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An index theorem for Wiener–Hopf operators

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Abstract

We study the multivariate generalisation of the classical Wiener–Hopf algebra, which is the C*-algebra generated by the Wiener–Hopf operators, given by convolutions restricted to convex cones. By the work of Muhly and Renault, this C*-algebra is known to be isomorphic to the reduced C*-algebra of a certain restricted action groupoid. It admits a composition series, and therefore, a 'symbol' calculus. Using groupoid methods, we obtain, in the framework of Kasparov's bivariant KK-theory, a topological expression of the index maps associated to these symbol maps in terms of geometric-topological data of the underlying convex cone. This generalises an index theorem by Upmeier concerning Wiener–Hopf operators on symmetric cones. Our result covers a wide class of cones containing polyhedral and homogeneous cones.

MSC: 47B35; 19K56

Keywords: Wiener–Hopf operator; Groupoid C*-algebra; Topological index; KK theory

1. Introduction

Let Ω be a closed, pointed, and solid convex cone in the n-dimensional real inner product space X. Wiener–Hopf operators are the bounded operators W_f on $\mathbf{L}^2(\Omega)$, defined by

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$$W_f\xi(x) = \int_\Omega f(x-y)\xi(y)\,dy \quad \text{for all } f \in \mathbf{L}^1(X),\ \xi \in \mathbf{L}^2(\Omega),\ x \in \Omega.$$

Classically, one considers $X = \mathbb{R}$ and $\Omega = \mathbb{R}_{\geqslant 0}$. Then it is a well-known fact that $1 + W_f$ is a Fredholm operator if and only if $1 + \hat{f}$ is everywhere non-vanishing, where $\hat{f}$ denotes Fourier transform. Then $1 + \hat{f}$ may be considered as a non-vanishing function on $\mathbb{S}^1$, and its winding number is the index of $1 + W_f$ (possibly up to a sign, depending on the normalisation of the Fourier transform).

These two facts may be phrased in a modern language, as follows. Let A be the C*-algebra, generated by W_f, $f \in \mathbf{L}^1(\mathbb{R})$. Then the map $W_f \mapsto \hat{f}$ extends to a morphism of C*-algebras, and we have a short exact sequence

$$0 \longrightarrow \mathbb{K}(\mathbf{L}^2(0,\infty)) \longrightarrow A \longrightarrow \mathcal{C}_0(\mathbb{R}) \longrightarrow 0,$$

$\mathbb{K}$ denoting the set of all compact operators and $\mathcal{C}_0$ that of the continuous functions vanishing at infinity. Moreover, the index of operators with given (matrix) symbols corresponds to the connecting map in operator K-theory, $K_c^1(\mathbb{R}) = K_1(\mathcal{C}_0(\mathbb{R})) \to K_0(\mathbb{K}) = \mathbb{Z}$, induced by this short exact sequence. Computing the index of Fredholm Wiener–Hopf operators (in fact, Fredholm matrices in the unitisation of A) amounts to the computation of this map.

The generalisation of these results to multivariate situations, where X has dimension $n \geqslant 2$ and Ω is a closed convex cone, has been a long-standing problem, and various authors have contributed to it over time. We shall only give the very briefest overview at this point and refer to the introduction of [2] and the literature cited there for a more in-depth account of these historical matters.

In the multivariate case, the above short exact sequence is replaced by a composition series, i.e. a filtration by closed ideals whose subquotients are continuous trace algebras, and the corresponding index maps are the d^1 differentials of the Atiyah–Hirzebruch homology spectral sequence in K-theory induced by this filtration. A composition series was obtained in the opposite case of polyhedral and symmetric (i.e. homogeneous and self-dual) cones by Muhly and Renault [22]; at this point, no index theorems were established. Upmeier [34] gave an alternative description of the composition series in the symmetric case and computed the index maps. (In fact, he considers the C*-algebras of Toeplitz operators on bounded symmetric domains and develops the corresponding theory for them. If the bounded symmetric domain has a Siegel realisation as a tube domain, then the Wiener–Hopf algebra A occurs as an ideal in the Toeplitz algebra, as in the one-variable case.)

This makes the class of symmetric cones the only class of cones for which a complete index theory of multivariate Wiener–Hopf operators has been developed. One obstacle to the generalisation of the finer K-theoretical results beyond symmetric cones seemed to be that Upmeier's work relies on the use of the theory of Jordan algebras, which is only available in this case. Due to the refinement of the theory of groupoid C*-algebras and operator K theory over the last 20 years, we have been able to follow the groupoid approach of Muhly and Renault, thereby obtaining an independent proof of Upmeier's results and simultaneously extending them to a larger class of cones which, in particular, allows the treatment of polyhedral and homogeneous (not necessarily symmetric) cones by the same methods.

Let us explain this approach. The C*-algebra generated by the Wiener–Hopf operators W_f, $f \in \mathbf{L}^1(X)$, is isomorphic to the reduced groupoid C*-algebra $C^*_r(\mathcal{W}_\Omega)$ of the 'Wiener–Hopf groupoid' $\mathcal{W}_\Omega$.

The groupoid $\mathcal{W}_\Omega$ is a certain restriction of an action groupoid, constructed as follows. The vector space X acts by translation on its space of closed subsets $\mathbb{F}(X)$, which is a compact metrisable space when endowed with the Fell topology. The set $-\Omega$ is a point of $\mathbb{F}(X)$, and the closure $\overline{X}$ of the orbit through this point under the action of X is compact. The orbit closure for the action of the subsemigroup $\Omega \subset X$ is denoted by $\overline{\Omega}$. Now, if $\overline{X} \rtimes X$ denotes the usual action groupoid given by the action of the group X on the space $\overline{X}$, the Wiener–Hopf groupoid is defined as the restriction $\mathcal{W}_\Omega = (\overline{X} \rtimes X)|\overline{\Omega}$ to the non-invariant subset $\overline{\Omega}$. We refer to [14,2] for the details of the construction.

The groupoid $\mathcal{W}_\Omega$ describes a lamination of the space $\overline{\Omega}$, and in [2], we have constructed a composition series of the algebra $\mathrm{C}^*_r(\mathcal{W}_\Omega)$ associated to the orbit type stratification for the groupoid's action, which we proceed to describe.

Recall that a subset F of a convex set C is a *face* if any segment in C whose midpoint belongs to F lies completely in F. Order the dimensions of the faces of the dual cone Ω^* increasingly,

$$\{0 = n_0 < n_1 < \cdots < n_d = n\} = \{\dim F \mid F \subset \Omega^* \text{ face}\}.$$

Let P_j be the set of n_{d-j}-dimensional faces of Ω^*. Any element of P_j, considered as a point of $\overline{\Omega}$, has the same orbit type. Here, recall that an *orbit type* of a groupoid $\mathcal{G}$ with structure maps r and s is the conjugacy class, under the action of $\mathcal{G}$, of an isotropy group $\mathcal{G}(x) = r^{-1}(x) \cap s^{-1}(x)$, $x \in \mathcal{G}^{(0)}$, of $\mathcal{G}$. The isotropies of the groupoid $\mathcal{W}_\Omega$ are linear subspaces of X and their conjugacy is classified by dimension. The orbit type of every point of P_j then turns out to be that of an $(n - n_{d-j})$-dimensional linear subspace of X.

Now, let $Y_j \subset \overline{\Omega}$ be the 'orbit type stratum' of P_j, i.e. the set of all points of $\overline{\Omega}$ with the same orbit type as the points of P_j. Moreover, *assume that the sets P_j are compact* for all j, in the space of all closed subsets of X, endowed with the Fell topology. Then the orbit type strata Y_j cover $\overline{\Omega}$, and naturally form fibre bundles over the spaces P_j.

We call cones whose sets of faces of fixed dimension are compact *facially compact*. This condition obtains for Ω^* if Ω is, e.g., polyhedral or symmetric (i.e. homogeneous and self-dual). In these special cases, the sets P_j are, respectively, finite sets and certain compact homogeneous spaces including, in particular, all spheres.

The strata Y_j are closed and invariant, and the unions $U_j = \bigcup_{i=0}^{j-1} Y_i$ are open and invariant. Thus, we obtain ideals $I_j = \mathrm{C}^*_r(\mathcal{W}_\Omega|U_j)$ of the Wiener–Hopf C^*-algebra $\mathrm{C}^*_r(\mathcal{W}_\Omega)$, and extensions

$$0 \twoheadrightarrow \mathrm{C}^*_r(\mathcal{W}_\Omega|Y_j) \twoheadrightarrow I_{j+1}/I_{j-1} = \mathrm{C}^*_r\big(\mathcal{W}_\Omega\big|(U_{j+1} \setminus U_{j-1})\big) \twoheadrightarrow \mathrm{C}^*_r(\mathcal{W}_\Omega|Y_j) \twoheadrightarrow 0.$$

Moreover, the fibre bundle structure of the strata induces Morita equivalences $\mathcal{W}_\Omega|Y_j \sim \Sigma_j$ where Σ_j is the 'co-tautological' vector bundle over the space P_j whose fibre at the face F is the orthogonal complement $F^\perp$. (Alternatively, Σ_j is the restriction of the isotropy group bundle of $\mathcal{W}_\Omega$ to the section P_j of the bundle $Y_j \to P_j$.) Thus, $\mathrm{C}^*_r(\mathcal{W}_\Omega)$ is solvable of length d, and its spectrum can be computed in terms of a suitable gluing of the bundles Σ_j. As a particular case, one obtains the classical Wiener–Hopf extension (associated to $X = \mathbb{R}$ and $\Omega = \mathbb{R}_{\geqslant 0}$).

The above extensions induce index maps $K^1_c(\Sigma_j) \to K^0_c(\Sigma_{j-1})$, given as the Kasparov product with the KK class

$$\partial_j \in KK^1\big(\mathrm{C}^*_r(\mathcal{W}_\Omega|Y_j), \mathrm{C}^*_r(\mathcal{W}_\Omega|Y_{j-1})\big)$$

representing the above extension. As already noted, these maps may also be viewed as the d^1 differentials of the Atiyah–Hirzebruch type homology spectral sequence induced by the filtration. In [2], we expressed the index map ∂_j as the family index of certain continuous Fredholm families of operators on a continuous field of Hilbert spaces over Σ_{j-1}.

In this paper we prove a formula for ∂_j which expresses the latter through topological data. This formula generalises Upmeier's [34] result for symmetric cones to a larger class of convex cones. Let us stress that our methods are different from his, and originate in the non-commutative geometry approach to foliation C*-algebras, benefiting from the recent progress in this field. Our result applies both to cones with a large number of automorphisms (such as homogeneous cones which need not be symmetric), and those with only few (such as polyhedral cones).

We proceed to describe our index formula. To stress the analogy, we deliberately adopt Upmeier's notation, despite the differences in our assumptions and methods. Assume that the cone Ω has a facially compact and locally smooth dual cone (compare Section 6 for the definition of 'locally smooth'). Consider the compact space $\mathcal{P}_j$ consisting of all pairs $(E, F) \in P_{j-1} \times P_j$ such that $E \supset F$. It has projections

$$P_{j-1} \xleftarrow{\ \xi\ } \mathcal{P}_j \xrightarrow{\ \eta\ } P_j$$

which need not be surjective unless $j = 1, d$ (although they are in the polyhedral and symmetric cases). The projection $\xi : \mathcal{P}_j \to P_{j-1}$ turns $\mathcal{P}_j$ into a fibrewise $\mathcal{C}^1$ manifold over the compact base $\xi(\mathcal{P}_j)$. Moreover, $\eta^*\Sigma_j$ is the trivial line bundle over $T\mathcal{P}_j \oplus \xi^*\Sigma_{j-1}$ if $T\mathcal{P}_j$ denotes the fibrewise tangent bundle. Then we have the following theorem.

Theorem 1. *The KK-theory element ∂_j representing the jth Wiener–Hopf extension is given by*

$$\boldsymbol{\zeta}_*\partial_j = \boldsymbol{\eta}^*[y \otimes \tau_j]$$

where $y \in KK^1(\mathbb{C}, S)$ represents the classical Wiener–Hopf extension, $\boldsymbol{\eta}$ is associated to the projection $\eta^*\Sigma_j \to \Sigma_j$, and $\boldsymbol{\zeta}$ is associated to the inclusion $\Sigma_{j-1}|\xi(\mathcal{P}_j) \subset \Sigma_{j-1}$. Here,

$$\tau_j \in KK\big(\mathrm{C}^*_r(T\mathcal{P}_j \oplus \xi^*\Sigma_{j-1}), \mathrm{C}^*_r\big(\Sigma_{j-1}|\xi(\mathcal{P}_j)\big)\big)$$

represents the Atiyah–Singer family index for $T\mathcal{P}_j \oplus \xi^*\Sigma_{j-1}$, considered as a vector bundle over $\Sigma_{j-1}|\xi(\mathcal{P}_j)$.

To illustrate, we first consider the special case $j = d$. Here, η is constant ($P_d = \{0\}$, $\Sigma_d = X$), ξ is the identity, and in particular, surjective. The fibres of ξ are points, so $T\mathcal{P}_{d-1} = 0$. The vector space X is turned into the trivial real line bundle over Σ_{d-1} by letting the fibre at $(E, u) \in \Sigma_{d-1}$ be the line spanned by the extreme ray E of Ω^*. We have that τ_{d-1} is the identity, so our index formula in this case is just $\partial_d = \boldsymbol{\eta}^* y$, which recovers the case of the classical Wiener–Hopf extension for $\Omega = \mathbb{R}_{\geqslant 0}$.

A more interesting special case is $j = 1$. Here, $P_0 = \{\Omega^*\}$ is the point, and Σ_0 the zero bundle over the point. So, ξ is constant, and η is the identity. The set $\mathcal{P}_1 = P_1$ consists of all maximal-dimensional proper faces of Ω^*. Their dual faces $\check{F}$ are exposed extreme rays of Ω. The tangent bundle TP_1 has at the face F the fibre $F^\perp \cap \check{F}^\perp$. It is important to note that for non-polyhedral cones, this space is usually non-zero, the simplest case being that of the three-dimensional Lorentz cone, where TP_1 is the tangent bundle of $\mathbb{S}^1$. In any case, for $j = 1$,

$\tau_1 \in KK(TP_1, \mathbb{C}) = K_0(TP_1)$ induces on K-theory the Atiyah–Singer index $K_c^0(TP_1) \to \mathbb{Z}$ for TP_1, and we have $\partial_1 = \boldsymbol{\eta}^*(y \otimes \tau_1)$.

In a forthcoming paper, the first-named author shows that for a polyhedral cone Ω, the complex formed by the index maps ∂_j and the cohomology groups of the vector bundles Σ_j is precisely the augmented cellular complex associated to the CW complex given by a polygonal section of Ω. This suggests an interesting connection between the combinatorics of polyhedra and the K-theory of subquotients of the Wiener–Hopf algebra.

Let us sketch our strategy of proof. We first observe that KK^1 elements representing extensions induced by restrictions of groupoids to open invariant subsets behave naturally in the category KK under proper groupoid homomorphisms, even if these do not induce $*$-morphisms of the groupoid C*-algebras.

We then construct an appropriate proper homomorphism φ to the restricted Wiener–Hopf groupoid $\mathcal{W}_\Omega|(U_{j+1} \setminus U_j)$ which defines the extensions giving rise to the index maps ∂_j. The domain of φ is the fibred product of the classical Wiener–Hopf groupoid $\mathcal{W}_{\mathbb{R}_{\geqslant 0}}$ and the tangent groupoid of a fibrewise $\mathcal{C}^1$ groupoid $\mathcal{D}_j$. Thus, the idea is that the extensions ∂_j are classical Wiener–Hopf extensions with a (possibly highly non-trivial) 'twist'.

Using naturality, we pull back ∂_j along φ. As is to be expected, the result is a Kasparov product $y \otimes \tau$ where y represents the classical Wiener–Hopf extension and $\tau = \tau_j$ is the 'Connes–Skandalis map' associated to the tangent groupoid. It remains to express $y \otimes \tau$ by topological means, but this can be done essentially by standard procedures.

To summarise, the construction of the groupoid $\mathcal{D}_j$ and the homomorphism φ constitutes the main step of the proof; the remainder of our paper consists in building up a little machinery.

Following the basic philosophy that an index formula should be the consequence of a naturality of transformations, applied to a particular homomorphism, we have organised our material as follows. Sections 2 to 5 consist in collecting tools which are applied only in Section 6 to prove our main theorem. Here, Sections 2 and 4 contain to our knowledge new constructions, whereas Sections 3 and 5 contain extensions of known techniques.

More precisely, in Section 2, we treat the naturality of extensions by expressing the mapping cone construction for groupoid C*-algebras by a groupoid construction. Section 3 concerns fibrewise differentiable groupoids. After recalling basic definitions, we study the tangent groupoid and introduce the (fibrewise) Connes–Skandalis map τ. In Section 4, we construct the 'cone' $\mathbb{W}\mathcal{G}$ over the tangent groupoid of a given fibrewise $\mathcal{C}^1$ groupoid $\mathcal{G}$, and use it to compute $y \otimes \tau$ as the extension of a groupoid C*-algebra; here, the naturality of extensions (from Section 2) also enters. We effect the computation of τ in topological terms in Section 5 by adapting Connes's familiar construction [8] of the classifying space for the tangent groupoid of a manifold. In Section 6, we finally consider the Wiener–Hopf groupoid. We construct the fibrewise $\mathcal{C}^1$ groupoid $\mathcal{D}_j$, and a proper homomorphism $\mathbb{W}\mathcal{D}_j \to \mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})$. An application of the methods previously established proves the index formula.

2. Groupoid extensions and naturality

2.1. Preliminaries

We will consider groupoids $\mathcal{G}$, usually locally compact (and Hausdorff), and often endowed with a (continuous left) Haar system $(\lambda^x)_{x \in \mathcal{G}^{(0)}}$. In the latter case, we will consider the full and reduced groupoid C*-algebras $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$, where we suppress the Haar system from the notation. We will use these concepts freely, and refer the reader to [27] for details.

To fix our notation and terminology, we collect some well-known facts on generalised morphisms and related matters. In what follows, let $\mathcal{G}, \mathcal{H}$ be locally compact (Hausdorff) groupoids whose source and range maps are open.

A *generalised morphism* from $\mathcal{G}$ to $\mathcal{H}$ is a locally compact space Z, together with maps $\mathcal{G}^{(0)} \xleftarrow{r} Z \xrightarrow{s} \mathcal{H}^{(0)}$ such that $\mathcal{G}$ acts from the left on Z relative r, $\mathcal{H}$ acts from the right on Z relative s, the actions commute, and $r : Z \to \mathcal{G}^{(0)}$ is a principal $\mathcal{H}$ fibration (which is to say that $\mathcal{H}$ acts properly and freely on Z, transitively on the fibres of r, and r is open and surjective). To fix terminology, a continuous homomorphism (i.e., a functor) $\mathcal{G} \to \mathcal{H}$ will be called a *strict morphism*. For these definitions, compare [33,31,20,15,9].

Strict morphisms $f : \mathcal{G} \to \mathcal{H}$ induce generalised morphisms Z_f, defined by $Z_f = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}$, with respect to $f : \mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$. The action of $\mathcal{H}$ is the obvious one, and the action of $\mathcal{G}$ is given by

$$\gamma.(x, \eta) = \bigl(r(\gamma), f(\gamma)\eta\bigr) \quad \text{for all } \gamma \in \mathcal{G}^x,\ \eta \in \mathcal{H}_{f(x)}.$$

Composition of generalised morphisms $Z : \mathcal{G} \to \mathcal{H}$, $W : \mathcal{H} \to \mathcal{I}$ is given by

$$W \circ Z = Z \times_{\mathcal{H}} W = (Z \times_{\mathcal{H}^{(0)}} W)/\mathcal{H},$$

where the action of $\mathcal{H}$ is diagonal:

$$(z, w).\tau = \bigl(z\tau, \tau^{-1}w\bigr) \quad \text{whenever } s(z) = r(\tau) = r(w).$$

This composition is compatible with the composition of strict morphisms, up to canonical isomorphism. Locally compact groupoids with equivalence classes of generalised morphisms form a category (without passing to equivalence classes, one obtains a 2-category); its isomorphisms are called *Morita equivalences*. In the sequel, we shall be somewhat lax in our use of terminology, and will not distinguish between generalised morphisms and their equivalence classes.

Following [32], we shall say that a generalised morphism Z is *locally proper* if the action of $\mathcal{G}$ is proper, and *proper* if in addition, all inverse images of compacts under $s : Z \to \mathcal{H}^{(0)}$ are $\mathcal{G}$-compacts. Equivalently, the induced map $\mathcal{G}\backslash Z \to \mathcal{H}^{(0)}$ is proper. The composition of proper generalised morphisms is proper, and any Morita equivalence is proper.

Generalising the construction of an equivalence bimodule given by Muhly–Renault–Williams [23], Tu [32, Theorem 7.8, Remark 7.17] has associated to any proper generalised morphism $Z : \mathcal{G} \to \mathcal{H}$ of locally compact groupoids with Haar systems a trivially graded right $\mathrm{C}^*_r(\mathcal{H})$-Hilbert module, on which $\mathrm{C}^*_r(\mathcal{G})$ acts by compact endomorphisms. In particular, this defines a Kasparov cycle $KK(Z) \in KK(\mathrm{C}^*_r(\mathcal{H}), \mathrm{C}^*_r(\mathcal{G}))$.

This correspondence is cofunctorial in the following sense. Let KK be the category whose objects are separable C^*-algebras, and whose Hom functor is $KK(-,-)$, with composition given by the Kasparov product. The map which associates to each locally compact groupoid with Haar system its reduced groupoid C^*-algebra, and to each proper generalised morphism Z as above the cycle $KK(Z)$, is a cofunctor. In particular, Morita equivalences give rise to KK equivalences.

2.2. The mapping cone groupoid

Let $\mathcal{G}$ be a locally compact σ-compact (Hausdorff) groupoid with Haar system (λ^u) and $U \subset \mathcal{G}^{(0)}$ an open invariant subset. Set $F = \mathcal{G}^{(0)} \setminus U$. As is well known [27, Proposition 4.5], [15, 2.4], [26, Proposition 2.4.2], there is a short exact sequence

$$0 \longrightarrow \mathrm{C}^*(\mathcal{G}|U) \xrightarrow{j} \mathrm{C}^*(\mathcal{G}) \xrightarrow{q} \mathrm{C}^*(\mathcal{G}|F) \longrightarrow 0 \tag{1}$$

where j is given by extension of compactly supported functions by zero, and q is the integrated version of the proper homomorphism given by the inclusion $\mathcal{G}|F \subset \mathcal{G}$. In particular, the KK theory class of q is the KK theory class induced, via the functor $KK(-)$ introduced above, by this inclusion.

Moreover, if the groupoid $\mathcal{G}|F$ is topologically amenable, the corresponding sequence of reduced groupoid C^*-algebras

$$0 \longrightarrow \mathrm{C}_r^*(\mathcal{G}|U) \xrightarrow{j} \mathrm{C}_r^*(\mathcal{G}) \xrightarrow{q} \mathrm{C}_r^*(\mathcal{G}|F) \longrightarrow 0$$

is also exact.

As does any extension of C^*-algebras, these sequences induce certain KK^1 elements, commonly constructed by considering mapping cones. To better understand these, we will in this section describe the mapping cones for the above sequences as the full or reduced C^*-algebra of a groupoid.

For brevity, we denote $I(U) = \mathrm{C}^*(\mathcal{G}|U)$, $I_r(U) = \mathrm{C}_r^*(\mathcal{G}|U)$. Within the corresponding groupoid C^*-algebras of $\mathcal{G}$, these are given as the closures of the image of $\mathcal{C}_c(U)$. We briefly recall that $I(-)$ and, under suitable conditions, $I_r(-)$, respect intersections and unions. This is probably well known; for the lack of a reference and to the reader's convenience, we give a proof.

Proposition 2. *Let $U_1, U_2 \subset \mathcal{G}^{(0)}$ be open and invariant. Then*

$$I(U_1) + I(U_2) = I(U_1 \cup U_2) \quad \textit{and} \quad I(U_1) \cap I(U_2) = I(U_1 \cap U_2).$$

Moreover, $I_r(U_1) + I_r(U_2) = I_r(U_1 \cup U_2)$, and if $\mathcal{G}|(U_1 \setminus U_2)$ is topologically amenable, then $I_r(U_1) \cap I_r(U_2) = I_r(U_1 \cap U_2)$.

Proof. Since $\mathcal{G}^{(0)}$ is normal, by Urysohn's lemma, there exists a partition of unity subordinate to the cover (U_1, U_2). Thus, $\mathcal{C}_c(U_1) + \mathcal{C}_c(U_2) = \mathcal{C}_c(U_1 \cup U_2)$, and there are canonical isomorphisms

$$\begin{aligned} I(U_1)/I(U_1) \cap I(U_2) &\cong \big(I(U_1) + I(U_2)\big)/I(U_2) \\ &= I(U_1 \cup U_2)/I(U_2) \cong \mathrm{C}^*\big(\mathcal{G}|(U_1 \setminus U_2)\big). \end{aligned}$$

Thus, we have a commutative diagram with exact lines,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(U_1)\cap I(U_2) & \longrightarrow & I(U_1) & \longrightarrow & I(U_1)/(I(U_1)\cap I(U_2)) & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
0 & \longrightarrow & I(U_1\cap U_2) & \longrightarrow & I(U_1) & \longrightarrow & \mathrm{C}^*(\mathcal{G}|U_1\setminus U_2) & \longrightarrow & 0.
\end{array}$$

Since the vertical arrows are isomorphisms, the kernels of the rightmost non-zero horizontal arrows coincide, so $I(U_1)\cap I(U_2) = I(U_1\cap U_2)$. If the groupoid $\mathcal{G}|(U_1\setminus U_2)$ is topologically amenable, the same argument applies on the level of reduced groupoid C*-algebras. □

We wish to express the KK^1 element associated to an extension of groupoid C*-algebras in groupoid terms. To that end, we recall the usual construction of the first Puppe sequence in KK theory. Given a $*$-morphism $q : A \to A''$ of separable C*-algebras, the mapping cone C_q is defined as the pullback of

$$A \xrightarrow{\;q\;} A'' \xleftarrow{\;e_0\;} CA''$$

where $CA'' = \mathcal{C}_0([0,1[) \otimes A''$ and $e_0(f) = f(0)$. The diagram

$$A \xleftarrow{\;0\;} SA'' \xrightarrow{\;\subset\;} CA''$$

induces a map $SA'' \to C_q$, where we write $SA = \mathcal{C}_0(]0,1[) \otimes A$ for the suspension; and the sequence

$$SA'' \longrightarrow C_q \longrightarrow A \xrightarrow{\;q\;} A''$$

is called a *mapping cone triangle*. By applying $KK(B,-)$, it gives rise to a long exact sequence of Abelian groups,

$$\cdots \dashrightarrow KK(B,SA) \xrightarrow{KK(B,Sq)} KK(B,SA'') \longrightarrow$$

$$\longrightarrow KK(B,C_q) \longrightarrow KK(B,A) \xrightarrow{KK(B,q)} KK(B,A'') \dashrightarrow \cdots$$

called the first Puppe sequence [6, Theorem 19.4.3].

If q is the quotient map of a semi-split extension (e.g. if $A'' = A/A'$ is nuclear), then C_q and $A' = \ker q$ are KK-equivalent [6, Theorem 19.5.5] via

$$\text{the map } A' \to C_q \text{ induced by } \; A \xleftarrow{\;q\;} A' \xrightarrow{\;0\;} CA''.$$

Thus, up to a KK-equivalence, the connecting map $\partial : KK(B,SA'') \to KK(B,A')$ is given by application of the functor $KK(B,-)$ to the natural map $SA'' \to C_q$, cf. [6, Theorem 19.5.7].

(In fact, more precisely, to its KK-theory class in $KK(SA'', C_q)$, but this amounts to the same [6, Proposition 18.7.1].) It is also given by the Kasparov product with the element representing the extension.

From this discussion, it appears to be desirable to give a description of C_q as a groupoid C^*-algebra in the case of the extension (1). To that end, consider the embedding $(0, \mathrm{id}) : \mathcal{G}|F \to [0,1[\times \mathcal{G}|F$ and form $\mathcal{C}_F = \mathcal{G} \cup_{\mathcal{G}|F} ([0,1[\times \mathcal{G}|F)$. Since $\mathcal{G}$ is the complement in $\mathcal{C}_F$ of the open subset $]0,1[\times \mathcal{G}|F$, and $[0,1[\times \mathcal{G}|F$ is the complement of the open subset $\mathcal{G}|U$, $\mathcal{G}$ and $[0,1[\times \mathcal{G}|F$ are closed in $\mathcal{C}_F$.

Let $\mathcal{C}_F^{(0)} = \mathcal{G}^{(0)} \cup_F ([0,1[\times F)$, and let $r, s : \mathcal{C}_F \to \mathcal{C}_F^{(0)}$ be induced by

$$\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathcal{G}^{(0)} \longrightarrow \mathcal{C}_F^{(0)} \longleftarrow [0,1[\times F \overset{r}{\underset{s}{\leftleftarrows}} [0,1[\times \mathcal{G}|F.$$

Then there is a continuous bijection $\mathcal{G}^{(2)} \cup_{(\mathcal{G}|F)^{(2)}} ([0,1[\times \mathcal{G})^{(2)} \to \mathcal{C}_F^{(2)}$ where as usual $\mathcal{G}^{(2)} = \mathcal{G} \times_{\mathcal{G}^{(0)}} \mathcal{G}$, etc. The images of $\mathcal{G}^{(2)}$ and $([0,1[\times \mathcal{G}|F)^{(2)}$ in $\mathcal{C}_F^{(2)}$ are closed, so that this map is in fact a homeomorphism. By this token, the operations of $\mathcal{G}$ and $[0,1[\times \mathcal{G}|F$ induce operations on $\mathcal{C}_F$, making the latter a topological groupoid.

Finally, the inclusions $\mathcal{G} \subset \mathcal{C}_F \supset [0,1[\times \mathcal{G}|F$ being proper, we have continuous maps

$$\mathcal{G}^{(0)} \xrightarrow{(\lambda^u)} \mathfrak{M}(\mathcal{G}) \longrightarrow \mathfrak{M}(\mathcal{C}_F) \longleftarrow \mathfrak{M}([0,1[\times \mathcal{G}) \xleftarrow{(\delta_t \otimes \lambda^u)} [0,1[\times F$$

of the spaces of Radon measures, endowed with their $\sigma(\mathfrak{M}, \mathcal{C}_c)$-topologies. Since these maps coincide on F, they induce a continuous map $\mu : \mathcal{C}_F^{(0)} \to \mathfrak{M}(\mathcal{C}_F)$ which can be seen to define a Haar system. We have established the following proposition.

Proposition 3. *The space $\mathcal{C}_F$ is a locally compact σ-compact groupoid with a Haar system, and the inclusions $\mathcal{G} \subset \mathcal{C}_F \supset [0,1[\times \mathcal{G}|F$ are proper homomorphisms.*

Theorem 4. *For the quotient map q in the extension* (1)*, and its mapping cone C_q, we have $C_q \cong \mathrm{C}^*(\mathcal{C}_F)$. If $\mathcal{G}|F$ is topologically amenable, and q is the quotient map of the corresponding short exact sequence of reduced groupoid* C^**-algebras, then the associated mapping cone is $C_q \cong \mathrm{C}^*_r(\mathcal{C}_F)$.*

Proof. Note that $\mathrm{C}^*([0,1[\times \mathcal{G}|F) \cong \mathcal{C}_0([0,1[) \otimes \mathrm{C}^*(\mathcal{G}|F)$. Hence, the commutative square of proper homomorphisms

$$\begin{array}{ccc} \mathcal{G}|F & \longrightarrow & [0,1[\times \mathcal{G}|F \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{C}_F \end{array}$$

preserves Haar systems, and thus integrates to the commutative square of $*$-morphisms

$$\begin{array}{ccc} \mathrm{C}^*(\mathcal{G}|F) & \xleftarrow{e_0} & C\mathrm{C}^*(\mathcal{G}|F) \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle p_2} \\ \mathrm{C}^*(\mathcal{G}) & \xleftarrow[p_1]{} & \mathrm{C}^*(\mathcal{C}_F). \end{array}$$

Thus, we obtain a $*$-morphism $p:\mathrm{C}^*(\mathcal{C}_F) \to C_q$ whose kernel is $\ker p_1 \cap \ker p_2$.

Now, $\mathcal{G} = \mathcal{C}_F|\mathcal{G}^{(0)}$ and $[0,1[\times\mathcal{G}|F = \mathcal{C}_F|(]0,1[\times F)$, so p_1 resp. p_2 are the quotient maps for extensions of type (1) for the groupoid $\mathcal{C}_F$ and the open invariant subsets $\mathcal{C}_F^{(0)} \setminus \mathcal{G}^{(0)} =]0,1[\times F$ and $\mathcal{C}_F^{(0)} \setminus ([0,1[\times F) = U$, respectively. Thus,

$$\ker p_1 \cap \ker p_2 = I\bigl(]0,1[\times F\bigr) \cap I(U) = I(\emptyset) = 0,$$

by Proposition 2.

As to the surjectivity, let $(a,f) \in C_q$, so $q(a) = f(0)$. There exists $b' \in \mathrm{C}^*(\mathcal{C}_F)$ such that $p_1(b') = a$. Then $p(b') = (a, f')$. We find $f'(0) = q(a) = f(0)$, so $f - f'$ belongs to $\ker e_0 = I(]0,1[\times F) = \ker p_1$. Hence, $f - f'$ can be considered as an element of $\mathrm{C}^*(\mathcal{C}_F)$, and $p_1(f - f') = 0$. Thus, setting $b = b' + f - f'$, we find $p(b) = (a, f') + (0, f - f') = (a, f)$. This shows that p is surjective.

The same argument goes through in the reduced case if $\mathcal{G}|F$ is topologically amenable, since $[0,1[\times\mathcal{G}|F$ is then also amenable. □

The groupoid expression of the mapping cone gives an easy proof of naturality.

Proposition 5. *Let $\mathcal{H}$ be another locally compact σ-compact groupoid, and $G \subset \mathcal{H}^{(0)}$ a closed invariant subset. Assume that $\varphi:\mathcal{H} \to \mathcal{G}$ is a strict morphism such that $\varphi(G) \subset F$. Let φ'' be the restriction of φ to $\mathcal{H}|G$, so that the right square in the following diagram commutes:*

$$\begin{array}{ccccccc} (\mathcal{G}|F)\times]0,1[& \longrightarrow & \mathcal{C}_F & \longleftarrow & \mathcal{G} & \longleftarrow & \mathcal{G}|F \\ {\scriptstyle \varphi''\times\mathrm{id}_{]0,1[}}\uparrow & & \vdots{\scriptstyle \psi} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi''} \\ (\mathcal{H}|G)\times]0,1[& \longrightarrow & \mathcal{C}_G & \longleftarrow & \mathcal{H} & \longleftarrow & \mathcal{H}|G. \end{array}$$

Here, the horizontal arrows pointing from left to right are open inclusions, and those pointing from right to left are closed inclusions. Then there exists a strict morphism ψ as indicated, which is proper if φ is, such that the entire diagram becomes commutative.

Proof. Indeed, simply set $\psi = \varphi \cup_{\mathcal{H}|G} \bar{\varphi}$ where $\bar{\varphi} = \varphi'' \times \mathrm{id}_{]0,1]}$. Then the diagram is commutative. The inclusions $\mathcal{G} \longrightarrow \mathcal{C}_F \longleftarrow \mathcal{G}|F$ are closed embeddings: hence, they are proper. If φ is proper, then so is $\bar{\varphi}$. If $K \subset \mathcal{C}_F$ is compact, then, identifying subsets of $\mathcal{H}$ and $\mathcal{H}|G \times]0,1]$ with their images in $\mathcal{C}_G$, $\psi^{-1}(K) = \varphi^{-1}(K) \cup \bar{\varphi}^{-1}(K)$, which is compact as the union of two compacts. Thus, ψ is proper. □

Let $U = \mathcal{G}^{(0)} \setminus F$ and $V = \mathcal{H}^{(0)} \setminus G$. By construction, the restriction of ψ to $\mathcal{H}|V$ is simply φ', which sends $\mathcal{H}|V \to \mathcal{G}|U$. We obtain the following corollary.

Corollary 6. *Retain the notation of Proposition 5 and let φ' be the restriction of φ to $\mathcal{H}|V$. If φ is proper and the groupoids $\mathcal{G}$ and $\mathcal{H}$ are amenable and have Haar systems, then the following diagram commutes in KK:*

$$\begin{array}{ccc} S\mathrm{C}_r^*(\mathcal{G}|F) & \xrightarrow{y\otimes\partial} & \mathrm{C}_r^*(\mathcal{G}|U) \\ {\scriptstyle SKK(\varphi')}\downarrow & & \downarrow{\scriptstyle KK(\varphi'')} \\ S\mathrm{C}_r^*(\mathcal{H}|G) & \xrightarrow{y\otimes\partial} & \mathrm{C}_r^*(\mathcal{H}|V). \end{array}$$

Here, S denotes suspension and the horizontal maps are the connecting maps in KK theory.

Proof. Applying the cofunctor $KK(-)$ introduced in Section 2.1, the following diagram commutes in the category KK:

$$\begin{array}{ccc} S\mathrm{C}_r^*(\mathcal{G}|F) & \longrightarrow & C_q = \mathrm{C}_r^*(\mathcal{C}_F) \\ {\scriptstyle SKK(\varphi')}\downarrow & & \downarrow{\scriptstyle KK(\varphi')} \\ S\mathrm{C}_r^*(\mathcal{H}|G) & \longrightarrow & C_p = \mathrm{C}_r^*(\mathcal{C}_G) \end{array}$$

where C_q and C_p are the mapping cones for

$$q : \mathrm{C}_r^*(\mathcal{G}) \to \mathrm{C}_r^*(\mathcal{G}|F) \quad \text{and} \quad p : \mathrm{C}_r^*(\mathcal{H}) \to \mathrm{C}^*(\mathcal{H}|G),$$

and the horizontal maps are natural. As observed in the discussion preceding the definition of $\mathcal{C}_F$, the connecting map for $\mathcal{G}$ is the inverse of the KK equivalence $\mathrm{C}_r^*(\mathcal{G}|U) \to C_q$, in turn induced by the open inclusion $\mathcal{G}|U \subset \mathcal{G}$, cf. [6, Theorem 19.5.7]. Similarly, this applies to $\mathcal{H}$. As noted above, $KK(\psi)$ pushes through these equivalences to the arrow $KK(\varphi')$. $\square$

3. Fibrewise differentiable groupoids

3.1. Basic definitions

In this section, we extend the concepts of continuous family manifolds and groupoids (of class $\mathcal{C}^{\infty,0}$), introduced by Paterson [24], to the case of finite differentiability class $\mathcal{C}^{q,0}$, $q < \infty$. This goes through without much ado, and we do not claim any originality in this respect. To our purposes, the interesting point is that the appropriate generalisation of Connes's tangent groupoid can be defined in the finite differentiability class $\mathcal{C}^{q,0}$, $q \geqslant 1$. Given Paterson's thorough treatment of the $q = \infty$ case, we need only sketch the elements of the theory for $q < \infty$. Continuous family groupoids have been studied in their own right; we refer the interested reader to [24,19].

Let Y be a paracompact topological space, and $A \subset Y \times \mathbb{R}^n$, $B \subset Y \times \mathbb{R}^m$ be open. Then a continuous and fibre-preserving map $f : A \to B$ is said to be of class $\mathcal{C}^{q,0}$, where $q \in \mathbb{N} \cup \infty$, if for any $U \times V \subset A$ and $U' \times V' \subset B$ where $U, U' \subset Y$ and $V \subset \mathbb{R}^n$, $V' \subset \mathbb{R}^m$ are open subsets and $f(U \times V) \subset U' \times V'$, the map

$$U \to U' \times \mathcal{C}^q(V, V') : y \mapsto f^y = f(y, \sqcup)$$

is well defined and continuous for the usual Fréchet topology on $\mathcal{C}^q(U, U')$. For $n = m$, the collection of all the locally defined $\mathcal{C}^{q,0}$ maps with $\mathcal{C}^{q,0}$ inverse defines a pseudogroup $\Gamma_Y^{q,0}(\mathbb{R}^n)$ of local homeomorphisms, cf. [16].

Let M, Y be paracompact locally compact Hausdorff spaces, and let the map $p : M \to Y$ be a continuous open surjection. Then (M, p) is called a manifold of class $\mathcal{C}^{q,0}$ over Y if it has an atlas compatible with $\Gamma_Y^{q,0}(\mathbb{R}^n)$. In this case, each of the fibres $M^y = p^{-1}(y)$ is a manifold of class $\mathcal{C}^q$, with dimension constant by definition. It is clear how to define the appropriate morphisms to turn the class of $\mathcal{C}^{q,0}$ manifolds over the fixed base Y into a category.

Moreover, given a continuous map $f : Z \to Y$, any Y-manifold (M, p) of class $\mathcal{C}^{q,0}$ pulls back to a Z-manifold f^*M of class $\mathcal{C}^{q,0}$. This enables one to define $\mathcal{C}^{q,0}$ maps between $\mathcal{C}^{q,0}$ over different bases, and gives rise to a sensible category of $\mathcal{C}^{q,0}$ manifolds with arbitrary base.

Similarly as above, we may define a pseudogroup $\mathrm{GL}_Y^{q,0}(\mathbb{R}^n, \mathbb{R}^k)$ of local homeomorphisms of $Y \times \mathbb{R}^n \times \mathbb{R}^k$ by considering maps f such that

$$f(y, a, x) = \bigl(y, f_y(a), L_{y,a}(x)\bigr) \quad \text{where } L_{y,a} \in \mathrm{GL}(k, \mathbb{R}).$$

This allows the definition of vector bundles of class $\mathcal{C}^{q,0}$. A trivial but striking consequence is that if E is a topological vector bundle over Y, and $p : M \to Y$ is a $\mathcal{C}^{q,0}$ manifold, then p^*E is naturally a $\mathcal{C}^{q,0}$ vector bundle over M.

Let $q \geqslant 1$ and (M, p) be a manifold over Y of class $\mathcal{C}^{q,0}$. Then we define the fibrewise tangent bundle TM as follows. Set-theoretically, TM is the (disjoint) union $TM = \bigcup_{y \in Y} TM^y$ where $M^y = p^{-1}(y)$, and the bundle projection is $\pi(y, x, \xi) = x$. Let $((U_\alpha, \phi_\alpha))$ be an atlas for (M, p), compatible with $\Gamma_Y^{p,0}(\mathbb{R}^n)$. Then let $\phi_\alpha^y = \phi_\alpha |(M^y \cap U_\alpha)$, and

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k : (x, \xi) \mapsto \bigl(\phi_\alpha(x), T_x \phi_\alpha^{p(x)} \xi\bigr).$$

(If $y = p(x)$ and $\xi = \dot{x}(0)$ where $x(t)$ is a $\mathcal{C}^1$ curve in M^y, $x(0) = x$, then $T_x \phi_\alpha^y \xi = \dot{z}(0)$ where $z(t) = \phi_\alpha \circ x(t) = \phi_\alpha^y \circ x(t)$.) Endow TM with the weakest topology turning all the ψ_α into homeomorphisms. Then $((\pi^{-1}(U_\alpha), \psi_\alpha))$ is the structure of a vector bundle over M of class $\mathcal{C}^{q-1,0}$ (where $q - 1 = \infty$ for $q = \infty$) and rank $k = \dim M^y$.

Assume (M, p) and (M', p') are spaces of class $\mathcal{C}^{q,0}$ where $q \geqslant 1$, and $f : M \to M'$ is a class $\mathcal{C}^{q,0}$ morphism. Then we may define a class $\mathcal{C}^{q-1,0}$ morphism $Tf : TM \to TM'$ in the obvious way, using fibrewise differentiation.

Definition 7. A groupoid $\mathcal{G}$ is said to be of class $\mathcal{C}^{q,0}$ over $\mathcal{G}^{(0)}$ if

(1) $(\mathcal{G}, r)$ and $(\mathcal{G}, s)$ are $\mathcal{G}^{(0)}$-manifolds of class $\mathcal{C}^{q,0}$,
(2) inversion is an isomorphism of class $\mathcal{C}^{q,0}$ between $(\mathcal{G}, r)$ and $(\mathcal{G}, s)$, and
(3) considering $\circ : (\mathcal{G}^{(2)}, \mathrm{pr}_1) \to (\mathcal{G}, r)$, $(\circ, r)$ is a morphism of class $\mathcal{C}^{q,0}$.

A $\mathcal{C}^{q,0}$ homomorphism of groupoids is a groupoid homomorphism $f:\mathcal{G}\to\mathcal{H}$ between $\mathcal{C}^{q,0}$ groupoids $\mathcal{G}$ and $\mathcal{H}$ such that that $(f, f|\mathcal{G}^{(0)})$ is a $\mathcal{C}^{q,0}$ morphism for both $(\mathcal{G},r)\to(\mathcal{H},r)$ and $(\mathcal{G},s)\to(\mathcal{H},s)$.

Let $q\geqslant 1$ and $\mathcal{G}$ be a groupoid of class $\mathcal{C}^{q,0}$. Considering $\mathcal{G}^{(0)}\subset\mathcal{G}$, we may take the restriction $A(\mathcal{G})=T\mathcal{G}|\mathcal{G}^{(0)}$, the so-called Lie algebroid of $\mathcal{G}$, as a vector bundle on $\mathcal{G}^{(0)}$.

3.2. The fibrewise tangent groupoid

Now we are ready to define the fibrewise tangent groupoid of a groupoid $\mathcal{G}$ of class $\mathcal{C}^{q,0}$. Set-theoretically, this is

$$\mathbb{T}\mathcal{G}=A(\mathcal{G})\times 0\cup\mathcal{G}\times]0,1].$$

The unit space is $(\mathbb{T}\mathcal{G})^{(0)}=\mathcal{G}^{(0)}\times[0,1]$, with source and range maps defined by

$$s(x,\xi,0)=(x,0),\quad s(\gamma,\varepsilon)=s(\gamma)\quad\text{and}\quad r(x,\xi,0)=(x,0),\quad r(\gamma,\varepsilon)=r(\gamma),$$

and composition given by

$$(x,\xi_1,0)(x,\xi_2,0)=(x,\xi_1+\xi_2,0)\quad\text{and}\quad(\gamma_1,\varepsilon)(\gamma_2,\varepsilon)=(\gamma_1\gamma_2,\varepsilon).$$

Consider the product topology on $\mathcal{G}^{(0)}\times[0,1]$. The topology of $\mathbb{T}\mathcal{G}$ is the weakest for which r and s are continuous, as well as the maps $\mathbb{T}f:\mathbb{T}\mathcal{G}\to\mathbb{R}$, defined by

$$\mathbb{T}f(x,\xi,0)=T_xf(\xi)\quad\text{and}\quad\mathbb{T}f(\gamma,\varepsilon)=\frac{f(\gamma)}{\varepsilon}$$

for any $\mathcal{C}^{q,0}$ map $f:\mathcal{G}\to\mathbb{R}$ which is 0 on $\mathcal{G}^{(0)}$.

Proposition 8. *For any groupoid $\mathcal{G}$ of class $\mathcal{C}^{q,0}$, $q\geqslant 1$, $\mathbb{T}\mathcal{G}$ is a groupoid of class $\mathcal{C}^{q,0}$.*

The proof is the same as in the $\mathcal{C}^{\infty,0}$ case, and we do not repeat it.

Remark 9.

(i) Note that the differentiability class of $\mathbb{T}\mathcal{G}$ is the same as for $\mathcal{G}$. This is due to the fact that only differentiability in the fibre direction is considered, and the fibres of $\mathbb{T}\mathcal{G}$ over $\mathcal{G}^{(0)}\times\varepsilon$, $\varepsilon>0$, are of class $\mathcal{C}^q$, whereas over $\mathcal{G}^{(0)}\times 0$, they are linear and hence of class $\mathcal{C}^\infty$.

(ii) The prime example of a $\mathcal{C}^{q,0}$ groupoid for which the tangent groupoid is considered is $\mathcal{G}=M\times_Y M$ where $p:M\to Y$ is a $\mathcal{C}^{q,0}$ manifold over Y, $q\geqslant 1$. Then r,s are the projections and composition is the same as for the pair groupoid. In this case, $A(\mathcal{G})=TM$, as is easy to see, so

$$\mathbb{T}(M\times_Y M)=TM\times 0\cup\big(M\times_Y M\times]0,1]\big),$$

with the weakest topology that makes source and range continuous, along with the maps $\tilde f:\mathbb{T}(M\times_Y M)\to\mathbb{R}$ defined for any $f\in\mathcal{C}^{q,0}(M,\mathbb{R})$ by

$$\tilde f(x,\xi,0)=T_xf(\xi)\quad\text{and}\quad\tilde f(x_1,x_2,\varepsilon)=\frac{f(x_1)-f(x_2)}{\varepsilon}.$$

Indeed, for any such f, $h_f : M \times_Y M \to \mathbb{R}$, $h_f(x_1, x_2) = f(x_1) - f(x_2)$ is a map of class $\mathcal{C}^{q,0}$.

For $Y = \mathrm{pt}$, $\mathbb{T}(M \times_Y M) = \mathbb{T}(M \times M)$ is Connes's tangent groupoid [8, § II.5] for the manifold M. In general, one may view $\mathbb{T}(M \times_Y M)$ as a 'family version' of this tangent groupoid.

The following result follows immediately from the triviality of the density bundle $|\Omega|(T^*\mathcal{G})$, cf. [24, proof of Theorem 1].

Proposition 10. *Let $\mathcal{G}$ be a groupoid of class $\mathcal{C}^{q,0}$, $q \geqslant 1$. Then $\mathcal{G}$ has a Haar system (λ^u) which is locally of the form $\lambda^u|U = \delta_u \otimes \alpha^u_U$, where α^u_U is absolutely continuous to Lebesgue measure on an open subset of $\mathbb{R}^n$.*

Proposition 11. *Let $\mathcal{G}$ be a groupoid of class $\mathcal{C}^{q,0}$, $q \geqslant 1$. Then $A(\mathcal{G})$ is topologically amenable. In particular, we have a short exact sequence*

$$0 \longrightarrow \mathcal{C}_0(]0,1], \mathrm{C}^*_r(\mathcal{G})) \longrightarrow \mathrm{C}^*_r(\mathbb{T}\mathcal{G}) \xrightarrow{e_0} \mathrm{C}^*_r(A(\mathcal{G})) \longrightarrow 0$$

of reduced groupoid C*-*algebras. Here, we denote by*

$$e_t : \mathrm{C}^*_r(\mathbb{T}\mathcal{G}) \to \mathrm{C}^*_r\big(\mathbb{T}\mathcal{G}|\mathcal{G}^{(0)} \times t\big)$$

the maps induced by restriction to the closed invariant subsets

$$\mathcal{G}^{(0)} \times t \subset \mathcal{G} \times [0,1] = (\mathbb{T}\mathcal{G})^{(0)}.$$

Proof. The groupoid $A(\mathcal{G})$ is amenable. Indeed, its isotropy groups $T_x\mathcal{G}^x$ are Abelian and hence amenable as groups. The principal groupoid associated to $A(\mathcal{G})$ is the graph of the identity on $\mathcal{G}^{(0)}$, so it is just the space $\mathcal{G}^{(0)}$. The latter is amenable by definition. Then [1, Corollary 5.3.33] gives the measurewise amenability of $A(\mathcal{G})$; the topological amenability follows from [1, Theorem 3.3.7] (the orbits in $A(\mathcal{G})^{(0)} = \mathcal{G}^{(0)}$ are points). The remaining claims follow from Section 2.2. □

Definition 12. The C*-algebra $\mathcal{C}_0(]0,1], \mathrm{C}^*_r(\mathcal{G}))$ being C*-contractible, the map e_0 is a KK equivalence, thus inducing an element

$$\tau = e_0^{-1} \otimes e_1 \in KK\big(\mathrm{C}^*_r\big(A(\mathcal{G})\big), \mathrm{C}^*_r(\mathcal{G})\big)$$

which we call the *Connes–Skandalis map*, cf. [15, Definition 3.2].

In fact, such a map can be introduced for any continuous field of C*-algebras over $[0,1]$ which is trivial over $]0,1]$. In our $\mathcal{C}^{1,0}$ groupoid setup, we shall show that the KK^1 class $y \otimes \tau$ represents an extension of groupoid C*-algebras.

4. The suspended Connes–Skandalis map

4.1. Suspension and cone on the tangent groupoid

Let $\mathcal{G}$ be a groupoid of class $\mathcal{C}^{1,0}$. To prove our index theorem, we shall have to compute $y \otimes \tau$ where τ is the Connes–Skandalis map associated to $\mathbb{T}\mathcal{G}$, and the element $y \in KK^1(S, \mathbb{C})$ represents the Wiener–Hopf extension, i.e., represents the standard extension of $\mathcal{W}_{\mathbb{R}_{\geqslant 0}}$. Whereas $x \otimes \tau$ (where $x = y^{-1}$) is easily evaluated without resorting to groupoid constructions (cf. [15, Remark 3.3.2]), we shall have to construct an auxiliary groupoid in order to compute the suspended Connes–Skandalis map $y \otimes \tau$.

Recall that $\mathcal{W}_{\mathbb{R}_{\geqslant 0}} = (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} \cup (\infty \times \mathbb{R})$ is the disjoint union of groupoids, with the topology given as a subspace of $[0, \infty] \times \mathbb{R}$. As a topological space, let $\mathbb{W}\mathcal{G} = \mathbb{T}\mathcal{G} \times_{[0,\infty]} \mathcal{W}_{\mathbb{R}_{\geqslant 0}}$ where the map $\mathbb{T}\mathcal{G} \to [0, \infty]$ is the composition of r (or s) with

$$\mathcal{G}^{(0)} \times [0, 1] \to [0, \infty]\colon \begin{cases} (x, \varepsilon) \mapsto \frac{1}{\varepsilon} - 1 & \varepsilon > 0, \\ (x, 0) \mapsto \infty & \text{otherwise,} \end{cases}$$

and $\mathcal{W}_{\mathbb{R}_{\geqslant 0}} \to [0, \infty]$ is the range projection. Define groupoid operations on $\mathbb{W}\mathcal{G}$ as follows:

$$r(\gamma, r_1, r_2 - r_1) = \bigl(r(\gamma), r_1\bigr), \qquad s(\gamma, r_1, r_2 - r_1) = \bigl(s(\gamma), r_2\bigr),$$
$$r(x, \xi, \infty, r) = (x, \infty) = s(x, \xi, \infty, r),$$

and

$$(\gamma_1, r_1, r_2 - r_1)(\gamma_2, r_2, r_3 - r_2) = (\gamma_1\gamma_2, r_1, r_3 - r_1),$$
$$(x, \xi_1, \infty, r_1)(x, \xi_2, \infty, r_2) = (x, \xi_1 + \xi_2, \infty, r_1 + r_2).$$

Proposition 13. *Given a class $\mathcal{C}^{1,0}$ groupoid $\mathcal{G}$, the space $\mathbb{W}\mathcal{G}$ is a locally compact groupoid such that $\mathbb{W}\mathcal{G}^{(0)} = \mathcal{G}^{(0)} \times [0, \infty]$. The subset $F = \mathcal{G}^{(0)} \times \infty$ is closed and invariant, and we have*

$$\mathbb{W}\mathcal{G}|F = A(\mathcal{G}) \times \mathbb{R} \quad \textit{and} \quad \mathbb{W}\mathcal{G}|U = \mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} \quad \textit{for } U = \mathbb{W}\mathcal{G}^{(0)} \setminus F.$$

Moreover, $\mathbb{W}\mathcal{G}$ carries a natural Haar system which may be chosen to induce on $\mathcal{G}$ any given Haar system associated to a positive section of the density bundle $|\Omega|(T^*\mathcal{G})$.

Proof. It is clear that $\mathbb{W}\mathcal{G}$ is a locally compact space, and it is also evidently a groupoid. We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{T}\mathcal{G}^{(2)} \times \mathcal{W}^{(2)}_{\mathbb{R}_{\geqslant 0}} & \xrightarrow{\ \circ\ } & \mathbb{T}\mathcal{G} \times \mathcal{W}_{\mathbb{R}_{\geqslant 0}} \\ \uparrow & & \uparrow \\ \mathbb{W}\mathcal{G}^{(2)} & \xrightarrow[\ \circ\ ]{} & \mathbb{W}\mathcal{G} \end{array}$$

where the vertical maps are self-evident, and the rightmost of these is a closed embedding. Thus, composition is continuous, and along the same lines, the continuity of the inverse is established. The projections r and s are continuous.

As to the existence of Haar systems, $\mathcal{W} =]-\infty, \infty] \rtimes \mathbb{R}$ is a $\mathcal{C}^{1,0}$ groupoid; in fact, the direct product $\mathcal{W} =]-\infty, \infty] \times \mathbb{R}$ of spaces is a $\mathcal{C}^{1,0}$ manifold over $]-\infty, \infty]$, and the operations are fibrewise those of the Lie group $\mathbb{R}$, independent of the fibre. Moreover, $\mathcal{W}_{\mathbb{R}_{\geqslant 0}}$ is the restriction of $\mathcal{W}$ to the non-invariant subset $[0, \infty]$ of $]-\infty, \infty]$.

Similarly as for $\mathbb{T}\mathcal{G}$, we may define a $\mathcal{C}^{1,0}$ groupoid $\mathcal{T} = A(\mathcal{G}) \times 0 \cup \mathcal{G} \times]0, \infty[$, by replacing $[0, 1]$ in the definition of the tangent groupoid by $[0, \infty[$. Choosing a homeomorphism $\phi : [0, \infty[\to]-\infty, \infty]$ which coincides on $[0, 1]$ with $\varepsilon \mapsto \frac{1}{\varepsilon} - 1$, we obtain a $\mathcal{C}^{1,0}$ groupoid $\mathcal{H} = \mathcal{T} \times_{]-\infty,\infty]} \mathcal{W}$ such that $\mathbb{W}\mathcal{G} = \mathcal{H}|(\mathcal{G}^{(0)} \times [0, \infty])$. In fact, if $f : \mathcal{G} \to \mathbb{R}^n$ is a $\mathcal{C}^{1,0}$ map such that (r, f) is a local chart, we may define $\psi_f : \mathcal{H} \to \mathbb{R}^{n+1}$ by

$$\psi_f(\tau) = \begin{cases} \left(\frac{f(\gamma)}{\phi^{-1}(r_1)}, r_2 - r_1\right), & \tau = (\gamma, r_1, r_2 - r_1),\ r_1 < \infty, \\ (T_x f(\xi), r), & \tau = (x, \xi, \infty, r). \end{cases}$$

Then (r, ψ_f) is a local chart for $\mathcal{H}$. Now, $\mathcal{H}$ has a Haar system given by a positive section of the density bundle, unique up to multiplication by such a density. Thus we may assume that this Haar system induces on $\mathcal{G}$ the given Haar system associated to the choice of a positive section of $|\Omega|(T^*\mathcal{G})$.

If $\lambda^{x,t}$, $(x, t) \in \mathcal{G}^{(0)} \times]-\infty, \infty]$, is a Haar system of $\mathcal{H}$, define an invariant system of positive Radon measures by $\mu^{x,t} = \lambda^{x,t}|\mathbb{W}\mathcal{G}^{x,t}$. Since $\mathbb{W}\mathcal{G}^{x,t}$ has dense interior in $\mathcal{H}^{x,t}$, the measures $\mu^{x,t}$ satisfy the support condition. The maps $x \mapsto \lambda^{x,t}$, for $t \in [0, \infty]$, are equicontinuous. Hence, the same is true for $x \mapsto \mu^{x,t}$. Since for fixed x, the characteristic functions of the interiors of $\mathbb{W}\mathcal{G}^{x,t}$ depend continuously in the topology of simple convergence on t, we find that $\mu^{x,t}$ satisfies the continuity axiom. The statement about the invariant subsets and the corresponding restricted groupoids is immediate. □

Corollary 14. *There is a short exact sequence*

$$0 \to \mathrm{C}^*_r(\mathcal{G}) \otimes \mathbb{K} \to \mathrm{C}^*_r(\mathbb{W}\mathcal{G}) \to S\mathrm{C}^*_r(A(\mathcal{G})) = \mathrm{C}^*_r(A(\mathcal{G})) \otimes \mathrm{C}^*_r(\mathbb{R}) \to 0. \qquad (**)$$

Proof. We need only remark that $A(\mathcal{G}) \times \mathbb{R}$ is an amenable groupoid, and that $(\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0}$ is isomorphic to the pair groupoid $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$, whose reduced C*-algebra is $\mathbb{K}$. □

To see that τ 'interpolates' between the Wiener–Hopf extension and the one constructed above, we need to construct the 'cone' $\mathbb{C}\mathcal{G}$ over the tangent groupoid. This is the content of the following proposition.

Proposition 15. *Let $\mathcal{G}$ be a groupoid of class $\mathcal{C}^{1,0}$. There exists a locally compact groupoid $\mathbb{C}\mathcal{G}$ over the 'triangle'*

$$\mathbb{C}\mathcal{G}^{(0)} = \mathcal{G}^{(0)} \times \Delta \quad \text{where } \Delta = [0, 1] \times [0, \infty]/[0, 1] \times \infty,$$

such that $U = [0, 1] \times \mathcal{G}^{(0)} \times]0, \infty]$ is an open invariant subset of $\mathbb{C}\mathcal{G}^{(0)}$,

$$\mathbb{C}\mathcal{G}|F = A(\mathcal{G}) \times \mathbb{R} \quad \text{and} \quad \mathbb{C}\mathcal{G}|U = \mathbb{T}\mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} \quad \text{where } F = \mathbb{C}\mathcal{G}^{(0)} \setminus U.$$

In addition, $\mathbb{C}\mathcal{G}$ carries a Haar system which induces on $\mathbb{T}\mathcal{G}$ any Haar system given by a fixed choice of a positive section of the latter groupoid's density bundle.

Proof. Let $\mathcal{H}$ be the $\mathcal{C}^{1,0}$ groupoid over $\mathcal{H}^{(0)} = \mathcal{G}^{(0)} \times]-\infty, \infty]$ constructed in the proof of Proposition 13, such that $\mathbb{W}\mathcal{G} = \mathcal{H}|(\mathcal{G}^{(0)} \times [0, \infty])$. We construct the 'partial' tangent groupoid $\mathcal{TH} = 0 \times A(\mathcal{G}) \times \mathcal{W} \cup]0, 1] \times \mathcal{H}$, the disjoint union of groupoids. We endow this set with the initial topology with respect to r, s, and the maps $\varrho_f : \mathcal{TH} \to \mathbb{R}^{n+1}$ defined for $\mathcal{C}^{1,0}$ charts (r, f) of $\mathcal{G}$, $f : \mathcal{G} \to \mathbb{R}^n$, as follows:

$$\varrho_f(\tau) = \begin{cases} \left(\frac{f(\gamma)}{\varepsilon+\phi^{-1}(r_1)}, r_2 - r_1\right), & \tau = (\varepsilon, \gamma, r_1, r_2 - r_1),\ \varepsilon > 0,\ r_1 < \infty, \\ (T_x f(\xi), r_2 - r_1), & \tau = (0, x, \xi, r_1, r_2 - r_1), r_1 < \infty, \\ (T_x f(\xi), r), & \tau = (\varepsilon, x, \xi, \infty, r),\ \varepsilon \in [0, 1]. \end{cases}$$

Then, for any such f, (r, ϱ_f) is a local $\mathcal{C}^{1,0}$ chart for $\mathcal{TH}$, turning the latter into a $\mathcal{C}^{1,0}$ groupoid.

Define on the unit space of $\mathcal{TH}$, $\mathcal{TH}^{(0)} = [0, 1] \times \mathcal{G}^{(0)} \times]-\infty, \infty]$, the following equivalence relation:

$$\begin{aligned} &(\varepsilon_1, x_1, r_1) \sim (\varepsilon_2, x_2, r_2) \\ &\quad \Leftrightarrow \quad x_1 = x_2 \quad \text{and} \quad \bigl(\min(r_1, r_2) < \infty \Rightarrow (\varepsilon_1, r_1) = (\varepsilon_2, r_2)\bigr), \end{aligned}$$

and denote its graph by S. Then S is a subgroupoid of the pair groupoid on $\mathcal{TH}^{(0)}$, and it acts on $\mathcal{TH}$ by

$$(s, t).\gamma = \begin{cases} \gamma, & t = r(\gamma) \notin [0, 1] \times \mathcal{G}^{(0)} \times \infty; \\ (\varepsilon_1, x, \xi, \infty, r) & \begin{cases} s = (\varepsilon_1, x, \infty),\ t = (\varepsilon_2, x, \infty), \\ \gamma = (\varepsilon_2, x, \xi, \infty, r). \end{cases} \end{cases}$$

Thus, S fixes γ whenever $r(\gamma) \notin [0, 1] \times \mathcal{G}^{(0)} \times \infty$, and on

$$r^{-1}\bigl([0, 1] \times \mathcal{G}^{(0)} \times \infty\bigr) = [0, 1] \times A(\mathcal{G}) \times \infty \times \mathbb{R},$$

the action of S fixes all but the first component, and there, it is the usual action of the pair groupoid over $[0, 1]$.

Let R denote the graph of the equivalence relation on $\mathcal{TH}$ defined by the action of S. Then R is a closed subset of $\mathcal{TH} \times \mathcal{TH}$, and the equivalence classes of R are compact. Therefore, $\mathcal{CH} = \mathcal{TH}/R$ is a locally compact space, and the associated canonical projection $\pi : \mathcal{TH} \to \mathcal{CH}$ is proper, by [7, Chapter I, § 10.4, Proposition 9]. Moreover, the charts ϱ_f are invariant for the action of S, and hence drop to $\mathcal{CH}$, thereby turning this space into a $\mathcal{C}^{1,0}$ manifold over the 'triangle' $\mathcal{CH}^{(0)} = \mathcal{TH}^{(0)}/S$. In fact, the operations of $\mathcal{CH}$ commute with the action of S, and since they are compatible with the charts ϱ_f, $\mathcal{CH}$ turns into a $\mathcal{C}^{1,0}$ groupoid.

In particular, $\mathcal{CH}$ has a Haar system induced by the choice of a positive density. It restricts to a Haar system for

$$\mathbb{C}\mathcal{G} = \mathcal{CH}|\bigl(\mathcal{G}^{(0)} \times \Delta\bigr) \quad \text{where } \Delta = [0, 1] \times [0, \infty]/[0, 1] \times \infty,$$

by the same argument as in the proof of Proposition 13.

Let $U' = [0,1] \times \mathcal{G}^{(0)} \times \mathbb{R}$. Because the quotient map $[0,1] \times [0,\infty] \to \Delta$ restricts to a homeomorphism on $[0,1] \times [0,\infty[$, the restriction of π to $\mathcal{TH}|U'$ has local sections and is injective. Hence, it is a homeomorphism onto its image. Moreover, $\mathcal{TH}|U'$ is R-saturated, so the image is open in $\mathcal{CH}$. Obviously, $\mathbb{C}\mathcal{G} \cap \pi(U') = U$, where U is the image of $[0,1] \times \mathcal{G}^{(0)} \times [0,\infty[$ in $\mathbb{C}\mathcal{G}^{(0)}$. Thus, the intersection of $\pi(\mathcal{TH}|U')$ with $\mathbb{C}\mathcal{G}$ is equal to $\mathbb{C}\mathcal{G}|U$, and

$$\mathbb{C}\mathcal{G}|U \cong \mathcal{TH}|U = \mathbb{T}\mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0},$$

where this isomorphism is the restriction of an isomorphism of $\mathcal{C}^{1,0}$ groupoids.

Now, $F = \mathbb{C}\mathcal{G}^{(0)} \setminus U$ equals the complement of U' in $\mathcal{TH}^{(0)}$. Since the action of S on $\mathcal{TH}|F$ identifies with the standard action of the pair groupoid on $[0,1]$, we have

$$\mathbb{C}\mathcal{G}|F = \mathcal{TH}|F \cong [0,1] \times A(\mathcal{G}) \times \mathbb{R}/[0,1] \times [0,1] \cong A(\mathcal{G}) \times \mathbb{R},$$

where the latter is an isomorphism of $\mathcal{C}^{1,0}$ groupoids which is fibrewise the identity. This proves our assertion. □

Corollary 16. *There is a short exact sequence*

$$0 \to \mathrm{C}^*_r(\mathbb{T}\mathcal{G}) \otimes \mathbb{K} \to \mathrm{C}^*_r(\mathbb{C}\mathcal{G}) \to S\mathrm{C}^*_r(A(\mathcal{G})) = \mathrm{C}^*_r(A(\mathcal{G})) \otimes \mathrm{C}^*_r(\mathbb{R}) \to 0.$$

Lemma 17. *If $\mathcal{G}$ is topologically amenable, then so are $\mathbb{W}\mathcal{G}$ and $\mathbb{C}\mathcal{G}$.*

Proof. Retain the previous notation. The obvious continuous surjection $p : \mathcal{TH} \to [0,1]$ factors through r and s. Then p is open when restricted to $\mathcal{TH}^{(0)} = [0,1] \times]-\infty,\infty] \times \mathcal{G}^{(0)}$, and r and s are open since $\mathcal{TH}$ carries a Haar system. Thus, p is open, and defines continuous field of groupoids in the sense of [18, Definition 5.2]. Hence, $\mathcal{TH}$ is topologically amenable if this is the case for the fibres of p, by [18, Corollary 5.6]. The fibre at 0 is $A(\mathcal{G}) \times \mathcal{W}_{\mathbb{R}_{\geqslant 0}}$, which is always amenable. The fibre at $\varepsilon > 0$ is isomorphic to $\mathcal{H}$, so $\mathcal{TH}$ is amenable if $\mathcal{H}$ is. By the same argument, $\mathcal{T}$ is amenable if $\mathcal{G}$ is. But $\mathcal{H}$ is the fibred product of $\mathcal{T}$ and $\mathcal{W}_{\mathbb{R}_{\geqslant 0}}$, so it is amenable if $\mathcal{G}$ is. So, in this case, both $\mathcal{TH}$ and $\mathbb{W}\mathcal{G}$, as a restriction of $\mathcal{H}$, are amenable. Since $\pi : \mathcal{TH} \to \mathcal{CH}$ is proper, the amenability of $\mathcal{CH}$, and hence, of its restriction $\mathbb{C}\mathcal{G}$, also follow. □

4.2. Computation of the suspended Connes–Skandalis map

Now, we can finally compute $y \otimes \tau$, as announced.

Theorem 18. *Let $\mathcal{G}$ be a topologically amenable, locally compact groupoid of class $\mathcal{C}^{1,0}$. Let $\tau \in KK(\mathrm{C}^*_r(A(\mathcal{G})), \mathrm{C}^*_r(\mathcal{G}))$ denote the Connes–Skandalis map for the tangent groupoid $\mathbb{T}\mathcal{G}$, and $y \in KK^1(S, \mathbb{C})$ represent the Wiener–Hopf extension. Then we have*

$$y \otimes \tau = \partial,$$

where $\partial \in KK^1(\mathrm{C}^*_r(A(\mathcal{G})) \otimes \mathrm{C}^*_r(\mathbb{R}), \mathrm{C}^*_r(\mathcal{G}))$ represents the extension $(**)$.

Proof. We retain the notations from the proof of Proposition 15. We have the commutative diagram of strict homomorphisms

$$\begin{array}{ccccc}
\mathbb{W}\mathcal{G} & & \xrightarrow{\phi_1} & & \mathbb{C}\mathcal{G} \\
\downarrow & & & & \downarrow \\
\mathcal{H} & \xrightarrow[(1,\mathrm{id})]{} & \mathcal{T}\mathcal{H} & \xrightarrow[\pi]{} & \mathcal{C}\mathcal{H}.
\end{array}$$

The vertical arrows are closed embeddings. The quotient map π is proper by the proof of Proposition 15. The restriction of π to $1 \times \mathcal{H}$ is injective, and thus, a closed embedding. Hence, the strict homomorphism ϕ_1 induced in the above diagram is a closed embedding, in particular, proper.

It is easy to compute that the following diagram of strict homomorphisms is commutative:

$$\begin{array}{ccccc}
\mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} & \longrightarrow & \mathbb{W}\mathcal{G} & \longleftarrow & A(\mathcal{G}) \times \mathbb{R} \\
{\scriptstyle i_1 \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi_1} & & \| \\
\mathbb{T}\mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} & \longrightarrow & \mathbb{C}\mathcal{G} & \longleftarrow & A(\mathcal{G}) \times \mathbb{R}.
\end{array}$$

Here $i_1 : \mathcal{G} \to \mathbb{T}\mathcal{G}$ is the inclusion at the fibre over 1, and thus induces the $*$-homomorphism $e_1 : \mathrm{C}^*_r(\mathbb{T}\mathcal{G}) \to \mathrm{C}^*_r(\mathcal{G})$. The groupoids involved being amenable by Lemma 17, we may apply Corollary 6 to obtain $\partial_{\mathbb{C}\mathcal{G}} \otimes e_1 = \partial$, where $\partial_{\mathbb{C}\mathcal{G}}$ represents the extension from Corollary 16.

Similarly, we have a commutative diagram of strict homomorphisms:

$$\begin{array}{ccccc}
A(\mathcal{G}) \times \mathcal{W}_{\mathbb{R}_{\geqslant 0}} & & \xrightarrow{\phi_0} & & \mathbb{C}\mathcal{G} \\
\downarrow & & & & \downarrow \\
A(\mathcal{G}) \times \mathcal{W} & \xrightarrow[(0,\mathrm{id})]{} & \mathcal{T}\mathcal{H} & \xrightarrow[\pi]{} & \mathcal{C}\mathcal{H}.
\end{array}$$

Again, the vertical arrows are closed embeddings, as is the restriction of π to $A(\mathcal{G}) \times \mathcal{W}$. Thus, ϕ_0, induced by the above diagram, is a proper strict homomorphism. We have a commutative diagram

$$\begin{array}{ccccc}
A(\mathcal{G}) \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} & \longrightarrow & A(\mathcal{G}) \times \mathcal{W}_{\mathbb{R}_{\geqslant 0}} & \longleftarrow & A(\mathcal{G}) \times \mathbb{R} \\
{\scriptstyle i_0 \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi_0} & & \| \\
\mathbb{T}\mathcal{G} \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0} & \longrightarrow & \mathbb{C}\mathcal{G} & \longleftarrow & A(\mathcal{G}) \times \mathbb{R}.
\end{array}$$

Here, $i_0 : A(\mathcal{G}) \to \mathbb{T}\mathcal{G}$ is the inclusion at the fibre over 0, and induces the $*$-homomorphism and KK equivalence $e_0 : \mathrm{C}^*_r(A(\mathcal{G})) \to \mathrm{C}^*_r(\mathbb{T}\mathcal{G})$. The upper line induces an extension which is represented by

$$\mathrm{id} \otimes y \in KK^1\big(\mathrm{C}^*_r\big(A(\mathcal{G})\big) \otimes \mathrm{C}^*_r(\mathbb{R}), \mathrm{C}^*_r\big(A(\mathcal{G})\big)\big).$$

Applying Corollary 6 entails $\partial_{\mathbb{C}\mathcal{G}} \otimes e_0 = \mathrm{id} \otimes y$. Hence,

$$y \otimes \tau = (\mathrm{id} \otimes y) \otimes \tau = \partial_{\mathbb{C}\mathcal{G}} \otimes e_1 = \partial,$$

which was our claim. □

5. Topological expression of the Connes–Skandalis map

5.1. Naturality of classifying spaces

In order to compute our index in topological terms, we shall be particularly interested in the Connes–Skandalis map in the following situation: Fix a manifold $p : M \to Y$ of class $\mathcal{C}^{1,0}$ over the locally compact, paracompact space Y, and assume that *p is closed*. Consider the category whose objects are groupoids $\mathcal{G}$ of class $\mathcal{C}^{1,0}$ over the fixed base $\mathcal{G}^{(0)} = M$ (cf. Definition 7) and whose arrows are the (strict) groupoid morphisms of class $\mathcal{C}^{1,0}$. In this category, the product of $\mathcal{G}$ with $M \times_Y M$ is

$$\mathcal{G} \times_M (M \times_Y M) = \bigl\{(\gamma, x_1, x_2) \bigm| s(\gamma) = x_1,\ p(x_1) = p(x_2)\bigr\},$$

with structure maps and groupoid composition given by

$$s(\gamma, x_1, x_2) = x_1, \quad r(\gamma, x_1, x_2) = x_2, \quad (\gamma_1, x_1, x_2)(\gamma_2, x_2, x_3) = (\gamma_1\gamma_2, x_1, x_3).$$

The Lie algebroid of $\mathcal{G} \times_M (M \times_Y M)$ is $A(\mathcal{G}) \oplus TM$, the direct sum of vector bundles over M, where TM is the fibrewise tangent bundle of M.

In particular, for any topological vector bundle $E \to Y$, the pullback p^*E is a class $\mathcal{C}^{1,0}$ groupoid $\mathcal{G}$ over M whose Lie algebroid is $p^*E \oplus TM$. It is instructive to note that the strict homomorphism

$$p^*E \times_M (M \times_Y M) \to E : (x_1, \xi, x_2) \mapsto \bigl(p(x_1), \xi\bigr)$$

is a Morita equivalence, although we shall not use this fact directly. From the previous sections, we have a Connes–Skandalis map

$$\tau \in KK\bigl(\mathrm{C}^*_r(p^*E \oplus TM), \mathrm{C}^*_r\bigl(p^*E \times_M (M \times_Y M)\bigr)\bigr).$$

In fact, this is a version ‘with coefficients’ of the topological family index in sense of Atiyah–Singer [4, Section 3]. This fact is well-known in the coefficient-free case (i.e. $E = 0$), although it is does not appear to be as well recorded in the literature, at least in this generality. The parameter- and coefficient-free version can be found in Connes’s book [8] or in [17]. We give a brief discussion of this fact, which also explains what we mean by a ‘topological family index’.

The basic idea is Connes’s realisation (explained in [17]) that the groupoids considered above act properly and freely on $\mathbb{R}^n$ (where the action is obtained along the lines of Atiyah–Singer’s construction), so that the Fourier transform can be used to write down K-theory isomorphisms between these groupoids and certain topological spaces explicitly. The details will become clear presently.

Let $\mathcal{G}$ be a locally compact groupoid with open range map, and let $h:\mathcal{G}\to V$ be a strict homomorphism, where V is some finite-dimensional real inner product space. Define the space $E(\mathcal{G},h)=\mathcal{G}^{(0)}\times V$. Then $\mathcal{G}$ acts on $E(\mathcal{G},h)$ from the right by

$$\bigl(r(\gamma),v\bigr)\gamma=\bigl(s(\gamma),v+h(\gamma)\bigr)\quad\text{for all } \bigl(r(\gamma),v,\gamma\bigr)\in E(\mathcal{G},h)\times_{\mathcal{G}^{(0)}}\mathcal{G}.$$

This action gives rise to the map

$$E(\mathcal{G},h)\times_{\mathcal{G}^{(0)}}\mathcal{G}\to E(\mathcal{G},h)\times E(\mathcal{G},h):\bigl(r(\gamma),v,\gamma\bigr)\mapsto\bigl(s(\gamma),v+h(\gamma),r(\gamma),v\bigr).$$

The action is free (resp. proper) if and only if $(r,h,s):\mathcal{G}\to\mathcal{G}^{(0)}\times V\times\mathcal{G}^{(0)}$ is injective (resp. proper).

Now, consider $B(\mathcal{G},h)=E(\mathcal{G},h)/\mathcal{G}$. This is a locally compact space, and as such, a locally compact (cotrivial) groupoid. Its (trivial) action on $E(\mathcal{G},h)$ is proper and free, and the quotient by this action is $E(\mathcal{G},h)$. Since the range map of $\mathcal{G}$ is open, the canonical projection $\pi_h:E(\mathcal{G},h)\to B(\mathcal{G},h)$ is open and surjective, by [32, Lemma 2.30]. The following lemma characterises when $E(\mathcal{G},h)$ is a Morita equivalence of $B(\mathcal{G},h)$ and $E(\mathcal{G},h)\rtimes\mathcal{G}$. The lemma's proof is straightforward and will therefore be omitted.

Lemma 19. *Let $\mathcal{G}$ be a locally compact groupoid with an open range map. Then the following statements are equivalent.*

(i) *The groupoid $\mathcal{G}$ is principal.*
(ii) *The space $\mathcal{G}^{(0)}$ defines a Morita equivalence $\mathcal{G}\to\mathcal{G}^{(0)}/\mathcal{G}$.*
(iii) *The canonical projection $\pi:\mathcal{G}\to\mathcal{G}^{(0)}/\mathcal{G}$ is proper as a generalised morphism.*
(iv) *The space $\mathcal{G}^{(0)}$ defines a generalised morphism.*

In this case, π is the inverse of $\mathcal{G}^{(0)}$, considered as a generalised morphism.

A simple consequence of Lemma 19 is the following naturality of $E(\mathcal{G},h)$: Let $(\mathcal{G},h)$ and $(\mathcal{G}',h')$ be given, where the groupoids $\mathcal{G}$ and $\mathcal{G}'$ are locally compact with open range maps, and h, h' are strict homomorphisms such that (r,h,s) and (r,h',s) are injective and proper. Then a *morphism of pairs* $\varphi:(\mathcal{G},h)\to(\mathcal{G}',h')$ is a strict homomorphism $\varphi:\mathcal{G}\to\mathcal{G}'$ such that $h'\circ\varphi=h$. Such a morphism of pairs φ gives rise to the continuous map

$$E(\varphi):E(\mathcal{G},h)\to E(\mathcal{G}',h'):(x,v)\mapsto\bigl(\varphi(x),v\bigr)$$

which intertwines the actions of $\mathcal{G}$ and $\mathcal{G}'$:

$$E(\varphi)\bigl(\bigl(r(\gamma),v\bigr)\gamma\bigr)=\bigl(r\bigl(\varphi(\gamma)\bigr),v\bigr)\varphi(\gamma)\quad\text{for all } \gamma\in\mathcal{G},\ v\in V.$$

Hence, $E(\varphi)$ induces a map $B(\varphi):B(\mathcal{G},h)\to B(\mathcal{G}',h')$. Moreover,

$$E(\varphi)\times\varphi:E(\mathcal{G},h)\rtimes\mathcal{G}\to E(\mathcal{G}',h')\rtimes\mathcal{G}'$$

is a strict homomorphism inducing $B(\varphi)$ under the straightforward identification $B(\mathcal{G},h)=E(\mathcal{G},h)/E(\mathcal{G},h)\rtimes\mathcal{G}$. If π_h and $\pi_{h'}$ are the canonical projections, we thus have $\pi_{h'}\circ$

$(E(\varphi)\times\varphi)=B(\varphi)\circ\pi_h$. By Lemma 17, we may take inverses, and the following diagram of generalised morphisms commutes:

$$\begin{array}{ccc}
E(\mathcal{G},h)\rtimes\mathcal{G} & \xrightarrow{E(\varphi)\times\varphi} & E(\mathcal{G}',h')\rtimes\mathcal{G}' \\
{\scriptstyle E(\mathcal{G},h)}\big\uparrow & & \big\uparrow{\scriptstyle E(\mathcal{G}',h')} \\
B(\mathcal{G},h) & \xrightarrow[B(\varphi)]{} & B(\mathcal{G}',h').
\end{array}$$

Since the vertical arrows are Morita equivalences, the horizontal arrows are always simultaneously proper as generalised morphisms. For instance, this is the case if φ is proper as a strict morphism, since the same is then true of $E(\varphi)\times\varphi$. However, in this case, $B(\varphi)$ need not be proper as a strict morphism. For the sake of brevity, we shall write $B(\varphi)^*$ instead of $KK(B(\varphi))$ for the induced KK cycle, even if $B(\varphi)$ is only proper as a generalised morphism.

For any strict homomorphism $h:\mathcal{G}\to V$ from the locally compact groupoid with Haar system $\mathcal{G}$ to the finite-dimensional real inner product space V, an action of V on $C^*_r(\mathcal{G})$ is given by

$$\alpha_h(v)(\varphi)(\gamma)=e^{2\pi i(v|h(\gamma))}\cdot\varphi(\gamma)\quad\text{for all } v\in V,\ \varphi\in\mathcal{C}_c(\mathcal{G}),\ \gamma\in\mathcal{G}.$$

Moreover, we have a $*$-isomorphism $\mathcal{F}_h: C^*_r(\mathcal{G})\rtimes_{\alpha_h}V\to C^*_r(E(\mathcal{G},h)\rtimes\mathcal{G})$,

$$\mathcal{F}_h(\varphi)\big(r(\gamma),v,\gamma\big)=\int_V e^{-2\pi i(v|w)}\varphi(\gamma,w)\,dw\quad\text{for all } \varphi\in\mathcal{C}_c(\mathcal{G}\times V),$$

as is well known (cf. [8, Proposition 7]).

Lemma 20. *The isomorphism $\mathcal{F}_h$ is natural in the following sense: Given a morphism of pairs $\varphi:(\mathcal{G},h)\to(\mathcal{G}',h')$ such that φ is proper and preserves Haar systems, the following diagram of $*$-morphisms commutes:*

$$\begin{array}{ccc}
C^*_r(\mathcal{G}')\rtimes_{\alpha_{h'}}V & \xrightarrow{\varphi^*\otimes\mathrm{id}} & C^*_r(\mathcal{G})\rtimes_{\alpha_h}V \\
{\scriptstyle \mathcal{F}_{h'}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_h} \\
C^*_r(E(\mathcal{G}',h')\rtimes\mathcal{G}') & \xrightarrow[(E(\varphi)\times\varphi)^*]{} & C^*_r(E(\mathcal{G},h)\rtimes\mathcal{G}).
\end{array}$$

Now, fix a real inner product space V with $\dim_{\mathbb{R}}V=2n$. Consider the category Grp/V whose objects are pairs $(\mathcal{G},h)$ where $\mathcal{G}$ is a locally compact groupoid with a Haar system and $h:\mathcal{G}\to V$ is a strict homomorphism such that (r,h,s) is injective and proper, and whose arrows are proper strict homomorphisms $\varphi:\mathcal{G}\to\mathcal{G}'$ preserving Haar systems and satisfying $h=h'\circ\varphi$. Then we have two cofunctors, KK and $KK\circ B$, $\mathrm{Grp}/V\to KK$, given by

$$KK(\mathcal{G},h)=C^*_r(\mathcal{G}),\qquad KK\big(\varphi:(\mathcal{G},h)\to(\mathcal{G}',h')\big)=\varphi^*=KK(\varphi),$$

and

$$(KK \circ B)(\mathcal{G}, h) = \mathcal{C}_0\big(B(\mathcal{G}, h)\big),$$
$$(KK \circ B)\big(\varphi : (\mathcal{G}, h) \to (\mathcal{G}', h')\big) = B(\varphi)^* = KK\big(B(\varphi)\big).$$

By naturality of Thom isomorphisms [6, Proposition 19.3.5], we have a natural isomorphism of functors $\sigma : KK \Rightarrow KK \circ B$ given by

$$\sigma_{\mathcal{G},h} = t_h \otimes \mathcal{F}_h^* KK\big(E(\mathcal{G}, h)\big) \quad \text{where } t_h = t_{\alpha_h} \in KK\big(\mathrm{C}_r^*(\mathcal{G}), \mathrm{C}_r^*(\mathcal{G}) \rtimes_{\alpha_h} V\big)$$

is the Thom element for the action α_h.

5.2. The Connes–Skandalis map is a topological family index

We make the following basic observation, which is essentially a minor reformulation of [3, p. 497]. Let $Z \to Y$ be a class $\mathcal{C}^{1,0}$ family, and $\pi_V : V \to Z$ a $\mathcal{C}^{1,0}$ vector bundle. Choose a $\mathcal{C}^{1,0}$ embedding $i_V : V \to Y \times \mathbb{R}^k$ compatible with a $\mathcal{C}^{1,0}$ embedding of Z. Form the normal bundle NV of V, which is a $\mathcal{C}^{1,0}$ vector bundle over Z. Then the trivial bundle $NV \times \mathbb{R}^k$ is homeomorphic to $\pi_V^*(NV \otimes \mathbb{C})$, and hence, a complex vector bundle over V. In particular, there is a Thom isomorphism $\varphi_V : K_c^*(V) \to K_c^*(NV \times \mathbb{R}^k)$.

Now, let $p : M \to Y$ be a $\mathcal{C}^{1,0}$ family, $\pi_E : E \to Y$ a topological vector bundle. We apply our observation to $Z = M$ and $V = TM \oplus p^*E$. Given embeddings $i_{TM} : TM \to Y \times \mathbb{R}^k$ and $i_E : E \to Y \times \mathbb{R}^\ell$, we may define an embedding $i_V : V \to E \times \mathbb{R}^{k+\ell}$ of V. Thus, we obtain

$$(NTM \oplus p^*NE) \times \mathbb{R}^{k+\ell} \approx \pi_{TM\oplus p^*E}^*\big((NTM \oplus p^*NE) \otimes \mathbb{C}\big),$$

and a Thom isomorphism

$$\varphi_{TM\oplus p^*E} : K_c^*(TM \oplus p^*E) \to K_c^*\big((NTM \oplus p^*NE) \times \mathbb{R}^{k+\ell}\big).$$

Similarly, taking $Z = Y$ and $V = E$, and considering the souped-up embedding $E \to Y \times \mathbb{R}^{2k+\ell}$ given by the embedding of the origin in $\mathbb{R}^{2k}$, we obtain a Thom isomorphism $\varphi_E : K_c^*(E) \to K_c^*(NE \times \mathbb{R}^{2k+\ell})$. Since NTM is open in $Y \times \mathbb{R}^k$, we find that $(NTM \oplus p^*NE) \times \mathbb{R}^{k+\ell}$ is open in $NE \times \mathbb{R}^{2k+\ell}$.

Thus, we obtain a topological family index map (with coefficients in E) à la Atiyah–Singer [4] as $\varphi_E^{-1} \circ \varphi_{TM\oplus p^*E}$. As usual, this object is (as a homomorphism of K-groups or an element of KK theory), independent of the choice of embeddings. We shall presently see that this index coincides with the Connes–Skandalis map for the previously considered $\mathcal{C}^{1,0}$ groupoid $\mathcal{G} = p^*E \times_M (M \times_Y M)$.

To this end, we apply the construction detailed in the previous section. We assume now that p is proper, moreover, that the embeddings i_{TM} and i_E are closed, and write $i_{TM} = (p, i)$ and $i_E = (\pi_E, j)$. Define a homomorphism $h_1 : \mathcal{G} = p^*E \times_M (M \times_Y M) \to \mathbb{R}^{2(k+\ell)}$ by

$$h_1(x_1, \eta, x_2) = \big(i(x_1) - i(x_2), 0, j\big(p(x_1), \eta\big), 0\big) \quad \text{for all } (x_1, \xi, x_2) \in \mathcal{G}.$$

This induces a strict homomorphism $h : \mathbb{T}\mathcal{G} \to \mathbb{R}^{2(k+\ell)}$,

$$h(\tau) = \begin{cases} (T_x i\xi, 0, j(p(x), \eta), 0), & \tau = (x, \xi, \eta) \in TM \oplus p^*E, \\ (\varepsilon^{-1}(i(x_1) - i(x_2)), 0, j(p(x_1), \eta), 0), & \tau = (x_1, \eta, x_2, \varepsilon) \in \mathcal{G} \times]0, 1]. \end{cases}$$

Then (r,h,s) is injective, and from the closedness of i, Ti, and j, it follows it is closed, and hence, proper. Consider the embeddings at $\varepsilon=0$ and $\varepsilon=1$,

$$TM\oplus p^*E = A(\mathcal{G}) \xrightarrow{\;i_0\;} \mathbb{T}\mathcal{G} \xleftarrow{\;i_1\;} \mathcal{G}$$

and define $h_0=h\circ i_0$. Write $\sigma=\sigma_{\mathbb{T}\mathcal{G},h}$, $\sigma_0=\sigma_{A(\mathcal{G}),h_0}$, and $\sigma_1=\sigma_{\mathcal{G},h_1}$. Then we have in KK

$$\sigma\otimes B(i_0)^*=e_{0*}\sigma_0 \quad \text{and} \quad \sigma\otimes B(i_1)^*=e_{1*}\sigma_1,$$

where e_0, e_1 are the evaluations on $\mathrm{C}^*_r(\mathbb{T}\mathcal{G})$ induced by the inclusions i_0, i_1. Moreover, as spaces,

$$B_0=B\big(A(\mathcal{G}),h_0\big)\approx(NTM\oplus p^*NE)\times\mathbb{R}^{k+\ell} \quad \text{and}$$
$$B_1=B(\mathcal{G},h_1)\approx NE\times\mathbb{R}^{2k+\ell}.$$

The second homeomorphism is given by

$$B_1=M\times\mathbb{R}^{2(k+\ell)}/\mathcal{G}\to NE\times\mathbb{R}^{2k+\ell},$$
$$[x,u_1,u_2,v_1,v_2]\mapsto\big(p(x),v_1+E_{p(x)},u_1+i(x),u_2,v_2\big),$$

and similarly for the first. As noted above, B_0 is open in B_1. Denote the open inclusion by ι. Since the proper strict homomorphisms $B(i_0)$ and $B(i_1)\circ\iota$ are homotopic, we find $\boldsymbol{\iota}^*B(i_0)^*=B(i_1)^*$ in KK, where $\boldsymbol{\iota}$ is the $*$-morphism $\mathcal{C}_0(B_0)\to\mathcal{C}_0(B_1)$ induced by ι.

From the definitions, it follows that σ_0 is the KK element representing the topological Thom isomorphism $\varphi_{TM\oplus p^*E}$. We need to see that after applying suitable KK equivalences, the same holds for σ_1.

To that end, define the strict homomorphism

$$\pi:\mathcal{G}\to\mathcal{G}'=\big(\mathbb{R}^n\ltimes\mathbb{R}^n\big)\times E:(x_1,\eta,x_2)\mapsto\big(i(x_1)-i(x_2),i(x_2),p(x_1),\eta\big).$$

Since i is a closed embedding, π is injective and proper. Moreover, it preserves Haar systems if we choose the measure on M defining the Haar system of $M\times_Y M$ to be given by the pull-back of the $\dim M^y$-dimensional Hausdorff measure on $\mathbb{R}^k$ along i. We denote the $*$-morphism $\mathrm{C}^*_r(\mathcal{G}')\to\mathrm{C}^*_r(\mathcal{G})$ induced by π by the letter $\boldsymbol{\pi}$.

The homomorphism

$$h_1':\mathcal{G}'\to\mathbb{R}^{2(k+\ell)}:(u,v,e)\mapsto\big(u,0,j(e),0\big)$$

is closed, (r,h_1',s) is injective, and $h_1=h_1'\circ\pi$. We find that

$$B\big(\mathcal{G}',h_1'\big)=\big(\big(Y\times\mathbb{R}^k\big)\times\mathbb{R}^{2(k+\ell)}\big)/\mathcal{G}'=NE\times\mathbb{R}^{2k+\ell}=B_1$$

via

$$B\big(\mathcal{G}',h_1'\big)\to NE\times\mathbb{R}^{2k+\ell}:[y,u,v_1,v_2,w_1,w_2]\mapsto(y,w_1+E_y,v_1,v_2,w_2),$$

and that $B(\pi)=\mathrm{id}_{B_1}$.

We have $\mathrm{C}^*_r(\mathcal{G}') = \mathrm{C}^*_r(E) \otimes \mathbb{K}$ where $\mathbb{K} = \mathrm{C}^*_r(\mathbb{R}^k \ltimes \mathbb{R}^k)$. Write $h'_1 = j \times \mathrm{pr}_1$. There is a canonical isomorphism

$$\mathrm{C}^*_r\big(\mathbb{R}^k \ltimes \mathbb{R}^k\big) \rtimes_{\alpha_{\mathrm{pr}_1}} \mathbb{R}^{2k} \to \mathcal{C}_0\big(\mathbb{R}^{2k}\big) \rtimes \mathbb{R}^{2k} = \mathbb{K}$$

where the latter action is regular (i.e. dual to the trivial action). Because the Thom element in $KK(\mathcal{C}_0(\mathbb{R}^{2k}), \mathcal{C}_0(\mathbb{R}^{2k}) \rtimes \mathbb{R}^{2k})$ is the Bott element [6, Example 19.3.4(ii)], composition of this isomorphism with $t_{\alpha_{\mathrm{pr}_1}}$ is the identity in $KK(\mathbb{K}, \mathbb{K}) = \mathbb{Z}$.

Thus, the Thom element $t'_1 = t_{\alpha_{h'_1}}$ defining $\sigma'_1 = \sigma_{\mathcal{G}', h'_1}$ satisfies $\pi^* t_1 = t'_1$ where $t_1 = t_{\alpha_{h_1}}$, and on the other hand, σ'_1 induces on K-theory the topological Thom isomorphism φ_E. On the level of KK theory,

$$\tau = \sigma_0 \otimes \sigma_1^{-1} = \sigma_0 \otimes \boldsymbol{\pi}_*\big(\sigma'_1\big)^{-1} = \boldsymbol{\pi}_*\big(\varphi_{TM \oplus p^*E} \otimes \varphi_E^{-1}\big).$$

Thus, we have proved the following theorem.

Theorem 21. *Let $E \to Y$ be a topological vector bundle and $p: M \to Y$ a class $\mathcal{C}^{1,0}$ manifold such that p is proper. Then the Connes–Skandalis map τ associated to the tangent groupoid of $\mathcal{G} = p^*E \times_M (M \times_Y M)$ is*

$$\tau = \boldsymbol{\pi}_*\big(\varphi_{TM \oplus p^*E} \otimes \varphi_E^{-1}\big)$$

where $\boldsymbol{\pi}: \mathrm{C}^*_r(E) \otimes \mathbb{K} \to \mathrm{C}^*_r(\mathcal{G})$ is a KK equivalence and $\varphi_{TM \oplus p^*E} \otimes \varphi_E^{-1}$ is the Atiyah–Singer topological family index for $p^*E \oplus TM$.

6. Proof of the Wiener–Hopf index formula

In this section, we shall prove the index formula for the Wiener–Hopf C*-algebra explained in the introduction. Recall from the introduction and from [2] that on the level of KK theory, this was an expression of the KK^1 element ∂_j representing the extension

$$0 \longrightarrow \mathrm{C}^*_r(\mathcal{G}|U) \longrightarrow \mathrm{C}^*_r(\mathcal{G}) \longrightarrow \mathrm{C}^*_r(\mathcal{G}|F) \longrightarrow 0$$

where $\mathcal{G} = \mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})$ is the restriction of the Wiener–Hopf groupoid to the locally closed invariant subset $U_{j+1} \setminus U_{j-1} = Y_j \cup Y_{j-1}$, and the closed invariant subset $F = Y_j = U_j \setminus U_{j-1}$ is the stratum of the Wiener–Hopf compactification $\overline{\Omega}$ corresponding to the orbit type of P_j, and consists of all points of $\overline{\Omega}$ lying above P_j, the set of n_{d-j}-dimensional faces.

Our proof proceeds in three steps:

(1) Construct a class $\mathcal{C}^{1,0}$ groupoid $\mathcal{D}_j$, which is of the form

$$\mathcal{D}_j = p^*E \times_M (M \times_Y M)$$

for some $\mathcal{C}^{1,0}$ manifold $M \to Y \subset P_{j-1}$ and some vector bundle $E \to Y$.

(2) Construct a proper strict homomorphism $\mathbb{W}\mathcal{D}_j \to \mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})$, where $\mathbb{W}\mathcal{D}_j$ is the 'suspended tangent groupoid' constructed in Section 4.1. By naturality, ∂_j is expressed in terms of the standard extension belonging to $\mathbb{W}\mathcal{D}_j$.

(3) Now we are in a position to apply the KK theoretical yoga explained above. By the results of 4.2, we relate the extension induced by $\mathbb{W}\mathcal{D}_j$ to the Connes–Skandalis map pertaining to the tangent groupoid of $\mathcal{D}_j$. Because of the particular form of the latter groupoid, the computations from Section 5 furnish the KK theoretical index formula.

6.1. Embedding the fibres of $\mathcal{P}_j$

Returning to the study of the Wiener–Hopf groupoid $\mathcal{W}_\Omega$, we shall take the first of the three steps mentioned in this section's introduction, comprising the proof of the Wiener–Hopf index formula.

Recall to that end the notation and the notions from [2]. In particular, Ω is a pointed, solid, closed convex cone in the finite dimensional Euclidean vector space X $(= \mathbb{R}^n)$. Let $\{0 = n_0 < n_1 < \cdots < n_d = n\}$ be the set of dimensions of the faces of the dual cone Ω^*, ordered increasingly (compare the introduction). Set

$$P_j = \{F \subset \Omega^* \mid \dim F = n_{d-j},\ F \text{ face of } \Omega^*\}.$$

We shall assume that the cone Ω^* is *facially compact*, i.e. all the spaces P_j of are compact in the topology induced from the space $\mathbb{F}(X)$ of all closed subsets of X, endowed with the Fell topology. Consider

$$\mathcal{P}_j = \big\{(E, F) \times P_{j-1} \times P_j \bigm| E \supset F\big\},$$

which is a compact subspace of $P_{j-1} \times P_j$. We have projections $\xi : \mathcal{P}_j \to P_{j-1}$ and $\eta : \mathcal{P}_j \to P_j$.

We shall prove that under suitable assumptions, $\mathcal{P}_j$ is in a natural way a $\mathcal{C}^{1,0}$ manifold with respect to the projection ξ onto the base $\xi(\mathcal{P}_j) \subset P_{j-1}$, and moreover, we will give a specific Euclidean embedding. It is important to stress that the mere existence of a $\mathcal{C}^{1,0}$ structure is not sufficient for our needs: to ultimately evaluate the Atiyah–Singer family index map associated to the groupoid to be constructed, a concrete Euclidean embedding has to be at hand.

The $\mathcal{C}^{1,0}$ manifold $\mathcal{P}_j$ forms the basic building block for our sought-for $\mathcal{C}^{1,0}$ groupoid $\mathcal{D}_j$. Indeed, as groupoids, $\mathcal{D}_j = \xi^* \Sigma_{j-1} \times_{P_{j-1}} (\mathcal{P}_j \times_{P_{j-1}} \mathcal{P}_j)$. Here, Σ_{j-1} is the vector bundle which appears in the construction of the composition series of the Wiener–Hopf C*-algebra (compare the introduction).

To introduce a $\mathcal{C}^{1,0}$ structure on $\mathcal{P}_j$, we first construct suitable Euclidean embeddings of the fibres of ξ, and show that these admit tangent spaces of fixed dimension at every point.

To that end, we adopt the following notation: For $A \subset X$, let $\langle A\rangle$ denote the linear span of A, $A^\perp$ the orthogonal complement, A^* the dual cone, and $A^\circledast = A^* \cap \langle A\rangle$ the relative dual cone. Then, for $(E, F) \in \mathcal{P}_j$, define

$$E_{1/2}(F) = F^\perp \cap \big(F^\perp \cap E^\circledast\big)^\perp.$$

We shall be using some convex geometry in the sequel, and refer the reader to [13, Chapter I] for a thorough treatment.

Remark 22. Although the notation $E_{1/2}(F)$ may seem somewhat arbitrary, we have chosen it to stress the analogy to the case of symmetric cones. We briefly explain its meaning in this case, and refer to [11] for details on this topic.

Indeed, assume that $\Omega = \Omega^*$ is a symmetric cone in the Euclidean vector space X, i.e. a self-dual cone whose interior admits a transitive action of a subgroup of $\mathrm{GL}(X)$. Then, X is in an essentially unique way a Euclidean Jordan algebra. If $c \in X$ is an idempotent, then we have the orthogonal 'Peirce decomposition'

$$X = X_0(c) \oplus X_{1/2}(c) \oplus X_1(c) \quad \text{where } X_\lambda(c) = \ker\bigl(L(c) - \lambda\bigr)$$

are the eigenspaces for the action of c on X by left multiplication, called 'Peirce spaces', and $\dim X_1(c)$ is called the rank of c.

The faces of Ω are of the form $E = \Omega \cap X_0(e)$, for idempotents $e \in X$. Now assume that $E \supsetneq F$ are faces such that F has minimal codimension in E. Then $E = \Omega \cap X_0(e)$, $F = \Omega \cap X_0(c)$ where e, c are idempotents with $c - e$ an idempotent of rank one. The dual face of F in the self-dual cone E is the extreme ray $E \cap X_1(c - e)$. The Euclidean Jordan algebra $\langle E \rangle = X_0(e)$ has the Peirce decomposition

$$X_0(e) = X_0(c) \oplus \bigl(X_0(e) \cap X_{1/2}(c - e)\bigr) \oplus X_1(c - e)$$

w.r.t. $c - e \in X_0(e)$. Here, $X_0(c) = \langle F \rangle$ is the linear span of the face F, $X_1(c - e)$ is the line spanned by dual cone of F in E, and $X_0(e) \cap X_{1/2}(c - e)$ is the intersection of the orthogonal complements of the two former spaces.

Moreover, the set of all proper faces F of E of minimal codimension corresponds exactly to the set S of rank one idempotents of the Euclidean Jordan algebra $X_0(e)$. The set S is the 'Shilov boundary' of the bounded domain whose Siegel realisation is $X_0(e) + i\Omega \cap X_0(e)$. It is a compact submanifold of $X_0(e)$, and the tangent space at an idempotent $f \in S$ is precisely $X_0(e) \cap X_{1/2}(f)$.

Returning to the general case of a no longer necessarily symmetric cone Ω, we shall see that under mild conditions, the Peirce decomposition that we have explained for the case of symmetric cones has a counterpart for any pair $(E, F) \in \mathcal{P}_j$, the fibre of ξ over E corresponds exactly to a compact set of generators of extreme rays in $E^\circledast$, and the space $E_{1/2}(F)$ can be interpreted as the tangent space of this set at the point corresponding to F. First, let us explain what kind of geometric conditions have to be imposed on Ω^*.

Definition 23. A face E of a cone C will be called *modular* if it contains a face F which is maximal dimensional relative C, i.e.

$$\dim F = \max\{\dim G \mid G \text{ a face of } C,\ \dim G < \dim E\}.$$

Remark 24. Applied to the cone $C = \Omega^*$: For $j \geqslant 1$, $\xi(\mathcal{P}_j)$ is the collection of a modular faces of dimension n_{d-j+1}.

Lemma 25. *Let E be a pointed cone, and $F \subsetneq E$ a maximal face. Then F is exposed in E (i.e. equals the intersection of E with a supporting hyperplane), and the relative dual face $F^\perp \cap E^\circledast \neq 0$ is an extreme ray of $E^\circledast$ if and only if it contains an exposed ray of $E^\circledast$. In this case, we denote the normalised generator of $F^\perp \cap E^\circledast$ by $e_F = e_F(E)$.*

Proof. There exists a face $F \subsetneq G \subset E$ such that F is exposed in G. Since F is maximal, $G = E$, and F is exposed in E. The dual face $F^\perp \cap E^\circledast$ is an exposed face of $E^\circledast$. So, if it is an extreme ray, then it is exposed.

Conversely, let $G \subset F^\perp \cap E^\circledast$ be an exposed ray of $E^\circledast$. Then $F \subset E \cap G^\perp$. Of course, $E \cap G^\perp \neq E$, since $\dim G^\perp < \dim E$. The maximality of F entails $F = E \cap G^\perp$. Since G is exposed in $E^\circledast$, it follows that $G = F^\perp \cap E^\circledast$, by [13, Proposition I.2.5], and hence, an extreme ray. □

Definition 26. Let E be a modular face of a cone C. We shall say that E is *smooth* if the relative interior of every face $F \subsetneq E$, maximal dimensional relative C, consists of regular points of E. Here, a point of E is called regular if it admits a unique supporting hyperplane, cf. [13, Definition I.2.24].

We shall say that a cone is *locally smooth* if all of its modular faces are smooth.

Remark 27. Let $E \subset C$ be a modular face. By Lemma 25 and [13, Proposition I.2.25], E is smooth if and only if for all faces $F \subset E$ maximal dimensional relative C, all the extreme rays of the relative dual face $F^\perp \cap E^\circledast$ are exposed in $E^\circledast$. In particular, if $E^\circledast$ is facially exposed (i.e. all faces are exposed), then E is smooth. (However, the condition that all extreme rays of a cone be exposed does not imply that all faces are exposed.)

Corollary 28. *Let the cone Ω be such that Ω^* is locally smooth. Then for each $1 \leqslant j \leqslant d$ there is a well-defined injection*

$$\mathcal{P}_j \to P_{j-1} \times X : (E, F) \mapsto \big(E, e_F(E)\big).$$

We conclude this subsection by a brief digression to gauge the generality of the condition of local smoothness.

Lemma 29. *The following classes of cones are locally smooth and have locally smooth dual cones:*

(i) *Polyhedral cones,*
(ii) *Lorentz cones, and*
(iii) *homogeneous cones, in particular, symmetric cones.*

Proof. (i) The dual of a polyhedral cone is polyhedral [13, Corollary I.4.4]. Any face of a polyhedral cone is polyhedral, and polyhedral cones are facially exposed.

(ii) The dual of a Lorentz cone is a Lorentz cone, and its non-zero proper faces are all exposed extreme rays, cf. [13, Proposition I.4.11].

(iii) The dual of a homogeneous cone is itself homogeneous [10, Satz 4.3]. Homogeneous cones are facially exposed [30, Theorem 3.6]. So, it remains to see that all faces of homogeneous cones are homogeneous. To see this, we recall Rothaus's [28] inductive construction of homogeneous cones. If U, V are finite-dimensional vector spaces, $K \subset V$ a closed convex cone, and $B : U \times U \to V$ is a bilinear map, then we say that B is *K-positive* if $B(u) = B(u, u) \in K \setminus 0$ for all $u \in U \setminus 0$. Given such data, the *Siegel cone*

$$C(K, B) = \big\{(u, v, t) \in U \times K \times \mathbb{R}_{\geqslant 0} \,\big|\, tv - B(u) \in K\big\}$$

is a closed convex cone in $U \times V \times \mathbb{R}$. If K is homogeneous, then the bilinear map B is called *homogeneous* if for some subset $G \subset GL(V)$ acting transitively on K°, and for all $g \in G$, there exist elements $g_U \in \mathrm{GL}(U)$ such that

$$gB(u, u') = B\big(g_U(u), g_U(u')\big) \quad \text{for all } u, u' \in U.$$

If K and B are homogeneous, then $C(K, B)$ is homogeneous. Conversely, any homogeneous cone C can be obtained by this procedure from a homogeneous cone K of dimension strictly less than $\dim C$. Thus, all homogeneous cones can be constructed inductively from the real half-line, cf. [35,28,30]. All faces of the half-line are homogeneous. So, we need to see that this property remains stable under the inductive step of the above construction.

By [30, Theorem 3.2], the set $\operatorname{ext} C(K, B)$ of generators of extreme rays for a homogeneous Siegel cone is given as follows:

$$\operatorname{ext} C(K, B) = \big\{(u, v, t) \in U \times K \times \mathbb{R}_{\geqslant 0} \bigm| tv = B(u),\ t = 0 \Rightarrow v \in \operatorname{ext} K\big\}.$$

Now suppose that $E \subset C(K, B)$ is a face, and let $F = (0 \times K \times 0) \cap E$, which defines a face of K. Then, by [30, proof of Theorem 3.6], either E contains only extreme generators of the form $(0, v, 0)$, $v \in \operatorname{ext} K$, or we have the equivalence $B(u) \in F \Leftrightarrow (u, B(u), 1) \in E$. Any cone is the positive linear span of its extreme generators, by [13, Theorem I.3.16]. Thus, in the former case, $E = F$, in which case we are done by our inductive hypothesis. In the latter case,

$$\begin{aligned}\operatorname{ext} E &= \big\{(u, v, t) \in \operatorname{ext} C(K, B) \bigm| B(u) \in F\big\} \\ &= \big\{(u, v, t) \in U \times F \times \mathbb{R}_{\geqslant 0} \bigm| tv = B(u),\ t = 0 \Rightarrow v \in \operatorname{ext} F\big\}.\end{aligned}$$

Then, define $U_F = \{u \in U \mid B(u) \in F\}$. This set is a linear subspace of U. Indeed, if $u, v \in U_F$, then

$$\frac{1}{2} \cdot \big(B(u + v) + B(u - v)\big) = B(u) + B(v) \in F + F = F.$$

Since $B(u + v), B(u - v) \in K$ and $F \subset K$ is a face, it follows that $B(u \pm v) \in F$, so $u \pm v \in U_F$. Since U_F is invariant under positive scalar multiples, it is indeed a linear subspace.

But then $B_F = B|(U_F \times U_F)$ is F-positive, and $C(F, B_F)$ makes sense. In fact, $\operatorname{ext} E$ is the set of extreme generators of $C(F, B_F)$ by our previous calculations, as soon as we have established that B_F is homogeneous. In that case, it will follow that $E = C(F, B_F)$, both being the positive linear spans of their extreme generators, thereby establishing the homogeneity of E.

So, let us check that B_F is homogeneous. By our inductive assumption, F is homogeneous, and we may choose a subset $G \subset \mathrm{GL}(\langle F \rangle)$ acting transitively on F°. Since B is homogeneous, to $g \in G$, there exists $g_U \in \mathrm{GL}(U)$ such that $gB(u, u') = B(g_U(u), g_U(u'))$. If $u \in U_F$, then

$$B\big(g_U(u)\big) = g\big(B(u)\big) \in g(F) = F,$$

so g_U leaves U_F invariant, and B_F is homogeneous. This proves our claim. □

Remark 30. Although Ω^* is certainly locally smooth in the most interesting cases, let us note in passing the curious asymmetry of this condition. Indeed, consider the three-dimensional, facially

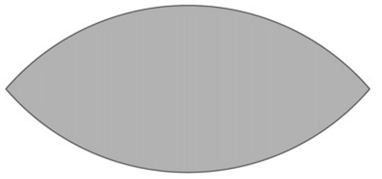

Fig. 1. The section of a three-dimensional cone which is not locally smooth but whose dual cone is.

exposed cone C which has the 'almond shaped' section illustrated in Fig. 1. C itself is not locally smooth, since it admits several supporting hyperplanes at the tips of the almond. The dual cone C^*, however, is locally smooth since its non-zero and proper modular faces are polyhedral.

6.2. *A $\mathcal{C}^{1,0}$ structure on $\mathcal{P}_j$*

Having verified that local smoothness of the dual cones obtains for large classes of cones, we shall in the sequel always *assume that Ω^* is locally smooth*. In particular, Lemma 25 allows for the definition of extremal generators $e_F(E) = e_F \in E^{\circledast}$ for any $(E, F) \in \mathcal{P}_j$, and by Corollary 28, this defines a set-theoretical embedding of the fibres of ξ.

We shall show that the embedding is in fact fibrewise bi-Lipschitz, and that the image is a closed $\mathcal{C}^{1,0}$ submanifold of $\mathcal{P}_j \times X$. To accomplish this, we apply a simple but effective idea used by Pugh in his dynamical systems proof of the Cairns–Whitehead Theorem [25, Proof of Proposition 3.3]. Namely, it is a basic fact from analysis that if the derivative of a Lipschitz map g (which exists almost everywhere due to Rademacher's Theorem) can be extended continuously, then g is $\mathcal{C}^1$. Thus, if $\xi : \mathcal{P}_j \to P_{j-1}$ is given a fibrewise Lipschitz manifold structure, it is sufficient to show that the family of the tangent spaces which exist almost everywhere in every Lipschitz chart, extends to a continuous map to an appropriate Grassmannian. It follows that the fibres have regular local $\mathcal{C}^1$ parametrisations.

The main part of the proof will in fact consist in establishing the Lipschitz structure.

Proposition 31. *The map*

$$\mathbf{e} : \mathcal{P}_j \to X : (E, F) \mapsto e_F(E)$$

is a closed embedding and bi-Lipschitz on every fibre of $\xi : \mathcal{P}_j \to P_{j-1}$, locally uniformly with respect to the fibres. Here, the metric on $\mathcal{P}_j$ is the box metric for a metric on the space $P = \bigcup_{i=0}^{d} P_i$ inducing the Fell topology on this set.

In the proposition's proof, we shall need a metric which defines the Fell topology on the set $\mathcal{C}$ of all closed convex cones in X. For all closed subsets $A, B \subset X$, define the excess functional $e(A, B) = \sup\{\operatorname{dist}(a, B) \mid a \in A\}$, and set

$$h(A, B) = \max\bigl(e(A \cap \mathbb{B}, B), e(B \cap \mathbb{B}, A)\bigr) \quad \text{for all } A, B \in \mathcal{C},$$

where $\mathbb{B}$ is the unit ball in X. If $a \in A \cap \mathbb{B}$, then the projection $b = \pi_B(a)$ onto B has $\|b\| \leqslant \|a\| \leqslant 1$, so $\operatorname{dist}(a, B) = \operatorname{dist}(a, B \cap \mathbb{B})$. Hence, $h = H$, where

$$H(A, B) = \max\bigl(e(A \cap \mathbb{B}, B \cap \mathbb{B}), e(B \cap \mathbb{B}, A \cap \mathbb{B})\bigr) \quad \text{for all } A, B \in \mathcal{C}.$$

Since $H(A,B)$ is the Hausdorff distance of $A\cap\mathbb{B}$ and $B\cap\mathbb{B}$, h defines a metric. By [5, Exercise 5.1.10, Lemma 7.2.6], the topology induced on $\mathcal{C}$ by the distance function h coincides with the subspace topology from $\mathbb{F}(X)$. Moreover, the Walkup–Wets Isometry Theorem [5, Theorem 7.2.9] shows that the map $C\mapsto C^*$ is an isometry for h.

Proof of Proposition 31. Observe that the map $\mathbf{e}$ has closed graph. In fact, given sequences $(E_k,F_k)\to(E,F)$ and $e_{F_k}(E_k)\to e$, then by [2, Proposition 2.2.8] and continuity of polarity [5, Corollary 7.2.12], we have $\langle E_k\rangle\to\langle E\rangle$, $F_k^\perp\to F^\perp$, and $E_k^*\to E^*$. From the definition of Painlevé–Kuratowski convergence (cf. [2, Section 2.1]), we conclude

$$e\in\langle E\rangle\cap E^*\cap F^\perp=\mathbb{R}_{\geqslant 0}\cdot e_F(E).$$

Since e is a unit vector, we have $e=e_F(E)$. Thus, $\mathbf{e}$ indeed has closed graph. Because $\mathcal{P}_j$ and the unit sphere of X are compact, it follows that $\mathbf{e}$ is continuous and closed.

Now, to the bi-Lipschitz continuity on every fibre. Fix $(E,F)\in\mathcal{P}_j$. Because $\mathbf{e}$ is continuous and $e_F(E)$ is a unit vector, there exists an open neighbourhood $U_{E,F}\subset\mathcal{P}_j$ of (E,F), such that

$$\bigl(e_{H_1}(G_1)\big|e_{H_2}(G_2)\bigr)>0\quad\text{for every }(G_j,H_j)\in U_{E,F},\ j=1,2.$$

Let $(G,H_j)\in U_{E,F}$, $j=1,2$. Writing $e_j=e_{H_j}(G)$, we obtain

$$h(H_1,H_2)=h\bigl(H_1^*\cap\langle G\rangle,H_2^*\cap\langle G\rangle\bigr)=\bigl\|e_1-(e_1|e_2)\cdot e_2\bigr\|=\sqrt{1-(e_1|e_2)^2},$$

because $C\mapsto C^*$ is an isometry and we may consider h relative to $\langle G\rangle$. Since $(e_1|e_2)\in[0,1]$, we have

$$\sqrt{2}\cdot\sqrt{1-(e_1|e_2)^2}\geqslant\|e_1-e_2\|=\sqrt{2-2(e_1|e_2)}\geqslant\sqrt{1-(e_1|e_2)^2}.$$

Thus, the map $\mathbf{e}$ is bi-Lipschitz when restricted to $U_{E,F}\cap\xi^{-1}(G)$, with Lipschitz constants independent of $G\in\xi(U_{E,F})$. By compactness, we obtain global Lipschitz conditions. □

Theorem 32. *The image of $\mathcal{P}_j$ under $(\xi,\mathbf{e})$ is a closed $\mathcal{C}^{1,0}$ submanifold of $\xi(\mathcal{P}_j)\times X$ (which is a $\mathcal{C}^{1,0}$ manifold over $\xi(\mathcal{P}_j)$ with respect to the first projection). This defines a $\mathcal{C}^{1,0}$ manifold structure on $\mathcal{P}_j$ over $\xi(\mathcal{P}_j)\subset P_{j-1}$ such that the fibrewise tangent space at $(E,F)\in\mathcal{P}_j$ is $E_{1/2}(F)$.*

The theorem's proof follows the basic strategy outlined at the beginning of the subsection and decomposes into a sequence of lemmata.

Lemma 33. *Let $(E,F),(E,F_k^i)\in\mathcal{P}_j$, $F_k^i\to F$, $i=0,1$, and $\varepsilon_k\to 0+$. Whenever the limit $v=\lim_k\varepsilon_k^{-1}\cdot(e_{F_k^0}-e_{F_k^1})$ exists, it lies in $E_{1/2}(F)$.*

Proof. First, we have $v\perp F$. Indeed, write $e_k^i=e_{F_k^i}$. If $f\in F$, then there exist $f_k^j\in F_k^i$ such that $f_k^i\to f$. In particular,

$$(-1)^{1-i}\cdot(v|f)=(-1)^{1-i}\cdot\lim_k\varepsilon_k^{-1}\cdot\bigl(e_k^0-e_k^1\big|f_k^i\bigr)=\lim_k\varepsilon_k^{-1}\cdot\bigl(e_k^{1-i}\big|f_k^i\bigr)\geqslant 0,$$

since $e_k^{1-i} \in E^{\circledast} \subset \Omega$ and $f_k^i \in F_k^i \subset E$, so $(v|f) = 0$. This shows that $v \perp F$, as desired. Moreover, it is clear that $v \in \langle E \rangle$.

It remains to prove that $v \perp e_F$. This is seen similarly, namely

$$(v|e_F) = \lim_k \varepsilon_k^{-1} \cdot \left(e_k^0 - e_k^1 \middle| e_k^i\right) = (-1)^i \cdot \lim_k \varepsilon_k^{-1} \cdot \left(1 - \left(e_k^0 \middle| e_k^1\right)\right),$$

and thus $(v|e_F) = -(v|e_F) = 0$. This proves our assertion. □

Lemma 34. *Given $(E, F) \in \mathcal{P}_j$, the map $p = p_{E_{1/2}(F)}$ is injective in a neighbourhood U of $e_F = e_F(E)$ in $\mathbf{e}(\xi^{-1}(E))$.*

Proof. To establish the injectivity of p, assume, seeking a contradiction, that for each neighbourhood $U \subset \mathbf{e}(\xi^{-1}(E))$ of e_F, there exist $y_1, y_2 \in U$, $y_1 \neq y_2$, such that $y_1 - y_2 \in E_{1/2}(F)^{\perp}$. Then there are sequences $y_j^k \in \mathbf{e}(\xi^{-1}(E))$, $j = 1, 2$, such that

$$0 < \left\| y_1^k - y_2^k \right\| \leqslant \frac{1}{k}, \quad y_1^k - y_2^k \perp E_{1/2}(F).$$

Passing to a subsequence, the limit $v = \lim_k \|y_1^k - y_2^k\|^{-1}(y_1^k - y_2^k)$ exists. Then $v \in E_{1/2}(F)^{\perp}$, and by Lemma 33, $v \in E_{1/2}(F)$. This is a contradiction, since $\|v\| = 1$. □

Lemma 35. *Let $K \subset \mathbb{R}^n$ be a compact subset, $B \subset \mathbb{R}^m$ a closed ball centred at the origin. Assume that $f, g : K \to B$ are continuous, and that for all $x \in K$, we have $f(x) = \lambda \cdot g(x)$ for some $\lambda > 0$ (depending on x). If g has convex fibres and $g(K) = B$, then $f(K)$ contains $\delta \cdot B$ for some $\delta > 0$.*

Proof. In fact, there is no restriction in assuming that B is the unit ball. The map g is continuous and open, and its fibres are compact convex sets. Thus, by Michael's Selection Theorem [21, Theorem 3.1′], g has a continuous section, s, say.

Let $x \in B$, $\|x\| = 1$. Then $(f \circ s)(0) = 0$ and $(f \circ s)([0, x]) \subset [0, x] = B \cap \mathbb{R}_{\geqslant 0} \cdot x$. Since $f \circ s$ has some non-zero value on $[0, x]$, because g does, we deduce that

$$\delta_x = \operatorname{diam}(f \circ s)\big([0, x]\big) > 0,$$

and $[0, \delta_x \cdot x] = (f \circ s)([0, x]) \subset f(K)$. It is clear that $x \mapsto \delta_x$ is lower semi-continuous and thus assumes its infimum. Thus, there exists $\delta > 0$ such that $[0, \delta x] \subset f(K)$ for all $x \in B$, $\|x\| = 1$. Hence, $\delta \cdot B \subset f(K)$. □

Remark 36. Let $C \subset \mathbb{R}^n$ be a closed convex set, $U_0 \subset \mathbb{R}^n$ a linear subspace, $u \in U_0^{\perp}$, $U = U_0 + u$, and $x \in \partial(C \cap U)$. If $y \in X$ is an inner normal to C in x, i.e. $(x|y) = \inf_{z \in C}(z|y)$, then $p_{U_0}(y)$ is an inner normal to $p_{U_0}(C \cap U) = C \cap U - u$ in $p_{U_0}(x) = x - u$. In fact, by the Projection Theorem, $(z|y) = (z - u|p_{U_0}(y))$.

Lemma 37. *Let $(E, F) \in \mathcal{P}_j$. The map $p = p_{E_{1/2}(F)} : \mathbf{e}(\xi^{-1}(E)) \to E_{1/2}(F)$ is open in a neighbourhood of $e_F = e_F(E)$.*

Proof. The map p is certainly open onto its image, so it suffices to show that the image contains some ball.

Denote by $\partial_{\max} E$ the set of all those boundary points of E which lie in the relative interior of a face of E which is maximal dimensional relative Ω^*. Then $\partial_{\max} E$ is an open subset of ∂E, and consists of regular points of E. In particular, the Gauß map $\nu : \partial_{\max} E \to X$ which to $x \in \partial_{\max} E$ associates the unique inner unit normal to E in x, is well defined. The fibre $\xi^{-1}(E)$ is precisely the quotient of $\partial_{\max} E$ by the equivalence relation induced by ν, and ν drops to the map $\xi^{-1}(E) \to \mathbf{e}(\xi^{-1}(E)) : G \mapsto e_G(E)$.

Hence, it is sufficient to see that the image of the restriction of $p \circ \nu$ to some neighbourhood of a point in the relative interior of F contains some ball. Fix $x \in F^\circ$ and let $e = e_F(E)$. Then the plane $\langle x, e\rangle$ intersects $E^{\circledast\circ} \cap E^\circ$ in some point h^* such that $(h^*|x) = 1$.

Let $S_0 = h^{*\perp}$ and $S = S_0 + h$ where $h = \|h^*\|^{-2} \cdot h^*$. Then $E \cap S$ is a compact convex base of E. Since $h \in \langle x, e\rangle \subset E_{1/2}(F)^\perp$, we have that $E_{1/2}(F) \subset S_0$.

Since $h \in E^{\circledast\circ}$ and $E^{\circledast\circ} \cap F^\perp = \emptyset$, $p_{S_0}|F^\perp$ is a linear isomorphism of $F^\perp$ onto $T_0 = p_{S_0}(F^\perp)$. Thus, $E_{1/2}(F) \subset T_0$ is a hyperplane, and p factors through p_{T_0}. Let $T = T_0 + \tilde{x} \subset S_0$. Then $\tilde{x} = p_{S_0}(x) = x - h$ is an exposed point of the compact convex set $C = (E - h) \cap T$. Indeed, $\tilde{e} = p_{S_0}(e)$ is an inner normal at $\tilde{x}$ by Remark 36, so $\tilde{x} + E_{1/2}(F) = \tilde{x} + S_0 \cap \tilde{e}^\perp$ is a supporting hyperplane for C, and $C \cap (\tilde{x} + E_{1/2}(F)) = \tilde{x}$.

Let $\partial_{\max} C = T \cap (\partial_{\max} E - h)$. Then the Gauß map $\nu_0 : \partial_{\max} C \to T_0$ which to any point y associates the unique unit inner normal to C in y, is well-defined. We claim that for any $y \in \partial_{\max} C$ sufficiently close to $\tilde{x}$, there exists $\lambda > 0$ such that $(p \circ \nu)(y + h) = \lambda \cdot (p \circ \nu_0)(y)$. If this is the case and the image of $p \circ \nu_0$ on some compact subset of $\partial_{\max} C$ contains some ball, then by Lemma 35, the same is true for the image of $p \circ \nu$. Indeed, the fibres of ν_0 are faces of C, and p is locally injective by Lemma 34.

By Remark 36, for all $y \in \partial_{\max} C$, we have $p_{T_0}(\nu(y + h)) = \lambda \cdot \nu(y)$ for some $\lambda \geqslant 0$, and we may apply p on both sides of the equation. If $y = \tilde{x} = x - h$, then $\nu(x) = e \in F^\perp$, and $p_{T_0}|F^\perp$ is an isomorphism, so $p_{T_0}(\nu(x)) \neq 0$. Thus, $\lambda > 0$ if y is sufficiently close to $\tilde{x}$.

We are now reduced to proving that $(p \circ \nu_0)(K)$ contains some ball for some compact neighbourhood $K \subset \partial_{\max} C$ of $\tilde{x}$. This follows similarly as in [12, Proof of Lemma 5.2]. Since we are not aware of an English translation of Fedotov's paper, we detail the argument.

Let $B \subset T_0$ be a ball centred at $\tilde{x}$ such that $K = B \cap \partial C$ is contained in $\partial_{\max} C$. Let $H \subset S_0$ be the affine hyperplane through $\tilde{x}$ which supports C. Since $\tilde{x}$ is exposed, there exists a half space H_+ containing $\tilde{x}$ in its interior and with edge H' parallel to H, such that $H_+ \cap C \subset B$.

Moreover, $H' \cap B \subset C$ is the base of a round (Lorentz) cone $W \subset T_0$ with apex $\tilde{x}$ and apex angle $0 < \alpha < \pi$, and spans an affine hyperplane parallel to the linear hyperplane $E_{1/2}(F) \subset T_0$. Its relative dual cone W' with apex 0 is a round cone with apex angle $\pi - \alpha$. Let V be the intersection of the relative interior of W' with the unit sphere in T_0. Then, by elementary trigonometry, $p(V)$ is precisely the open ball around zero in $E_{1/2}(F)$ with radius $\cos \frac{\alpha}{2} > 0$.

We claim $V \subset \nu_0(K)$. In fact, if $y \in V$, then the affine hyperplane H_y through $\tilde{x}$ with normal y intersects W in its apex $\tilde{x}$. Therefore, $H_y \cap H'$ is disjoint from $B \cap H' = W \cap H'$. In other words, $H_y \cap C \subset H_+$. If $H_y = H$, then $y = \nu_0(\tilde{x})$. Otherwise, H_y intersects C in the interior of $H_+ \cap C$, and the supporting hyperplane H'_y of C with inner normal y also has $H'_y \cap C \subset H_+$. Thus, $H'_y \cap C \subset B \cap \partial C = K$, and $y \in \nu_0(K)$. This proves the lemma. □

Remark 38. The article by Fedotov referred to in the proof of Lemma 37 is certainly known to experts in convexity, and R. Schneider cites it in his monograph [29].

Proof of Theorem 32. By Lemmata 34 and 37, $\mathbf{e}(\xi^{-1}(E))$ is a topological manifold of dimension $k = \dim E_{1/2}(F) = n_{d-j-1} - n_{d-j} - 1$. Moreover, the injectivity of p shows that $E_{1/2}(F)^{\perp}$ is transverse to $e_F(E) \in \mathbf{e}(\xi^{-1}(E))$ in the sense of Whitehead [36, p. 155f]. As explained there, $p : \mathbf{e}(\xi^{-1}(E)) \to E_{1/2}(F)$ has a local Lipschitz inverse on some ball $B \subset E_{1/2}(F)$. Therefore, $\mathbf{e}(\xi^{-1}(E))$ is in fact a Lipschitz manifold.

With respect to the Lipschitz chart p^{-1}, $\mathbf{e}(\xi^{-1}(E))$ has tangent spaces on a set of full k-dimensional Hausdorff measure in $p^{-1}(B)$. These tangent spaces have dimension k, and thus, by Lemma 33, in the $e_G(E) \in p^{-1}(B)$, the tangent space is $E_{1/2}(G)$. But the map $e_G(E) \mapsto E_{1/2}(G)$ is well defined and continuous with values in the Grassmannian of k-planes in $\langle E\rangle$. By the argument given at the beginning of this subsection, it follows that p^{-1} is a regular $\mathcal{C}^1$ map. Since p and the derivative of p^{-1} depend continuously on (E, F), it follows that $\mathcal{P}_j$ is indeed a closed $\mathcal{C}^{1,0}$ submanifold. □

6.3. Construction of a $\mathcal{C}^{1,0}$ groupoid and a proper homomorphism

We now proceed to the second step in the proof of our index theorem, as explained above. Since $\xi : \mathcal{P}_j \to \xi(\mathcal{P}_j) \subset P_{j-1}$ is a $\mathcal{C}^{1,0}$ manifold over $\xi(\mathcal{P}_j)$, we may consider the vector bundle $\varrho : \Sigma_{j-1} \to P_{j-1}$, and apply the construction of Section 5.1 to

$$Y = \xi(\mathcal{P}_j), \quad M = \mathcal{P}_j, \quad p = \xi, \quad \text{and} \quad E = \Sigma_{j-1}|\xi(\mathcal{P}_j),$$

in the notation used there. Thus,

$$\mathcal{D}_j = \xi^*\Sigma_{j-1} \times_{\mathcal{P}_j} (\mathcal{P}_j \times_{P_{j-1}} \mathcal{P}_j) = \bigl\{(E, u, F_1, F_2) \bigm| (E, F_i) \in \mathcal{P}_j,\ u \in E^{\perp}\bigr\}$$

is a $\mathcal{C}^{1,0}$ groupoid over $\mathcal{D}_j^{(0)} = \mathcal{P}_j$. Its Lie algebroid is $\xi^*\Sigma_{j-1} \oplus T\mathcal{P}_j$. Observe that since

$$F^{\perp} = E^{\perp} \oplus E_{1/2}(F) \oplus \mathbb{R}\cdot e_F \quad \text{for all } (E, F) \in \mathcal{P}_j,$$

there is an isomorphism

$$A(\mathcal{D}_j) \times \mathbb{R} \cong \eta^*\Sigma_j : (E, F, u \oplus v, r) \mapsto \bigl(E, F, u + v + r \cdot e_F(E)\bigr)$$

of topological vector bundles over $\mathcal{P}_j$, in particular, of topological groupoids.

We now consider the groupoid $\mathbb{W}\mathcal{D}_j$ and define

$$\varphi : \mathbb{W}\mathcal{D}_j \to \mathcal{W}_{\Omega}|(U_{j+1} \setminus U_{j-1})$$

by

$$\varphi(\tau) = \begin{cases} (E, r_1, u + r_2 e_{F_2}(E) - r_1 e_{F_1}(E)) & \begin{cases} \tau = (E, u, F_1, F_2, r_1, r_2 - r_1) \\ \in \mathcal{D}_j \times (\mathbb{R} \rtimes \mathbb{R})|\mathbb{R}_{\geqslant 0}; \end{cases} \\ (F, 0, u + v + r e_F(E)) & \begin{cases} \tau = (E, F, u \oplus v, \infty, r) \\ \in A(\mathcal{D}_j) \times \infty \times \mathbb{R}. \end{cases} \end{cases}$$

Proposition 39. *The map φ is a proper strict morphism.*

Proof. Recall that $\mathcal{W}_\Omega \subset \overline{\Omega} \times X$ is a closed embedding, so we may check the continuity of φ component-wise. Equally, $\mathbb{W}\mathcal{D}_j \subset \mathbb{T}\mathcal{D}_j \times \mathcal{W}_{\mathbb{R}_{\geqslant 0}}$ is a closed embedding. Now,

$$\mathcal{P}_j \to P_{j-1} \times X : (E, F, v) \mapsto (E, e_F)$$

is a $\mathcal{C}^{1,0}$ chart. The corresponding $\mathcal{C}^{1,0}$ map on $\mathcal{D}_j$ is given by

$$f : \mathcal{D}_j \to \mathcal{P}_j \times X : (E, u, F, G) \mapsto (E, F, u + e_G - e_F).$$

Hence, let $(E, F, u \oplus v) \in \xi^* \Sigma_{j-1} \oplus T\mathcal{P}_j = A(\mathcal{D}_j)$. Observe

$$r(E, F, u \oplus v) = (E, F) \equiv (E, 0, F, F) \in \mathcal{D}_j.$$

Since $E_{1/2}(F) = T_{e_F}\mathbf{e}(\xi^{-1}(E))$, there exist $F_\varepsilon \in P_j$, $F_\varepsilon \subset E$, such that

$$F = \lim_{\varepsilon \to 0+} F_\varepsilon \quad \text{and} \quad v = \lim_{\varepsilon \to 0+} \frac{e_{F_\varepsilon} - e_F}{\varepsilon}.$$

Moreover, $\gamma(\varepsilon) = (E, \varepsilon \cdot u, F, F_\varepsilon)$ is a $\mathcal{C}^1$ curve in $r^{-1}(E, F, v)$ representing the tangent vector $(E, F, u + v)$. We find

$$T_{(E,F)} f(u + v) = (f \circ \gamma)'(0) = (E, F, u + v).$$

By the definition of the topology on the tangent groupoid, the second component of φ is continuous.

As to the continuity of the first component, we need to see that

$$-F^* = \lim_{\varepsilon \to 0+} \frac{e_{F_{1\varepsilon}}}{\varepsilon} - E_\varepsilon^* \quad \text{in } \overline{\Omega}$$

if $(E, F, u \oplus v) = \lim_{\varepsilon \to 0+} (E_\varepsilon, u_\varepsilon, F_{1\varepsilon}, F_{2\varepsilon}, \varepsilon)$ in $\mathbb{T}\mathcal{D}_j$. In fact, we have already seen that $F = \lim_{\varepsilon \to 0+} F_{1\varepsilon}$, and $E = \lim_{\varepsilon \to 0+} E_\varepsilon$. By continuity of polarity [5, Corollary 7.2.12], $E^* = \lim_{\varepsilon \to 0+} E_\varepsilon^*$. Then

$$\begin{aligned} \lim_{\varepsilon \to 0+} \frac{e_{F_{1\varepsilon}}}{\varepsilon} - E_\varepsilon^* &= \lim_{\varepsilon \to 0+} \frac{e_F}{\varepsilon} - E^* \\ &= \lim_{\lambda \to \infty} \lambda \cdot e_F - E^* = \overline{\mathbb{R}_{\geqslant 0} \cdot e_F - E^*} = -F^*, \end{aligned}$$

because e_F generates the relative dual face of F in $E^{\circledast}$, and by [2, Lemma 2.2.4]. Therefore, φ is continuous, and it is trivial to check that it indeed is a homomorphism.

To see that φ is proper, note first that $\mathcal{W}_\Omega|Y_j$ is closed in $\mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})$. For any $K \subset \mathcal{W}_\Omega|Y_j$, we have $\varphi^{-1}(K) \subset A(\mathcal{D}_j) \times \mathbb{R}$. But the restriction of φ to this set is the composition of the projection $\eta^* \Sigma_j \to \Sigma_j$, which is proper, with the closed embedding $\Sigma_j \subset \mathcal{W}_\Omega|Y_j$. Hence, $\varphi^{-1}(K)$ is compact if K is.

Similarly, if $K \subset \mathcal{W}_\Omega|(U_{j+1} \setminus U_{j-1})$ is compact and completely contained in $\mathcal{W}_\Omega|Y_{j-1}$, then there exists $1 \leqslant R < \infty$ such that

$$\max\bigl(\bigl\|\lambda\bigl(r(\omega)\bigr)\bigr\|, \bigl\|\lambda\bigl(s(\omega)\bigr)\bigr\|\bigr) \leqslant R \quad \text{for all } \omega \in K,$$

where we recall from [2, Theorem 9] that $\lambda : \bar{\Omega} \to X$ is defined by $\lambda(F, x) = \lambda(x - F^*) = x$ and its restriction to Y_{j-1} is continuous. Furthermore,

$$L = \{u \in X \mid \exists E \in P_{j-1},\ v_1, v_2 \in E^{\circledast}\colon\ (E, v_1, u + v_2 - v_1) \in K\}$$

is compact. Hence,

$$\varphi^{-1}(K) \subset P_{j-1} \times L \times P_j \times P_j \times [0, R] \times [-R, R]$$

is compact in $\mathbb{WD}_j$.

It remains to consider a sequence

$$\omega_k = \bigl(E_k, r_k^1 \cdot e_{F_k^1}, u_k + r_k^2 e_{F_k^2} - r_k^1 e_{F_k^1}\bigr)$$

converging to $(F, 0, u + w) \in \Sigma_j$, $u \perp F$, $w \in F^{\circledast}$, and to exhibit a subsequence of $(E_k, u_k, F_k^1, F_2^2, r_k^1, r_k^2 - r_k^1)$ converging to $(E, F, u \oplus v, \infty, r)$ for some $E \in P_{j-1}$, $F \subset E$, $u \in E_{1/2}(F)$, $r \in \mathbb{R}$, such that $w = v + r \cdot e_F$.

In fact, by compactness of $\mathcal{P}_j \times_{P_{j-1}} \mathcal{P}_j$, by passing to a subsequence, we may assume that $(E_k, F_k^1, F_k^2) \to (E, F', F'')$. Moreover, since

$$\lim_k r_k^1 \cdot e_{F_k^1} - E_k^* \to -F^* \quad \text{and} \quad \dim F < \dim E = \dim E_k,$$

the sequence $r_k^1 \cdot e_{F_k^1}$ is unbounded by [2, Lemma 13(iii)], so $r_k^1 \to \infty$. Because $E_k \to E$ and we have $\dim E_k = \dim E$, we obtain $E_k^\perp \to E^\perp$. Thus,

$$u_k + r_k^2 \cdot e_{F_k^2} - r_k^1 \cdot e_{F_k^1} \to u + w \quad \text{implies} \quad u_k \to u,$$

whence in turn $r_k^2 \cdot e_{F_k^2} - r_k^1 \cdot e_{F_k^1} \to w$. Because

$$\bigl\|r_k^2 \cdot e_{F_k^2} - r_k^1 \cdot e_{F_k^1}\bigr\|^2 = \bigl(r_k^2 - r_k^1 \cdot (e_{F_k^2} | e_{F_k^1})\bigr)^2 + \bigl(r_k^1 \cdot \bigl(1 - (e_{F_k^2} | e_{F_k^1})\bigr)\bigr)^2,$$

passing to a subsequence, we may assume $r_k^2 - r_k^1 \to r \in \mathbb{R}$, and $e_{F_k^2} - e_{F_k^1} \to 0$, so $F' = F''$. Now,

$$-F^* = \lim_k r_k^1 \cdot e_{F_k^1} - E_k^* = \lim_{\lambda \to \infty} \lambda \cdot e_{F'} - E^*.$$

This implies that $e_{F'}$ lies in the relative interior of $E^{\circledast} \cap F^\perp = \mathbb{R}_{\geqslant 0} \cdot e_F$, and hence $e_F = e_{F'}$, which finally gives $F = F'$. Since $(r_k^2 - r_k^1) \cdot e_{F_k^2} \to r \cdot e_F$, the limit $v = \lim_k r_k^1 \cdot (e_{F_k^2} - e_{F_k^1})$ exists. Necessarily, $v \in E_{1/2}(F)$, by Lemma 33. We conclude $w = v + r \cdot e_F$. Thus, we have established the required relation

$$\bigl(E_k, u_k, F_k^1, F_k^2, r_k^1, r_k^2 - r_k^1\bigr) \to (E, F, u \oplus v, \infty, r) \quad \text{in } \mathbb{WD}_j,$$

and thereby, that φ is proper. $\square$

6.4. Proof of the main theorem

As a corollary of the construction of the proper strict morphism φ, we obtain the topological Wiener–Hopf index formula on the level of operator KK theory. To that end, let

$$\varphi_0 : A(\mathcal{D}_j) \times \mathbb{R} = \eta^* \Sigma_j \to \Sigma_j \subset \mathcal{W}_\Omega | Y_j$$

and

$$\varphi_1 : \mathcal{D}_j \times (\mathbb{R} \rtimes \mathbb{R}) | \mathbb{R}_{\geqslant 0} \to \mathcal{W}_\Omega | Y_{j-1}$$

be the corresponding restrictions of φ.

Theorem 40. *We have the following expression for ∂_j:*

$$\partial_j \otimes KK(\varphi_1) = KK(\varphi_0) \otimes y \otimes \tau_j \quad \text{in } KK^1\big(\mathrm{C}^*_r(\mathcal{W}_\Omega | Y_j), \mathrm{C}^*_r(\mathcal{D}_j)\big).$$

Here, τ_j is the Connes–Skandalis map associated to the tangent groupoid $\mathbb{T}\mathcal{D}_j$, and the element $y \in KK^1(S, \mathbb{C})$ is associated to the classical Wiener–Hopf extension.

Proof. Consider the strict morphism $\varphi : \mathbb{W}\mathcal{D}_j \to \mathcal{W}_\Omega | (U_{j+1} \setminus U_{j-1})$ from Proposition 39. Applying Corollary 6, we obtain

$$\partial_j \otimes KK(\varphi_1) = KK(\varphi_0) \otimes \partial,$$

where ∂ represents the extension for $\mathbb{W}\mathcal{D}_j$ from Corollary 14. Now, the assertion follows from Theorem 18. □

Consider the embeddings

$$i_{\mathcal{P}_j} : \mathcal{P}_j \to P_{j-1} \times X^2 : (E, F) \mapsto (E, e_F, 0),$$
$$i_{\Sigma_{j-1}} : \Sigma_j \to P_{j-1} \times X^2 : (E, u) \mapsto (E, u, 0),$$

and the associated topological family index $\varphi_{T\mathcal{P}_j \oplus \xi^* \Sigma_{j-1}} \otimes \varphi^{-1}_{\Sigma_{j-1} | \xi(\mathcal{P}_j)}$ (cf. Section 5).

Recall from the introduction or [2, Proposition 16] that the inclusion of Σ_j in $\mathcal{W}_\Omega | Y_j$ is a Morita equivalence, and similarly for Σ_{j-1}. By this token, the element $\partial_j \in KK^1(\mathrm{C}^*_r(\mathcal{W}_\Omega | Y_j), \mathrm{C}^*_r(\mathcal{W}_\Omega | Y_{j-1}))$ may be pulled back to an element of $KK^1(\mathrm{C}^*_r(\Sigma_j), \mathrm{C}^*_r(\Sigma_{j-1}))$ which we denote by the same letter. Applying Theorem 21, we obtain the following corollary.

Corollary 41. *Let $\boldsymbol{\eta}$ be the $*$-morphism induced by the projection $\eta^* \Sigma_j \to \Sigma_j$ (which is proper), and similarly let $\boldsymbol{\zeta}$ be induced by the closed embedding $\zeta : \Sigma_{j-1} | \xi(\mathcal{P}_j) \to \Sigma_{j-1}$. Then*

$$\boldsymbol{\zeta}_* \partial_j = \boldsymbol{\eta}^* \big[y \otimes \varphi_{T\mathcal{P}_j \oplus \xi^* \Sigma_{j-1}} \otimes \varphi^{-1}_{\Sigma_{j-1} | \xi(\mathcal{P}_j)} \big]$$

in $KK^1(\mathrm{C}^*_r(\Sigma_j), \mathrm{C}^*_r(\Sigma_j | \xi(\mathcal{P}_j)))$. If $\xi : \mathcal{P}_j \to P_{j-1}$ is surjective, i.e. every n_{d-j+1}-dimensional face of the cone Ω^* contains an n_{d-j}-dimensional face, then $\boldsymbol{\zeta}$ is the identity.

Proof. The assertion follows from Theorems 40 and 21 by noting that $\eta^* = KK(\varphi_0)$, and that φ_1 drops to ζ through π. □

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