X|$, and by $N(P)$ the number of vertices on a directed path P .

All graph theoretical terms not defined here can be found in [1].

We will give now some complexity results for some special classes of graphs.

Theorem 1. *Let G_M be a mixed graph having the following properties:*

- (1) for all $x_i \in X$, there exists $x_j \in X$ such that $(x_i, x_j) \in U$ or $(x_j, x_i) \in U$;
- (2) for all maximal directed paths P in G_M , $N(P) = \gamma(G_M^0)$ or $N(P) = \gamma(G_M^0) - 1$.

Then deciding whether $\gamma(G_M) = \gamma(G_M^0)$ or $\gamma(G_M) > \gamma(G_M^0)$ can be done in polynomial time.

Proof. We transform the problem into a *2SAT* problem which is known to be polynomially solvable [2]. Denote by \mathcal{P} the set of vertices belonging to a path P with $N(P) = \gamma(G_M^0)$.

- (1) to each vertex $x \in \mathcal{P}$ with $\text{in}(x) = r$, we associate a variable x_r and a clause (x_r) ;
- (2) to each vertex $x \notin \mathcal{P}$ with $\text{in}(x) = r$, we associate two variables x_r and x_{r+1} ;
- (3) to each path $P = (x^0, x^1, \dots, x^{\gamma(G_M^0)-2})$ with $N(P) = \gamma(G_M^0) - 1$, we associate the clauses $(x_i^i \vee x_{i+1}^i)$, $(\bar{x}_i^i \vee \bar{x}_{i+1}^i)$, for $i = 0, 1, \dots, \gamma(G_M^0) - 2$, and the clause $(\bar{x}_{j+1}^j \vee \bar{x}_{j+1}^{j+1})$, for $j = 0, 1, \dots, \gamma(G_M^0) - 3$;
- (4) to each edge $[x, y] \in E$ such that $x \in \mathcal{P}$, $y \notin \mathcal{P}$ and $\text{in}(x) = \text{in}(y) = r$ (resp. $\text{in}(x) = \text{in}(y) + 1 = r + 1$), we associate the clause $(\bar{x}_r \vee \bar{y}_r)$ (resp. $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$);
- (5) to each edge $[x, y] \in E$ such that $x, y \notin \mathcal{P}$ and $\text{in}(x) = \text{in}(y) = r$ (resp. $\text{in}(x) = \text{in}(y) + 1 = r + 1$), we associate the clauses $(\bar{x}_r \vee \bar{y}_r)$, $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$ (resp. $(\bar{x}_{r+1} \vee \bar{y}_{r+1})$);
- (6) to each edge $[x, y] \in E$ such that $x, y \in \mathcal{P}$ and $\text{in}(x) = \text{in}(y) = r$, we associate the clause $(\bar{x}_r \vee \bar{y}_r)$.

Suppose that an instance of *2SAT* is true. If a variable x_r is set to be ‘true’, then we will color the corresponding vertex x with color r , i.e. $c(x) = r$. Notice that each vertex $x \in \mathcal{P}$ will be colored with $c(x) = \text{in}(x)$ (see (1)) and each vertex $x \notin \mathcal{P}$ will be colored with $c(x) = \text{in}(x)$ or $c(x) = \text{in}(x) + 1$ (see (2) and (3)). Thus, the coloring uses at most $\gamma(G_M^0)$ colors. The clauses in (1) and (3) ensure that for all $(x, y) \in U$ we have $c(x) < c(y)$ and the clauses in (1), (4), (5) and (6) ensure that for all $[x, y] \in E$, $c(x) \neq c(y)$. So we conclude that $\gamma(G_M) = \gamma(G_M^0)$.

Suppose now that $\gamma(G_M) = \gamma(G_M^0)$. Notice that in that case each vertex $x \in \mathcal{P}$ will be colored with $c(x) = \text{in}(x)$ and each vertex $x \notin \mathcal{P}$ will be colored with $c(x) = \text{in}(x)$ or $c(x) = \text{in}(x) + 1$. For each variable x_r occurring in the formula, set it to the value ‘true’ if $c(x) = r$ and false otherwise. It is easy to see that each clause will be satisfied. \square

The previous theorem has the following consequence:

Corollary 2. *Let G_M be a mixed graph having the following properties:*

- (1) *for all $x_i \in X$, there exists $x_j \in X$ such that $(x_i, x_j) \in U$ or $(x_j, x_i) \in U$;*
- (2) $\gamma(G_M^0) \leq \gamma(G_M) \leq \gamma(G_M^0) + 1$;
- (3) *for all maximal directed paths P in G_M , $N(P) = \gamma(G_M^0)$ or $N(P) = \gamma(G_M^0) - 1$.*

Then the mixed chromatic number of G_M can be determined in polynomial time.

Now we will focus on mixed bipartite graphs $G_M = (V_1, V_2, U, E)$. From [3], we know that for mixed bipartite graphs $\gamma(G_M^0) \leq \gamma(G_M) \leq \gamma(G_M^0) + 1$. As already mentioned before, Rote has shown with an elementary construction that deciding whether for such a graph the mixed chromatic number is $\gamma(G_M^0)$ or $\gamma(G_M^0) + 1$ is *NP*-complete [8]. We will strengthen this by proving that it is *NP*-complete even if G_M is planar or if G_M has maximum degree 3 and we will also give some polynomially solvable cases.

Our decision problem on mixed bipartite graphs is the following:

2.1. Mixed bipartite graph coloring (MBGC(G_M, γ_0))

Instance: Mixed bipartite graph $G_M = (X_1, X_2, U, E)$ with $\gamma(G_M^0) = \gamma_0 \geq 2$.

Question: Can G_M be properly colored using at most γ_0 colors?

In order to prove our results, we will often use the precoloring extension problem, which is the following:

2.2. Precoloring extension (PrExt(G, p))

Instance: A positive integer p and a graph G some of whose vertices are precolored using at most p colors.

Question: Can the precoloring of G be extended to a proper coloring of G using at most p colors?

We first derive a simple refinement of the complexity result for MBGC($G_M, 3$) of [8].

Theorem 3. *MBGC($G_M, 3$) is *NP*-complete, even if G_M is planar.*

Proof. To show *NP*-completeness of MBGC($G_M, 3$), we use a reduction from PrExt($G, 3$) on a planar bipartite graph. This problem is proven to be *NP*-complete in [6].

Let $G = (X_1, X_2, E)$ be a planar bipartite graph. Let $X \subseteq X_1 \cup X_2$ be the set of precolored vertices using at most 3 colors (color 0, 1 and 2). Denote by c the proper precoloring. For every vertex $x \in X$, we do the following:

- if $c(x) = 0$, add vertices y_x^1, y_x^2 and a directed path $(x, y_x^1), (y_x^1, y_x^2)$,
- if $c(x) = 1$, add vertices y_x^0, y_x^2 and a directed path $(y_x^0, x), (x, y_x^2)$,
- if $c(x) = 2$, add vertices y_x^1, y_x^2 and a directed path $(y_x^0, y_x^1), (y_x^1, x)$.

We obtain a mixed planar bipartite graph G_M with $\gamma_0 = 3$. We can easily observe that PrExt($G, 3$) on G is equivalent to MBGC($G_M, 3$) on G_M . This shows the *NP*-completeness of MBGC($G_M, 3$) even if G_M is planar as G_M can be obtained from G in polynomial time. \square

We will now give our main statement:

Theorem 4. *MBGC($G_M, 3$) is *NP*-complete when G_M has maximum degree 3.*

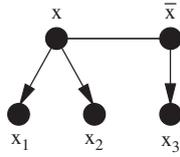


Fig. 1. The variable gadget.

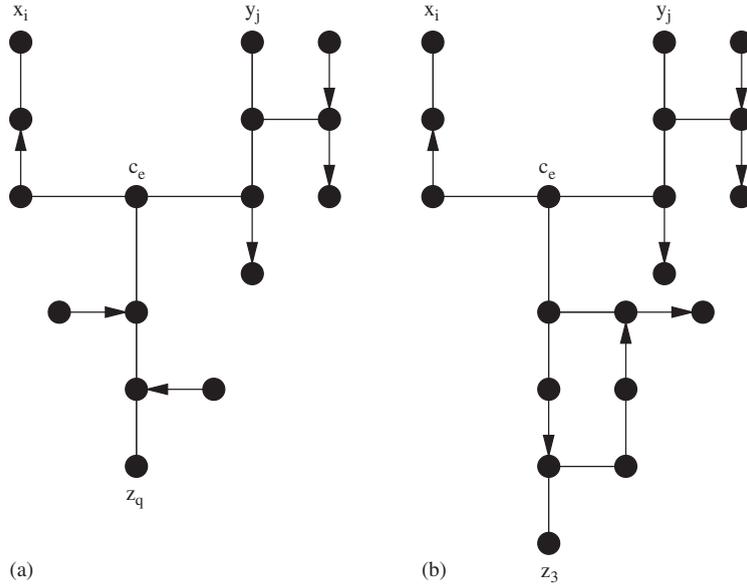


Fig. 2. The clause gadget for (a) $c_e = (x \vee y \vee z)$ and (b) $c_e = (x \vee y \vee \bar{z})$.

Proof. We will use a transformation from the 3SAT problem which is known to be NP-complete even if each variable appears at most three times, each literal at most twice and each clause contains two or three literals [7]. Notice that we can assume that whenever a literal appears twice, this literal is positive.

To each variable x , we associate the variable gadget shown in Fig. 1.

To each clause $c_e = (x \vee y \vee z)$, where x has its i th occurrence, y its j th occurrence and z its q th occurrence, $i, j, q \in \{1, 2\}$, we associate the variable gadget shown in Fig. 2 (a).

To each clause $c_e = (x \vee y \vee \bar{z})$, where x has its i th occurrence, y its j th occurrence, $i, j \in \{1, 2\}$, we associate the variable gadget shown in Fig. 2(b).

To each clause $c_e = (x \vee \bar{y} \vee \bar{z})$, where x has its i th occurrence, $i \in \{1, 2\}$, we associate the variable gadget shown in Fig. 3 (a).

To each clause $c_e = (\bar{x} \vee \bar{y} \vee \bar{z})$, we associate the variable gadget shown in Fig. 3(b).

To each clause c_e containing only two literals, say l', l'' , we associate the clause gadget that we would associate to the clause $(t_e \vee l' \vee l'')$, where t_e is a new variable, but for t_e we introduce the variable gadget shown in Fig. 4 instead of the variable gadget of Fig. 1.

The mixed graph we obtain has maximum degree 3. Notice that for every pair of vertices c_e, c_f , a chain from c_e to c_f always has even length. Thus, there is no odd cycle in G_M , i.e. G_M is bipartite. Furthermore, the length of a longest directed path is 2.

Also notice that if we want to color the vertices of G_M using only colors 0, 1 and 2, in each clause gadget associated to clause c_e the vertices $x_i, y_j, z_q, i, j, q \in \{1, 2, 3\}$, must not all be colored with color 2 otherwise there would be no more color left for vertex c_e . On the other hand, when at least one of the vertices x_i, y_j, z_q has color 1, the clause gadget can be colored properly with colors 0, 1 and 2.

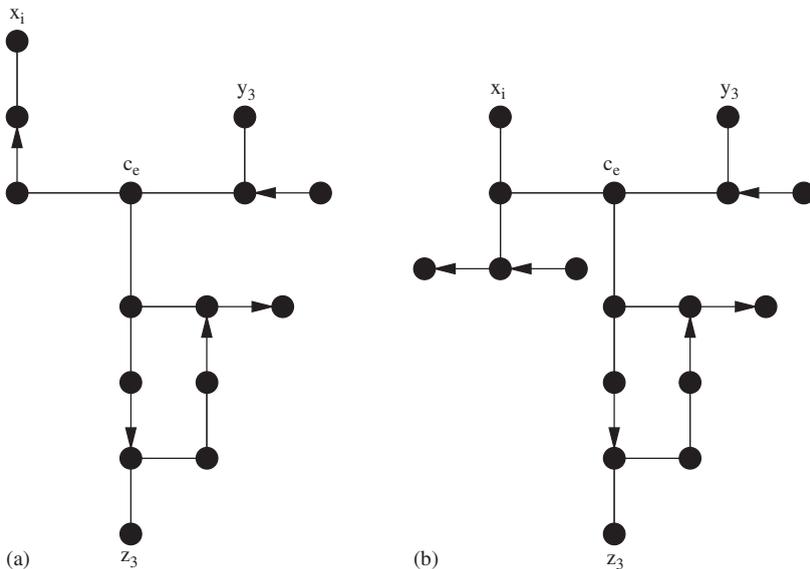


Fig. 3. The clause gadget for (a) $c_e = (x \vee \bar{y} \vee \bar{z})$ and (b) $c_e = (\bar{x} \vee \bar{y} \vee \bar{z})$.



Fig. 4. Variable gadget for t_e .

Suppose there is a truth assignment such that the formula is true. For each variable x which has value ‘true’, color vertex x with color 0, vertices x_1, x_2, \bar{x} with color 1 and vertex x_3 with color 2. For each variable y which has value ‘false’, color vertex \bar{y} with color 0, vertices y, y_3 with color 1 and vertices y_1, y_2 with color 2. As in each clause there is at least one variable which has value ‘true’, there is in each clause gadget at least one of the vertices x_i, y_j, z_q which has color 1, and thus, as we mentioned before, the clause gadget can be colored properly using colors 0, 1 and 2. So $\text{MBGC}(G_M, 3)$ has a positive answer.

Suppose now $\text{MBGC}(G_M, 3)$ has a positive answer. In this case, in each variable gadget corresponding to a variable x , one of the vertices x, \bar{x} has color 0 and one has color 1. Set variable x to ‘true’ if x has color 0 and set it to ‘false’ otherwise. As in each clause gadget at least one of the vertices x_i, y_j, z_q has color 1, there is at least one variable in the corresponding clause which will be set to the value ‘true’. Thus, we have a truth assignment such that the formula is true. \square

We will give now some cases where mixed graph coloring in bipartite graphs can be solved in polynomial time.

Theorem 5. $\text{MBGC}(G_M, 2)$ can be solved in polynomial time.

Proof. Let $G_M = (X_1, X_2, U, E)$ be a mixed bipartite graph such that $\gamma_0 = 2$. Add two new vertices y_0 and y_1 to G_M such that y_0 is adjacent to each vertex x with $\text{in}(x) = 0$ and $\text{out}(x) \neq 0$ and y_1 is adjacent to each vertex x with $\text{in}(x) = 1$. Color y_0 with color 1 and y_1 with color 0. Consider all arcs as edges. We obtain an undirected graph G with some vertices precolored properly with colors 0 and 1. Observe that the answer for $\text{MBGC}(G_M, 2)$ on G_M is ‘yes’ if and only if the answer for $\text{PrExt}(G, 2)$ on the precolored graph G is ‘yes’. By [4] we know that $\text{PrExt}(G, 2)$ is polynomially solvable. This completes the proof as G can be obtained from G_M in polynomial time. \square

Let us consider now a special class of graphs. A k -tree is a graph defined recursively as follows: a k -tree on k vertices consists of a clique on k vertices (k -clique); given any k -tree T_n on n vertices, we construct a k -tree on $n + 1$ vertices

by adjoining a new vertex v_{n+1} to T_n , which is made adjacent to each vertex of some k -clique of T_n and nonadjacent to the remaining $n - k$ vertices. G is called a *partial k -tree* if G is a subgraph of a k -tree.

Theorem 6. $\text{MBGC}(G_M, \gamma_0)$ is polynomially solvable if G_M is a partial k -tree, k fixed, and if for all maximal directed paths P in G_M , $N(P) = \gamma_0$ or $N(P) = \gamma_0 - 1$.

Proof. Let G_M be a mixed bipartite partial k -tree, k fixed. We shall transform the problem in a $\text{PrExt}(G, \gamma_0)$ problem, where G is an unoriented bipartite partial k -tree, which is shown to be polynomially solvable for fixed k [5]. Let \mathcal{P} be the set of vertices belonging to a path P with $N(P) = \gamma_0$.

- (1) for all $x \in \mathcal{P}$, color x with color $c(x) = \text{in}(x)$ (if there is a conflict, STOP, $\gamma(G_M) = \gamma_0 + 1$);
- (2) for all $x \notin \mathcal{P}$, x incident to an arc and $\text{in}(x) = q$, add $\gamma_0 - 2$ pairwise nonadjacent vertices $w_0, w_1, \dots, w_{q-1}, w_{q+2}, \dots, w_{\gamma_0-1}$ such that each of them is adjacent to x ; color each vertex w_i with color $c(w_i) = i$.

Now consider all arcs in G_M as edges. We get an unoriented bipartite graph G containing some precolored vertices. It is easy to see that G is also a partial k -tree, for this same fixed k .

Suppose now that the answer for $\text{PrExt}(G, \gamma_0)$ is ‘yes’. In that case for all $x \in \mathcal{P}$, $c(x) = \text{in}(x)$ and for all $x \notin \mathcal{P}$, x incident to an arc and $\text{in}(x) = q$, $c(x) = q$ or $c(x) = q + 1$. Furthermore, for all $(x, y) \in U$ we have that $c(x) < c(y)$ because of the precolored vertices w_i . Thus, the answer for $\text{MBGC}(G_M, \gamma_0)$ is ‘yes’.

Suppose that the answer for $\text{MBGC}(G_M, \gamma_0)$ is ‘yes’. This implies that for all $x \in \mathcal{P}$, $c(x) = \text{in}(x)$ and for all $x \notin \mathcal{P}$, x incident to an arc and $\text{in}(x) = r$, $c(x) = r$ or $c(x) = r + 1$. So the coloring is a proper extension of the precoloring of G . \square

Corollary 7. $\text{MBGC}(G_M, 3)$ can be solved in polynomial time if G_M is a partial k -tree, k fixed.

3. Conclusion

We considered the mixed graph coloring problem which consists in coloring the vertices of a mixed graph G_M such that for every edge $[x_i, x_j]$, $c(x_i) \neq c(x_j)$ and for every arc (x_p, x_q) , $c(x_p) < c(x_q)$. We gave some complexity results on that problem for some special classes of graphs. In particular, we showed that it is NP -complete for planar bipartite graphs and for bipartite graphs having maximum degree 3. Further research is needed to extend complexity results to some other classes of graphs.

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