Discrete Mathematics 207 (1999) 65–72

DISCRETE MATHEMATICS

Magic rectangles revisited

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Received 6 May 1998; revised 16 December 1998; accepted 4 January 1999

Abstract

Magic rectangles are a generalization of magic squares that have been recently investigated by Bier and Rogers (European J. Combin. 14 (1993) 285–299); and Bier and Kleinschmidt (Discrete Math. 176 (1997) 29–42). In this paper, we present a new, simplified proof of the necessary and sufficient conditions for a magic rectangle to exist. We also show that magic rectangles, under the natural multiplication, have a unique factorization as a product of irreducible magic rectangles.

MSC: 05B15

Keywords: Magic squares; Magic rectangles

1. Introduction

Magic rectangles are a natural generalization of the magic squares which have long intrigued mathematicians and the general public. A magic (m,n)-rectangle R is an $m\times n$ array in which the first mn positive integers are placed so that the sum over each row of R is constant and the sum over each column of R is another (different if $m\neq n$) constant. They were first studied a century ago by Harmuth who proved in [5,6] that

Theorem 1. *For* $m,n>1$, *there is a magic* (m,n)*-rectangle* R *if and only if* $m\equiv n \bmod 2$ *and* $(m,n)\neq(2,2)$.

Recently, Sun [7], Bier and Rogers [2], and Bier and Kleinschmidt [3] have published modern proofs of Harmuth's result. In [4], the concept of magic rectangles was generalized to n-dimensions and several existence theorems were proven.

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PII: S0012-365X(99)00041-2

In this paper, we present an elementary proof of Theorem 1. We use the ideas of [3,4], to give a very simple construction of the even magic rectangles. We also give a new construction of the odd magic rectangles. Our method is entirely theoretical and does not depend on any experimental constructions of magic rectangles (see [3, Section 2.7]). This last point was important philosophically for the work in [4] as the experimental calculation of magic rectangles in higher dimensions is computationally unfeasible. Finally, in Section 5, we extend Adler's result [1] that magic squares have a unique prime factorization to the case of magic rectangles.

2. Properties of centrally symmetric rectangles

Let $R=(r_{ij})$ be a magic (m,n)-rectangle. Since the row sums of R are $n(mn+1)/2$ and the column sums of R are $m(mn+1)/2$ and both are integral, we have

Lemma 1. *If R is a magic (m,n)-rectangle, then $m \equiv n \bmod 2$.*

Lemma 1 allows the set of magic rectangles to be divided into the set of odd and even rectangles. Inspection quickly shows that an even magic $(2,2)$-rectangle does not exist. To show the existence of the other even magic rectangles, we introduce the closely related concept of centrally symmetric (m,n)-rectangles.

Definition 1. Let $x > -1$ and let R be an even $m \times n$ rectangular array whose entries are the numbers $\pm(x+1),\ldots,\pm(x+mn/2)$. R is a centrally symmetric (m,n)-rectangle of type x if the sum of all the rows and columns is zero. Additionally, if R has an equal number of positive and negative numbers in each row and column, we say that R is balanced.

If R is an even magic (m,n)-rectangle then by subtracting $(mn+1)/2$ from each entry of R, we obtain a centrally symmetric (m,n)-rectangle of type $-1/2$. Similarly, every centrally symmetric (m,n)-rectangle of type $-1/2$ determines an even magic (m,n)-rectangle. Thus, we can use the existence of centrally symmetric (m,n)-rectangles to prove results about magic (m,n)-rectangles. The following propositions are proved in [4].

Proposition 1. *If a balanced centrally symmetric (m,n)-rectangle exists, then a magic (m,n)-rectangle exists.*

Proposition 2. *If balanced centrally symmetric (m_i,n)-rectangles exist for $i=1,2$, then a balanced centrally symmetric (m_1+m_2,n)-rectangle exists.*

Proposition 3. *Suppose a magic* (m_1, n)*-rectangle and a balanced centrally symmetric* (m_2, n)*-rectangle exist. Then a magic* $(m_1 + m_2, n)$*-rectangle exists.*

3. Even magic rectangles

Using the concept of a centrally symmetric rectangle, we can quickly prove the existence of even magic rectangles. Our tools are the balanced centrally symmetric (2,4)-rectangle

$$A = \begin{pmatrix} 1 & -2 & -3 & 4 \\ -1 & 2 & 3 & -4 \end{pmatrix}$$

and the magic (2,6)-rectangle

$$B = \begin{pmatrix} 1 & 11 & 3 & 9 & 8 & 7 \\ 12 & 2 & 10 & 4 & 5 & 6 \end{pmatrix}.$$

Proposition 4. *Let* $n > 2$ *be an even integer. Then a magic* $(2, n)$*-rectangle exists.*

Proof. We induct on n. The existence of rectangles A and B shows that we need only prove the proposition for $n \geqslant 8$. Assume we know that a magic $(2, m)$-rectangle exists for all even $m < n$. Then we know a magic $(2, n-4)$-rectangle R exists. By Proposition 3, we can add R and A together to form a magic $(2, n)$-rectangle.

Proposition 5. *Let* m *and* n *be even positive integers with* $(m, n) \neq (2, 2)$*. Then a magic* (m, n)*-rectangle exists.*

Proof. By Proposition 4, we can assume that $n > 2$. Using A and Proposition 2, induction shows that a balanced centrally symmetric $(m, 4)$-rectangle R exists. Thus a magic $(m, 4)$-rectangle exists and we can assume that $n > 4$. Now assume that a magic (m, n')-rectangle exists for all even $n' < n$. Then a magic $(m, n-4)$-rectangle S exists. By Proposition 2, adding R and S together gives a magic (m, n)-rectangle.

4. Odd magic rectangles

In this section, we present a simplified proof that odd magic 2-rectangles exist. Our proof is inspired by the centrally symmetric rectangles of [3], but eliminates the need for the explicit calculation of examples. We first prove:

Proposition 6. *For $n>1$ an odd integer, there exists a magic $(3,n)$-rectangle R such that one row of R contains all the integers from $n+1$ to $2n-1$, with the exception of $(3n+1)/2$.*

Our construction of R will use smaller rectangles as building blocks. Define the 3×2 matrices

$$B_+(i)=\begin{pmatrix} i+1 & n+1-i \\ \dfrac{3n+1}{2}+i & \dfrac{3n+1}{2}-i \\ 3n-2i & 2n+2i \end{pmatrix},$$

$$B_-(i)=\begin{pmatrix} 3n-2i & 2n+2i \\ \dfrac{3n+1}{2}+i & \dfrac{3n+1}{2}-i \\ i+1 & n+1-i \end{pmatrix}.$$

The column sums of both matrices are $3(3n+1)/2$. The row sums of $B_+(i)$ are $n+2, 3n+1$, and $5n$ respectively and the row sums of $B_-(i)$ are $5n$, $3n+1$, and $n+2$. Hence when a 3×4 matrix is formed from the matrices $B_+(i)$ and $B_-(j)$, it has constant row and column sums.

Proof of Proposition 6. Since a magic $(3,3)$-square exists, we can assume that $n>3$. We first consider the case $n\equiv 1 \bmod 4$. Let

$$A=\begin{pmatrix} 1 & 2n & \dfrac{n+3}{2} \\ 3n & \dfrac{n+1}{2} & n+1 \\ \dfrac{3n+1}{2} & 2n+1 & 3n-1 \end{pmatrix}$$

and let R be the $3\times n$ rectangle obtained by glueing to A the $(n-5)/4$ matrices $B_+(i)$, for $i=1,\dots,(n-5)/4$ and the $(n-1)/4$ matrices $B_-(j)$, for $j=(n-1)/4,\dots,(n-3)/2$. By construction, the column sums of R are constant. The rows sums of R are also constant since glueing A to $B_-(i)$, and glueing $B_+(i)$ to $B_-(j)$ gives rectangles whose row sums are constant. A straightforward calculation shows that the entries of R are precisely the integers from 1 to $3n$. Hence R is a magic rectangle.

We now consider the case when $n\equiv 3 \bmod 4$. Let $m=[(n+1)/3]$. Then $n-3m=0$ or ± 1. We let

$$A=\begin{pmatrix} 1 & \dfrac{n+3}{2} \\ 3n & n+1 \\ \dfrac{3n+1}{2} & 3n-1 \end{pmatrix},$$

$$B = \begin{cases} \begin{pmatrix} m+1 & 2n+1 & 2n+2m \\ \dfrac{3n+1}{2}+m & \dfrac{n+1}{2} & \dfrac{3n+1}{2}-m \\ 3n-2m & 2n & n+1-m \end{pmatrix} & \text{if } n-3m=1, \\ \begin{pmatrix} m+1 & 2n & 2n+2m \\ \dfrac{3n+1}{2}+m & \dfrac{n+1}{2} & \dfrac{3n+1}{2}-m \\ 3n-2m & 2n+1 & n+1-m \end{pmatrix} & \text{if } n-3m=0, \\ \begin{pmatrix} 3n-2m & 2n+1 & n+1-m \\ \dfrac{3n+1}{2}+m & \dfrac{n+1}{2} & \dfrac{3n+1}{2}-m \\ m+1 & 2n & 2n+2m \end{pmatrix} & \text{if } n-3m=-1. \end{cases}$$

In all three cases, when we glue A and B with the matrix $B_-(1)$, we have a 3×7 matrix whose row sums are all equal. We now construct our magic $(3,n)$-rectangle R by adding the $(n-7)/4$ blocks $B_+(i)$, $2 \leqslant i \leqslant (n-3)/4$ and the $(n-7)/4$ blocks $B_-(i)$, $(n+1)/4 \leqslant i \leqslant (n-3)/2$, $i \neq m$ to A and B. We note that we omit the block $B_-(m)$ since its entries already appear in A and that we know $m = [(n+1)/3] \geqslant (n+1)/4$. The respective row and column sums of our rectangle are constant. Again, it is straightforward to see that the entries in R are the integers from 1 to $3n$. □

Proposition 7. *For odd numbers $m,n > 1$, a magic (m,n)-rectangle exists.*

Proof. We can assume that $m \leqslant n$. Then the existence of a magic (m,n)-rectangle follows from the more general proposition:

Proposition 8. *Suppose $n \geqslant m > 1$ are odd integers and $d = \frac{1}{2}(m-1)$. Then there exists a magic (m,n)-rectangle R that contains the $n-2d$ integers $d(n+1) \leqslant x \leqslant d(n-1)+n, x \neq dn+(n+1)/2$, in a single row.*

To construct R, we introduce the following matrices. Let

$$C_+(i) = \begin{pmatrix} i & n+1-i \\ mn+1-i & mn-n+i \end{pmatrix},$$

$$C_-(i) = \begin{pmatrix} mn+1-i & mn-n+i \\ i & n+1-i \end{pmatrix},$$

and

$$D_n = \begin{cases} \begin{pmatrix} (n+1)/2 \\ mn+(1-n)/2 \end{pmatrix} & \text{if } n \equiv 1 \bmod 4, \\ \begin{pmatrix} mn+(1-n)/2 \\ (n+1)/2 \end{pmatrix} & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Proof of Proposition 8. By the proof of Proposition 6, we can assume that $m \geqslant 5$. We will prove the general case by induction on m. Assume that the proposition is true for $m-2$ and let S be the desired magic $(m-2, n)$-rectangle. By adding n to each entry of S, we obtain a rectangle R. The entries of R are the integers $[n+1, mn-n]$ and one row of R, row i, contains the integers $x \neq dn+(n+1)/2$ from $dn+d-1$ to $dn+n+1-d$. We now obtain a magic (m, n)-rectangle by adding two rows containing the integers in $[1, n]$ and $[mn+1-n, mn]$ to R. By interchanging the columns of R, we can assume that the integers $dn+d-1$ and $dn+n+1-d$ in row i appear in the first and second columns of R. We now let

$$A=\begin{pmatrix} d-1 & d & n+2-d & n+1-d \\ mn+2-d & mn+1-d & mn-n+d-1 & mn-n+d \end{pmatrix}.$$

Now, there are $(n-5)/2$ positive integers less than $n/2$ which are not equal to $d-1$ or d. If $n \equiv 1 \bmod 4$, divide these numbers into a collection of $(n-1)/4$ integers a_i and a group of $(n-9)/4$ integers b_i. If $n \equiv 3 \bmod 4$, divide these integers into a collection of $(n-3)/4$ integers a_i and a group of $(n-7)/4$ numbers b_i. Let R_1 be the rectangle obtained by appending to the bottom of R the two rows formed by glueing in order the rectangles A, the $C_-(a_i)$, the $C_+(b_i)$, and D_n. The column sums of R_1 are all equal. However, the sums of the last two rows of R_1 are not equal. The sum of row $m-1$ is easily checked to be $mn-n$ less than the sum of the last row. But we can correct this problem by exchanging the positions of $d-1$ and $dn+d-1$ in the first column of R_1 and the positions of $mn+1-d$ and $dn+n+1-d$ in the second column of R_1. This does not change the column sums. And the sum of row i of R is unchanged as $dn+d-1$ and $dn+n+1-d$ lie in the same row i of R and their sum $2dn+n=mn$ equals $(d-1)+(mn+1-d)$. Now, the row sums of the last two rows of R_1 both equal the desired $n(mn+1)/2$. Hence R_1 is a magic (m, n)-rectangle and the integers from $dn+d$ to $dn+n-d$, with the exception of $dn+(n+1)/2$, lie in row i. So the induction hypothesis is satisfied and the proposition has been proven. □

5. Unique prime factorization

Proposition 3 shows that in certain cases one can add two magic rectangles together to form a third magic rectangle. We now recall a classical method for multiplying magic rectangles. If $A=(a_{ij})$ is a magic (m_1, n_1)-rectangle and $B=(b_{ij})$ is a magic (m_2, n_2)-rectangle, then we define the magic (m_1m_2, n_1n_2)-rectangle $C=A*B$ by letting $C=(c_{ij})$, where

$$c_{ij}=a_{i_0j_0}+m_1n_1(b_{i_1j_1}-1),$$

where $i=m_1i_1+i_0$, with $0 \leqslant i_0 < m_1$ and $j=n_1j_1+j_0$, with $0 \leqslant j_0 < n_1$. C can be shown to be a magic rectangle [2]. For this multiplication, the magic $(1,1)$-rectangle e is the identity element. The multiplication's definition immediately shows that we have left and right cancellation laws:

Proposition 9. *Suppose A, B, and C are magic rectangles. If $A*B=A*C$, then $B=C$. If $A*C=B*C$, then $A=B$.*

We call A an irreducible magic rectangle if A cannot be expressed as a product $B*C$ of two magic rectangles B, $C \neq e$. Since the size of the factors of A must be smaller than A, we can factor every magic rectangle as a product of irreducible magic rectangles. In [1], Adler showed that for magic squares this decomposition is unique. We now show the analogous proposition for magic rectangles:

Proposition 10. *A magic rectangle has a unique decomposition as a product of irreducible magic rectangles.*

Proof. We will derive a contradiction by supposing that the magic rectangle A can be written in two different ways as the product of irreducible magic rectangles. Let $\prod_{i=1}^{n} R_i$ and $\prod_{i=1}^{m} S_i$ be the two different factorizations. Using the left cancellation law, we can assume that $R_1 \neq S_1$. However, this contradicts Proposition 11 and the proposition is proven.

Proposition 11. *Let A, B, C, and D be magic rectangles with $A*B=C*D$. Suppose that A and C are irreducible. Then $A=C$.*

Proof. Let $A=(a_{ij})$ be a magic (m_1,n_1)-rectangle and $C=(c_{ij})$ a magic (m_2,n_2)-rectangle. We can assume that $m_1n_1 \leqslant m_2n_2$. Suppose that the integer 1 appears in B at the (b_1,b_2)-th place. If $A*B=(\alpha_{ij})$, then $a_{ij}=\alpha_{m_1b_1+i,m_2b_2+j}$ and thus a copy A_0 of A occurs in $A*B$. Similarly there is a copy C_0 of C inside of $C*D$. Since these are equal, and A and C consist of the first m_1n_1 (resp. m_2n_2) positive integers, we must have that $A_0 \subset C_0$. If $m_1n_1=m_2n_2$, then $A_0 \subset C_0$ implies that $m_1=m_2$ and $n_1=n_2$. And since the entries agree, we see that $A_0=C_0$ and $A=C$.

Hence we can assume that $m_1n_1<m_2n_2$. Let A_k be the translate of A in $A*B$ obtained by adding km_1n_1 to all the entries of A. Now a similar argument as above shows that C_0 is covered by the translates $A_0,\ldots,A_t$ of A, where $A_t \cap C_0 \neq \emptyset$. Since the entries of C_0 and each translate A_k are consecutive positive integers, the first t translates of A must be contained in C_0. The last translate A_t may or may not be contained in C_0. If $A_t \subset C_0$, then the rectangle C_0 is tiled by the $t+1$ translates of A. By the geometry of $A*B$, these tiles are obtained by horizontal and vertical translations of A_0. Thus, m_2 (resp. n_2) must be a multiple of m_1 (resp. n_1). We then have $C=A*E$, where $E=(e_{ij})$ is defined by $c_{m_1i+i_0,n_1j+j_0}=a_{i_0,j_0}+m_1n_1(e_{ij}-1)$. Since C and A are magic rectangles, it is straightforward to show that E is a magic rectangle. But this contradicts the irreducibility of C.

So we can assume that $A_t \not\subset C_0$. But the only way that A_t can be the only translate of A not contained in C_0 is if either $m_1=m_2$ or $n_1=n_2$. Without loss of generality, we can assume $n_1=n_2$. Let $r>0$ be the remainder of the quotient m_2/m_1. Then the translate A_t will have r of its rows contained in C_0 and m_1-r of its rows not contained in C_0. Since C_0 consists of the first m_2n_2 positive integers, all the entries of the r rows of A_t contained in C_0 will be less than the entries of the rows of A_t not in C_0. Hence the sums of the rows of A_t will not be the same. But since A_t is a translate of the magic rectangle A, its row sums must be equal. We have a contradiction. Hence our assumption that $m_1n_1 < m_2n_2$ is incorrect and $A = C$. The proposition is proved.

Acknowledgements

The author would like to thank Dan Shapiro for sparking the author's interest in this subject, and to Allan Adler and the referees for their suggestions.

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