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On conservative and supermagic graphs

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ABSTRACT

A graph is called supermagic if it admits a labelling of the edges by pairwise different consecutive positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. A graph G is called conservative if it admits an orientation and a labelling of the edges by integers $\{1, \dots, |E(G)|\}$ such that at each vertex the sum of the labels on the incoming edges is equal to the sum of the labels on the outgoing edges. In this paper we deal with conservative graphs and their connection with the supermagic graphs. We introduce a new method to construct supermagic graphs using conservative graphs. Inter alia we show that the union of some circulant graphs and regular complete multipartite graphs are supermagic.

1. Introduction

We consider finite multigraphs without loops and isolated vertices. In a graph, no multiple edges are allowed. If G is a multigraph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into the set of positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into the set of positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v, e) f(e) \quad \text{for every } v \in V(G), \tag{1}$$

where $\eta(v, e)$ is equal to 1 when e is an edge incident with vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into the set of positive integers is called a *magic labelling* of G for an *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G). \tag{2}$$

A magic labelling f of G is called a *supermagic labelling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) whenever there exists a supermagic (magic) labelling of G.

A bijection f from $E(G)$ into $\{1, 2, \dots, |E(G)|\}$ is called a *degree-magic labelling* (or only a *d-magic* labelling) of a graph G if its index-mapping f^* satisfies

$$f^*(v) = \frac{1 + |E(G)|}{2} \deg(v) \quad \text{for all } v \in V(G). \tag{3}$$

A d-magic labelling f of G is called *balanced* if for all $v \in V(G)$ it holds

$$|\{e \in E(G) : \eta(v, e) = 1, f(e) \le \lfloor |E(G)|/2 \rfloor\}| = |\{e \in E(G) : \eta(v, e) = 1, f(e) > \lfloor |E(G)|/2 \rfloor\}|. \tag{4}$$

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We say that a graph G is *degree-magic* (*balanced degree-magic*) (or only *d-magic*) when there exists a d-magic (balanced d-magic) labelling of G.

The concept of magic graphs was introduced by Sedláček [13]. Supermagic graphs were introduced by Stewart [17]. There is by now a considerable number of papers published on magic and supermagic graphs; we single out [5,12,9,6,10,14,11] as being more particularly relevant to the present paper, and refer the reader to [8] for comprehensive references. The concept of degree-magic graphs was introduced in [3] as some extension of supermagic regular graphs.

The basic properties of degree-magic graphs have been introduced in [3]. Let us recall those, which we shall use in the next.

Proposition 1. *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

Proposition 2. *Let G be a balanced d-magic graph. Then every vertex of G has an even degree and every component of G has an even size.*

Proposition 3. *Let G be a graph obtained from a graph H by an identification of two vertices whose distance is at least three. If H is (balanced) d-magic then G is also a (balanced) d-magic graph.*

Proposition 4. *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is d-magic and H_2 is balanced d-magic then G is a d-magic graph. Moreover, if H_1 and H_2 are both balanced d-magic then G is a balanced d-magic graph.*

In the paper we introduce some constructions of balanced degree-magic (and also supermagic) labellings for a large family of graphs.

2. Conservative multigraphs

To find a balanced d-magic labelling of some graph it is convenient to use another (easier) labelling. In this section we define such labelling and present its basic properties.

A directed multigraph $D(M)$ obtained from an undirected multigraph M by assigning a direction to each edge is called an *orientation* of M. Let μ be a mapping from $V(M) \times E(M)$ into $\{-1, 0, 1\}$ defined by

$$\mu(v, e) = \begin{cases} 1 & \text{if the arc } e \text{ comes to the vertex } v \text{ in } D(M),\\ -1 & \text{if the arc } e \text{ goes out of the vertex } v \text{ in } D(M),\\ 0 & \text{otherwise.}\end{cases}$$

Evidently, for all $u, v \in V(M)$, $u \neq v$, $e \in E(M)$ it holds:

(i) $\eta(v, e) = |\mu(v, e)|$,
(ii) if $\eta(u, e) + \eta(v, e) = 2$ then $\mu(u, e) + \mu(v, e) = 0$,

and conversely, any mapping $\mu : V(M) \times E(M) \to \{-1, 0, 1\}$ satisfying (i) and (ii) describes some orientation of M.

An *Eulerian orientation* of M is an orientation of its edges with the property that each vertex has the same number of incoming and outgoing arcs. Therefore,

$$\sum_{e\in E(M)} \mu(v, e) = 0 \quad \text{for every } v \in V(M).$$

Note that a multigraph admits an Eulerian orientation if and only if each of its components is an Eulerian multigraph.

Let a multigraph M, a mapping $\mu : V(M) \times E(M) \to \{-1, 0, 1\}$, and a bijection $f : E(M) \to \{1, 2, \ldots, |E(M)|\}$ be given. A pair (μ, f) is called a *conservative labelling* of M if μ describes an orientation of M and it holds

$$\sum_{e\in E(M)} \mu(v, e)f(e) = 0 \quad \text{for every } v \in V(M) \tag{5}$$

(i.e., f satisfies Kirchhoff's Current Law at each vertex of M). A conservative labelling (μ, f) of M is called *Eulerian* if μ defines an Eulerian orientation of M. We say that a multigraph M is *conservative* (*Eulerian conservative*) whenever there exists a conservative (an Eulerian conservative) labelling of M.

The concept of conservative graphs was introduced in [1]. A topological interpretation of these graphs was given in [19]. The basic properties of conservative graphs have been established in [1] (see also [7,9]). Let us state the next proposition, that we will find useful next in the paper.

Proposition 5. *Let H_1 and H_2 be edge-disjoint submultigraphs of a multigraph M which form its decomposition. If H_1 is conservative and H_2 is Eulerian conservative then M is a conservative multigraph. Moreover, if H_1 and H_2 are both Eulerian conservative then M is an Eulerian conservative multigraph.*

Therefore, the family of Eulerian conservative multigraphs is closed under the edge-disjoint union. It is easy to see that the conservative graphs are also closed under the edge-bijective homomorphism, i.e., it holds.

Proposition 6. *Let M be a multigraph obtained from a multigraph H by an identification of two nonadjacent vertices. If H is (Eulerian) conservative then M is also (Eulerian) conservative.*

In [1], it is proved that each component of a conservative multigraph is 3-edge-connected. Moreover, we can prove the following necessary condition for bipartite conservative multigraphs.

Theorem 1. *Let M be a bipartite conservative multigraph. Then either $|E(M)| \equiv 0$ or $|E(M)| \equiv 3 (\bmod 4)$.*

Proof. Let (μ, f) be a conservative labelling of a bipartite multigraph M and let $U \subset V(M)$ be a part of M. For $\kappa \in \{-1, 1\}$ let

$$E_\kappa = \bigcup_{u \in U} \{e \in E(M) : \mu(u, e) = \kappa\}.$$

Evidently, $E_{-1} \cap E_1 = \emptyset$ and $E_{-1} \cup E_1 = E(M)$. Since

$$\sum_{e \in E(M)} \mu(u, e)f(e) = \sum_{e \in E_{-1}} \mu(u, e)f(e) + \sum_{e \in E_1} \mu(u, e)f(e),$$

according to (5), we get

$$\sum_{e \in E_{-1}} \mu(u, e)f(e) = -\sum_{e \in E_1} \mu(u, e)f(e),$$

for every vertex $u \in U$. Therefore,

$$\sum_{e \in E_{-1}} f(e) = -\sum_{u \in U} \sum_{e \in E_{-1}} \mu(u, e)f(e) = \sum_{u \in U} \sum_{e \in E_1} \mu(u, e)f(e) = \sum_{e \in E_1} f(e).$$

Then, $\sum_{e \in E(M)} f(e) = \frac{|E(M)|}{2}(1 + |E(M)|)$ is an even integer, which implies the assertion. □

A spanning submultigraph H of a multigraph M is called a *half-factor* of M whenever $\deg_H(v) = \deg_M(v)/2$ for every vertex $v \in V(M)$. Note that a spanning submultigraph of M with edge set $E(M) - E(H)$ is also a half-factor of M.

In [1] it was proved that a multigraph, which is decomposable into two Hamilton cycles, is Eulerian conservative. Now we are able to prove an extension of this result.

Theorem 2. *Let H_1 and H_2 be edge-disjoint half-factors of a multigraph M. If H_1 and H_2 are Eulerian multigraphs, then M is an Eulerian conservative multigraph.*

Proof. Since H_1 and H_2 are edge-disjoint half-factors of M, $|E(H_1)| = |E(H_2)|$ and $E(M) = E(H_1) \cup E(H_2)$. Put $m := |E(M)|/2$ and choose a vertex $v \in V(M)$. The submultigraph H_i, $i \in \{1, 2\}$, is Eulerian; therefore, there is an ordering $e_1^i, e_2^i, \ldots, e_m^i$ of $E(H_i)$ which forms an Eulerian trail of H_i starting (and finishing) at v. Define an orientation of M by orienting each edge in accordance with its direction of the Eulerian trail. Evidently, it is an Eulerian orientation. Denote by μ the function describing this orientation and consider a bijection f from $E(M)$ onto $\{1, 2, \ldots, 2m\}$ defined by

$$f(e_j^i) = \begin{cases} j & \text{if } i = 1 \text{ and } j \in \{1, 2, \ldots, m\}, \\ 1 + 2m - j & \text{if } i = 2 \text{ and } j \in \{1, 2, \ldots, m\}. \end{cases}$$

Since each of the Eulerian trails passes through a vertex $u \in V(M)$ exactly $\deg_M(u)/4$ times and $f(e_m^1) - f(e_1^1) = m - 1$, $f(e_m^2) - f(e_1^2) = 1 - m$, $f(e_r^1) - f(e_{r+1}^1) = -1$, $f(e_r^2) - f(e_{r+1}^2) = 1$ for each $r \in \{1, \ldots, m-1\}$, $\sum_{e \in E(H_1)} \mu(u, e)f(e) = -\sum_{e \in E(H_2)} \mu(u, e)f(e)$ for every $u \in V(M)$. Thus,

$$\sum_{e \in E(M)} \mu(u, e)f(e) = \sum_{e \in E(H_1)} \mu(u, e)f(e) + \sum_{e \in E(H_2)} \mu(u, e)f(e) = 0.$$

Therefore, (μ, f) is an Eulerian conservative labelling. □

Let n, m and $a_1 < \cdots < a_m \le \lfloor n/2 \rfloor$ be positive integers. An undirected graph with the set of vertices $\{v_0, \ldots, v_{n-1}\}$ and the set of edges $\{v_i v_{i+a_j} : 0 \le i < n, 1 \le j \le m\}$, the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, \ldots, a_m)$. It is easy to see, that the circulant graph $C_n(a_1, \ldots, a_m)$ is a regular graph of degree r, where $r = 2m - 1$ when $a_m = n/2$, and $r = 2m$ otherwise. The circulant graph $C_n(a_1, \ldots, a_m)$ has $d = \gcd(a_1, \ldots, a_m, n)$ connected components (see [4]), which are isomorphic to $C_{n/d}(a_1/d, \ldots, a_m/d)$. Since a connected circulant graph of degree 4 has a decomposition into two Hamilton cycles (see [2]), according to Proposition 5 and Theorem 2, we immediately have

Corollary 1. *Any circulant graph of degree four is Eulerian conservative.*

The union of m disjoint copies of a multigraph M is denoted by mM.

Let ${}^kM$ denote the multigraph obtained by replacing each edge uv of a multigraph M with k edges joining u and v. Thus, $V({}^kM) = V(M)$ and $E({}^kM) = \cup_{e\in E(M)}\{(e,1),(e,2),\dots,(e,k)\}$, where an edge (e,i), $1 \le i \le k$, is incident with a vertex v in ${}^kM$ whenever e is incident with v in M. We characterize Eulerian conservative multigraphs ${}^kM$, $k \ge 3$, but first we prove two auxiliary results.

Lemma 3. *The multigraph $m\,{}^3K_2$ is conservative if and only if either $m \equiv 0$ or $m \equiv 1 (\mathrm{mod}\ 4)$.*

Proof. Suppose that $m\,{}^3K_2$ is conservative. It is a bipartite multigraph of size $3m$. Therefore, the necessary condition follows from Theorem 1.

On the other hand, let m be an integer such that either $m \equiv 0$ or $m \equiv 1 (\mathrm{mod}\ 4)$. In this case, Skolem [16] proved that it is possible to distribute the numbers $1, 2, \dots, 2m$ into m pairs (a_j, b_j) such that $b_j - a_j = j$ for $j = 1, 2, \dots, m$.

Denote the vertices of $m\,{}^3K_2$ by $u_1, \dots, u_m, v_1, \dots, v_m$ in such a way that its edge set is $E(m\,{}^3K_2) = \cup_{j=1}^{m}\cup_{i=1}^{3}\{(u_jv_j, i)\}$. Consider mappings $\mu : V(m\,{}^3K_2) \times E(m\,{}^3K_2) \to \{-1, 0, 1\}$ and $f : E(m\,{}^3K_2) \to \{1, 2, \dots, 3m\}$ given by

$$\mu(x,(u_jv_j,i)) := \begin{cases} -1 & \text{if } x = u_j \text{ and } i \in \{1,2\},\\ 1 & \text{if } x = v_j \text{ and } i \in \{1,2\},\\ -1 & \text{if } x = v_j \text{ and } i = 3,\\ 1 & \text{if } x = u_j \text{ and } i = 3,\\ 0 & \text{if } x \notin \{u_j, v_j\}, \end{cases}$$

$$f((u_jv_j,i)) := \begin{cases} j & \text{if } i = 1,\\ m + a_j & \text{if } i = 2,\\ m + b_j & \text{if } i = 3, \end{cases}$$

for each $j \in \{1, 2, \dots, m\}$. Clearly, f is a bijection,

$$\sum_{e\in E(m\,{}^3K_2)} \mu(u_j,e)f(e) = -j - (m + a_j) + (m + b_j) = 0,$$

and

$$\sum_{e\in E(m\,{}^3K_2)} \mu(v_j,e)f(e) = -\sum_{e\in E(m\,{}^3K_2)} \mu(u_j,e)f(e) = 0.$$

Therefore, (μ, f) is a conservative labelling of $m\,{}^3K_2$. □

Lemma 4. *Let M be a multigraph with all vertices of even degree. Then ${}^2M$ is an Eulerian conservative multigraph.*

Proof. According to Proposition 5, it is sufficient to consider that M is a connected multigraph. In this case, for $i \in \{1, 2\}$, the spanning submultigraph H_i of ${}^2M$ with edges $\{(e,i) : e \in E(M)\}$ is an Eulerian half-factor of ${}^2M$. Therefore, the result immediately follows from Theorem 2. □

Theorem 5. *Let $k \ge 3$ be an integer and let M be a multigraph. ${}^kM$ is Eulerian conservative if and only if the following statements hold:*

(i) *if k is odd then every vertex of M has even degree,*
(ii) *if M is bipartite then either $k|E(M)| \equiv 0$ or $k|E(M)| \equiv 3 (\mathrm{mod}\ 4)$.*

Proof. A multigraph admits an Eulerian orientation if and only if every vertex has an even degree. Thus, by Theorem 1, it is easy to see that the statements (i) and (ii) are necessary for ${}^kM$ to be Eulerian conservative.

On the other hand, put $m := |E(M)|$ and consider the following cases.

A. Let $k = 3$. Since every vertex of the multigraph M has an even degree, there is a mapping $\nu : V(M)\times E(M) \to \{-1, 0, 1\}$ which describes an Eulerian orientation of M, i.e., $\sum_{e\in E(M)} \nu(v,e) = 0$ for every $v \in V(M)$.

If m is odd then we define a mapping $\mu : V({}^3M) \times E({}^3M) \to \{-1, 0, 1\}$ by $\mu(v,(e,i)) := \nu(v,e)$ for all $e \in E(M)$ and $i \in \{1, 2, 3\}$. Evidently, μ describes an Eulerian orientation of ${}^3M$. Let ξ be a bijection from $E(M)$ to $\{1, 2, \dots, m\}$. Consider a mapping $f : E({}^3M) \to \{1, 2, \dots, 3m\}$ given by

$$f((e,i)) := \begin{cases} \xi(e) & \text{if } i = 1,\\ \xi(e) + \dfrac{3m+1}{2} & \text{if } i = 2 \text{ and } \xi(e) \le \dfrac{m-1}{2},\\ \xi(e) + \dfrac{m+1}{2} & \text{if } i = 2 \text{ and } \xi(e) > \dfrac{m-1}{2},\\ 3m - 2\xi(e) + 1 & \text{if } i = 3 \text{ and } \xi(e) \le \dfrac{m-1}{2},\\ 4m - 2\xi(e) + 1 & \text{if } i = 3 \text{ and } \xi(e) > \dfrac{m-1}{2}. \end{cases}$$

Clearly, f is a bijection and $\sum_{i=1}^{3} f((e,i)) = \frac{9m+3}{2}$ for all $e \in E(M)$. Hence

$$\sum_{(e,i)\in E(^3M)} \mu(v,(e,i))f((e,i)) = \frac{9m+3}{2} \sum_{e\in E(M)} \nu(v,e) = 0.$$

Therefore, (μ, f) is an Eulerian conservative labelling of ^3M.

If $m \equiv 0 \pmod 4$ then $m\,^3K_2$ is a conservative multigraph by Lemma 3. The graph ^3M is obtained from $m\,^3K_2$ by a sequence of identifications of some vertices. According to Proposition 6, the multigraph ^3M is conservative. Moreover, each copy of $^3K_2$ (i.e., a component of $m\,^3K_2$) corresponds to an edge of M in this case. Consider an orientation of ^3M by orienting the edge of any copy of $^3K_2$ with the maximal label in accordance with the direction of the corresponding edge of $D(M)$, where $D(M)$ is an Eulerian orientation of M. The remaining two edges of the copy of $^3K_2$ are oriented conversely, thus it is not difficult to see, that the considered orientation of ^3M is Eulerian. Hence, ^3M is an Eulerian conservative multigraph.

If $m \equiv 2 \pmod 4$ then according to (ii), M is not bipartite. Thus, there is an odd cycle C in M. Let H be a submultigraph of M induced by edges $E(M) - E(C)$. Evidently, both multigraphs C and H have an odd number of edges. Hence, the multigraphs ^3C and ^3H are Eulerian conservative. The multigraph ^3M is decomposable into two edge-disjoint submultigraphs isomorphic to ^3C and ^3H. Therefore, by Proposition 5, ^3M is an Eulerian conservative multigraph.

B. Let $3 < k \equiv 1 \pmod 2$. Then there is a positive integer t such that $k = 2t + 3$. In this case, ^3M is Eulerian conservative by the previous case, and ^2M is also Eulerian conservative by Lemma 4. Since the multigraph ^kM is decomposable into a submultigraph isomorphic to ^3M and t submultigraphs isomorphic to ^2M, by Proposition 5, it is Eulerian conservative.

C. Let $k \equiv 0 \pmod 4$. Then there is a positive integer t such that $k = 4t$. According to Theorem 2, the multigraph $^kK_2$ is Eulerian conservative because it is decomposable into two edge-disjoint Eulerian submultigraphs isomorphic to $^{2t}K_2$. The multigraph ^kM is decomposable into m edge-disjoint submultigraphs isomorphic to $^kK_2$ and by Proposition 5, it is Eulerian conservative.

D. Let $k = 6$. Since the multigraph 2^6K_2 is isomorphic to $^3(2^2K_2)$, by case A, it is Eulerian conservative. By Proposition 6, ^6G is Eulerian conservative for each multigraph $G \in \mathcal{F}_2 := \{2K_2, P_3, {}^2K_2\}$ ($\mathcal{F}_2$ is a family of multigraphs with exactly two edges). Consequently, according to Proposition 5, ^6M is Eulerian conservative whenever m is even.

If m is odd, then M is not bipartite because of (ii). Thus, there is an odd cycle C in M. The multigraph ^6C is decomposable into two edge-disjoint Eulerian half-factors isomorphic to ^3C. Thus, Theorem 2 implies that ^6C is an Eulerian conservative multigraph. Let H be a submultigraph of M induced by edges $E(M) - E(C)$. Evidently, H has an even number of edges and so the multigraph ^6H is Eulerian conservative. Moreover, ^6M is decomposable into two edge-disjoint submultigraphs isomorphic to ^6C and ^6H. Hence, by Proposition 5, ^6M is an Eulerian conservative multigraph.

E. Let $6 < k \equiv 2 \pmod 4$. Then there is a positive integer t such that $k = 4t + 6$. The multigraphs ^{4t}M and ^6M are Eulerian conservative by C and D. According to Proposition 5, ^kM is Eulerian conservative. $\square$

For $k = 2$, we conjecture as in what follows.

Conjecture. *Let M be a multigraph. Then ^2M is an Eulerian conservative multigraph if and only if the following statements hold:*

(i) *every component of M is 2-edge connected,*
(ii) *if M is bipartite then it has an even number of edges.*

3. Degree-magic graphs

The *subdivision graph* $S(M)$ of a multigraph M is a bipartite graph with vertex set $V(S(M)) = V(M) \cup E(M)$, where ve is an edge of $S(M)$ whenever $v \in V(M)$ is incident to $e \in E(M)$ in M. In other words, the subdivision of a multigraph is a graph obtained by inserting a vertex of degree 2 into every edge of the original multigraph. A correspondence between Eulerian conservative graphs and balanced degree-magic subdivision graphs is established in the following result.

Theorem 6. *A multigraph M is Eulerian conservative if and only if its subdivision graph $S(M)$ is balanced degree-magic.*

Proof. Put $m := |E(M)|$. First suppose that there is an Eulerian conservative labelling (ν, t) of M. Define the mapping g from $E(S(M))$ into $\{1, 2, \ldots, 2m\}$ by

$$g(ve) := \begin{cases} t(e) & \text{if } \nu(v,e) = -1, \\ 1 + 2m - t(e) & \text{if } \nu(v,e) = 1. \end{cases}$$

Evidently, g is a bijection. If $u \in V(M)$ then u is also a vertex of $S(M)$ and it holds:

$$\begin{aligned} g^*(u) &= \sum_{ue\in E(S(M))} g(ue) = \sum_{\substack{e\in E(M)\\ \nu(u,e)=-1}} t(e) + \sum_{\substack{e\in E(M)\\ \nu(u,e)=1}} (1+2m-t(e)) \\ &= \frac{\deg_M(u)}{2}(1+2m) - \sum_{e\in E(M)} \nu(u,e)t(e) = \frac{1+2m}{2}\deg_{S(M)}(u), \end{aligned}$$

$$\begin{aligned} |\{x \in E(S(M)) : \eta(u,x) = 1, g(x) \le m\}| &= |\{e \in E(M) : \nu(u,e) = -1\}| = |\{e \in E(M) : \nu(u,e) = 1\}| \\ &= |\{x \in E(S(M)) : \eta(u,x) = 1, g(x) > m\}|. \end{aligned}$$

Similarly, if $e \in E(M)$ then e is also a vertex of $S(M)$ of degree 2. The labels of edges incident to e are $t(e)$ and $1+2m-t(e)$, thus it holds:

$$g^*(e) = 1+2m = \frac{1+|E(S(M))|}{2}\deg_{S(M)}(z),$$
$$|\{x \in E(S(M)) : \eta(e,x)=1, g(x) \le m\}| = 1 = |\{x \in E(S(M)) : \eta(u,x)=1, g(x) > m\}|.$$

Therefore, g is a balanced d-magic labelling of $S(M)$.

On the other hand, suppose that h is a balanced d-magic labelling of $S(M)$. For any edge $e = uw \in E(M)$ and any vertex $v \in V(M)$ put

$$f(e) := \min\{h(ue), h(we)\},$$
$$\mu(v,e) := \begin{cases} 0 & \text{if } v \notin \{u,w\}, \\ 1 & \text{if } v \in \{u,w\} \text{ and } h(ve) > m, \\ -1 & \text{if } v \in \{u,w\} \text{ and } h(ve) \le m. \end{cases}$$

In this way, given a bijection $f : E(M) \to \{1,2,\dots,m\}$ and a mapping $\mu : V(M) \times E(M) \to \{-1,0,1\}$ describing an orientation of M. Since h is balanced, it follows that

$$\sum_{e\in E(M)} \mu(v,e) = |\{ve \in E(S(M)) : h(ve) > m\}| - |\{ve \in E(S(M)) : h(ve) \le m\}| = 0,$$

for each $v \in V(M)$. Thus, the considered orientation is Eulerian. Moreover, for each edge $e \in E(M)$, $h^*(e) = 1+2m$ since h is d-magic. Hence, $h(ve) = f(e)$ when $\mu(v,e) = -1$, and $h(ve) = 1+2m-f(e)$ when $\mu(v,e) = 1$. For any vertex $v \in V(M)$, we get

$$\frac{1+2m}{2}\deg_M(v) = h^*(v) = \sum_{e\in E(M)} |\mu(v,e)|h(ve) = \sum_{\substack{e\in E(M)\\ \nu(u,e)=-1}} f(e)$$
$$+ \sum_{\substack{e\in E(M)\\ \nu(u,e)=1}} (1+2m-f(e)) = (1+2m)\frac{\deg_M(v)}{2} - \sum_{e\in E(M)} \mu(v,e)f(e).$$

This implies $\sum_{e\in E(M)} \mu(v,e)f(e) = 0$, i.e., (μ, f) is an Eulerian conservative labelling of M. □

Let $H_1, H_2, \dots, H_m$ be pairwise edge-disjoint subgraphs of a graph G which form its decomposition. If all H_i are isomorphic to P_3, then we say that $\mathcal{P} = \{H_1, H_2, \dots, H_m\}$ is a P_3-decomposition of G. Denote by $M(G,\mathcal{P})$ the multigraph whose vertex set consists of all pendant vertices of H_i, and whose edge set consists of m edges, where each of them joins the pendant vertices of H_i, $i \in \{1,\dots,m\}$. It is easy to see that the graph G is obtained from the subdivision graph $S(M(G,\mathcal{P}))$ by a sequence of identifications of some vertices. Thus, according to the previous theorem and Proposition 3, we immediately have the following statement.

Corollary 2. *Let $\mathcal{P}$ be a P_3-decomposition of a graph G. If $M(G,\mathcal{P})$ is an Eulerian conservative multigraph, then G is a balanced d-magic graph.*

Note that the previous result describes a method to construct balanced d-magic (and also supermagic) graphs using Eulerian conservative multigraphs. Applying this method to circulant graphs we have

Corollary 3. *Any circulant graph of degree $8k$ is balanced d-magic.*

Proof. According to Proposition 4 it is sufficient to consider a circulant graph $G = C_n(a_1,a_2,a_3,a_4)$ of degree 8. Denote by J_i^r, $i \in \{0,1,\dots,n-1\}$, $r \in \{1,2\}$, the subgraph of G induced by edges $\{v_{i-a_{2r}}v_i, v_iv_{i-a_{2r-1}}\}$ (indices are taken modulo n). Clearly, it is isomorphic to P_3. It is not difficult to check that $\mathcal{P} = \cup_{i=0}^{n-1}\{J_i^1, J_i^2\}$ is a P_3-decomposition of G and $M(G,\mathcal{P})$ is isomorphic to either $C_n(a_2-a_1, a_4-a_3)$ (if $a_2-a_1 \ne a_4-a_3$) or ${}^2C_n(a_2-a_1)$ (if $a_2-a_1 = a_4-a_3$). By Corollary 1 or Lemma 4, the multigraph $M(G,\mathcal{P})$ is Eulerian conservative. Thus, G is a balanced d-magic graph. □

A *complete k-partite graph* is a graph whose vertices can be partitioned into $k \ge 2$ disjoint classes $V_1,\dots,V_k$ such that two vertices are adjacent whenever they belong to distinct classes. If $|V_i| = n_i$, $i = 1,\dots,k$, then the complete k-partite graph is denoted by $K_{n_1,\dots,n_k}$. If $n_i = p$ for all i, then it is denoted by $K_{k[p]}$ (or only K_k, when $p = 1$).

Corollary 4. *Let m, n be even integers such that $2 \le m \le n$ and $m \ge 4$ when $n \equiv 2 \pmod 4$. Then the complete bipartite graph $K_{m,n}$ is balanced degree-magic.*

Proof. The graph $K_{2,n}$ is isomorphic to $S({}^nK_2)$. Hence, according to Theorems 5 and 6, $K_{2,n}$ is a balanced d-magic graph for any $n \equiv 0 \pmod 4$. $K_{2t,r}$ is decomposable into t edge-disjoint subgraphs isomorphic to $K_{2,r}$, therefore by Proposition 4, $K_{m,n}$ is balanced d-magic when either $n \equiv 0$ or $m \equiv 0 \pmod 4$.

Let $V_1 = \{u_1, \ldots, u_6\}$ and $V_2 = \{v_1, \ldots, v_6\}$ be parts of $K_{6,6}$. Denote by $I_k^{i,j}$ ($I_{i,j}^k$), $i \neq j$, the subgraph of $K_{6,6}$ induced by edges $\{u_iv_k, v_ku_j\}$ ($\{v_iu_k, u_kv_j\}$). Clearly, it is isomorphic to P_3. It is not difficult to check that $\mathcal{P} = \{I_1^{1,2}, I_2^{1,2}, I_3^{1,3}, I_4^{1,3}, I_5^{2,3}, I_6^{2,3}, I_{5,6}^1, I_{3,4}^2, I_{1,2}^3\} \cup_{i=4}^6 \{I_{1,2}^i, I_{3,4}^i, I_{5,6}^i\}$ is a P_3-decomposition of $K_{6,6}$ and $M(K_{6,6}, \mathcal{P})$ is isomorphic to ${}^2(K_3 \cup 3^2K_2)$. Thus, according to Lemma 4 and Corollary 2, $K_{6,6}$ is balanced d-magic.

Finally, the graph $K_{4t+6,4r+6}$, $t \geq 1, r \geq 0$, is decomposable into balanced d-magic subgraphs isomorphic to $K_{6,6}$, $K_{4t,6}$ and $K_{4t+6,4r}$ (if $r > 0$). By Proposition 4, $K_{4t+6,4r+6}$ is a balanced d-magic graph. $\square$

In [3], there are characterized balanced degree-magic complete bipartite graphs. Now we extend this characterization.

Theorem 7. *Let $k \geq 2$, $n_1 \geq \cdots \geq n_k$ be positive integers. The graph $K_{n_1,\ldots,n_k}$ is a balanced d-magic graph if and only if the following conditions are satisfied:*

(i) $n_1 \equiv \cdots \equiv n_k (\mathrm{mod}\ 2)$,
(ii) *if* $n_1 \equiv 1(\mathrm{mod}\ 2)$ *then* $k \equiv 1(\mathrm{mod}\ 4)$,
(iii) *if* $k = 2$ *and* $n_2 = 2$ *then* $n_1 \equiv 0(\mathrm{mod}\ 4)$,
(iv) *if* $k = 5$ *and* $n_5 = 1$ *then* $n_1 \geq 3$.

Proof. Put $K := K_{n_1,\ldots,n_k}$ and $n := n_1 + \cdots + n_k$ for shortening.

Suppose that K is a balanced d-magic graph. The degree of each vertex in V_i is $n - n_i$. By Proposition 2, $n - n_i \equiv 0(\mathrm{mod}\ 2)$ for all $i = 1, \ldots, k$. This implies condition (i). If n_1 is odd, then n and all n_i are odd. Consequently, k is odd, too. According to Proposition 2, we have

$$\binom{k}{2} = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} 1 \equiv \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_in_j = |E(K)| \equiv 0(\mathrm{mod}\ 2).$$

Condition (ii) follows. Using Theorems 5 and 6 we get (iii) because the complete bipartite graph $K_{n_1,2}$ is isomorphic to $S({}^{n_1}K_2)$. In [18], Stewart proved that K_5 is not supermagic, thus by Proposition 1, (iv) is satisfied.

On the other hand, assume to the contrary that K is a complete multipartite graph with a minimum number of vertices, which satisfies (i)–(iv), and still K is not balanced d-magic. Then by Corollary 4, $k > 2$. Consider the following cases.

A. $n \equiv 0(\mathrm{mod}\ 2)$. By (i), all n_i are even in this case.

A1. Suppose that $k \geq 6$. K is decomposable into three edge-disjoint subgraphs isomorphic to K_{n_1,n_2,n_3}, $K_{n_4,\ldots,n_k}$ and $K_{n_1+n_2+n_3,n_4+\cdots+n_k}$. By the minimality of K and Corollary 4, they are balanced d-magic. Thus, by Proposition 4, K is balanced d-magic, a contradiction.

A2. Suppose that $n_1 \geq 4$. In this case, K is decomposable into two edge-disjoint subgraphs isomorphic to $K_{n_2,\ldots,n_k}$ and $K_{n_1,n_2+\cdots+n_k}$. According to the minimality of K and Corollary 4, they are balanced d-magic. Thus, by Proposition 4, K is balanced d-magic, a contradiction.

A3. Suppose that $n_1 = 2$ and $k = 5$. As in the previous case, K is decomposable into two edge-disjoint balanced d-magic subgraphs isomorphic to $K_{4[2]}$ and $K_{2,8}$. Thus, by Proposition 4, K is a balanced d-magic graph, a contradiction.

A4. Suppose that $n_1 = 2$ and $k = 4$. Let $V_i = \{u_i^1, u_i^2\}$, for $1 \leq i \leq 4$, be parts of K. Denote by $I_i^{j,r}$, $i \neq j$, the subgraph of K induced by edges $\{u_i^1u_j^r, u_j^ru_i^2\}$. Clearly, it is isomorphic to P_3. It is not difficult to check that $\mathcal{P} = \cup_{i=1}^3 \{I_i^{4,1}, I_i^{4,2}, I_i^{i+1,1}, I_i^{i+1,2}\}$ (indices $i + 1$ are taken modulo 3) is a P_3-decomposition of K and $M(K, \mathcal{P})$ is isomorphic to 3^4K_2. Thus, according to Theorem 5 and Corollary 2, K is balanced d-magic, a contradiction.

A5. Suppose that $n_1 = 2$ and $k = 3$. A balanced d-magic labelling of $K_{3[2]}$ is described below by giving the labels of edges u_iu_j in the following matrix.

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	–	–	11	10	4	1
u_2	–	–	3	2	9	12
u_3	11	3	–	–	7	5
u_4	10	2	–	–	6	8
u_5	4	9	7	6	–	–
u_6	1	12	5	8	–	–

Thus, K has no even order.

B. $n \equiv 1(\mathrm{mod}\ 2)$. By (i), all n_i are odd, and by (ii), $k \equiv 1(\mathrm{mod}\ 4)$.

B1. Assume that $n_1 \geq 5$. In this case K is decomposable into two edge-disjoint subgraphs isomorphic to $K_{n_1-4,n_2,\ldots,n_k}$ and $K_{4,n_2+\cdots+n_k}$. According to the minimality of K and Corollary 4, they are balanced d-magic except for $k = 5$, $n_1 = 5$ and $n_2 = 1$. In the exceptional case, K is decomposable into edge-disjoint balanced d-magic subgraphs isomorphic to $K_{3,1,1,1,1}$ and $K_{2,4}$. Thus, by Proposition 4, K is balanced d-magic, a contradiction.

B2. Assume that $n_1 = n_2 = 3$. Denote by H_r the graph with vertices $\cup_{i=1}^3 \{u_i, v_i\} \cup \{w_1, \ldots, w_r\}$ and edges $\cup_{i=1}^3 \{u_1v_i, u_2v_i\} \cup \{u_3v_1, u_3v_2\} \cup \cup_{j=1}^r \{u_1w_j, u_2w_j, v_1w_j, v_2w_j\}$. It is not difficult to check that subgraphs of H_r induced by $\{u_1v_i, v_iu_2\}$, $1 \leq i \leq 3$, $\{v_1u_3, u_3v_2\}$, $\{u_1w_1, w_1v_1\}$, $\{u_2w_1, w_1v_2\}$, $\{u_1w_2, w_2v_2\}$, $\{u_2w_2, w_2v_1\}$ and $\{u_1w_j, w_ju_2\}$, $\{v_1w_j, w_jv_2\}$, $3 \leq j \leq r$, form a P_3-decomposition $\mathcal{Q}$ of H_r. An Eulerian conservative labelling of $M(H_3, \mathcal{Q})$ is depicted in Fig. 1. The multigraph $M(H_{s+2}, \mathcal{Q})$

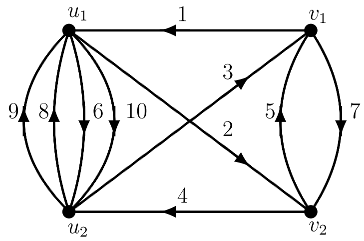

Fig. 1. An Eulerian conservative labelling of $M(H_3, \mathcal{Q})$.

is decomposable into submultigraphs isomorphic to $M(H_s, \mathcal{Q})$ and ${}^4K_2$. Therefore, by Proposition 5, $M(H_r, \mathcal{Q})$ is an Eulerian conservative multigraph for each odd integer $r \geq 3$. Hence, H_r is balanced d-magic for each odd $r \geq 3$.

The graph K is decomposable into two edge-disjoint subgraphs isomorphic to $K_{1,1,n_3,\ldots,n_k}$ and $H_{n_3+\cdots+n_k}$. According to the minimality of K, they are balanced d-magic except for $k = 5$ and $n_3 = 1$. Thus, by Proposition 4, K is balanced d-magic (a balanced d-magic labelling of $K_{3,3,1,1,1}$ is described below in the matrix), a contradiction.

–	–	–	2	9	1	28	26	27
–	–	–	12	23	30	18	6	4
–	–	–	29	8	21	3	25	7
2	12	29	–	–	–	11	19	20
9	23	8	–	–	–	17	14	22
1	30	21	–	–	–	10	16	15
28	18	3	11	17	10	–	13	24
26	6	25	19	14	16	13	–	5
27	4	7	20	22	15	24	5	–

B3. Assume that $n_1 = 3$ and $n_2 = 1$. In this case, K is decomposable into two edge-disjoint subgraphs isomorphic to K_k and $K_{2,k-1}$. According to the minimality of K and Corollary 4, they are balanced d-magic for $k > 5$. Thus, by Proposition 4, K is balanced d-magic, a contradiction. Similarly, for $k = 5$, a balanced d-magic labelling of $K_{3,1,1,1,1}$ is described below in the following matrix.

–	–	–	7	1	18	12
–	–	–	13	6	2	17
–	–	–	9	15	10	4
7	13	9	–	11	14	3
1	6	15	11	–	8	16
18	2	10	14	8	–	5
12	17	4	3	16	5	–

B4. Assume that $n_1 = 1$. In this case $K = K_k$, where $k = 4t + 1$, $t > 1$, because of (ii) and (iv). Let $V(K) = \{u_1, \ldots, u_k\}$ be the vertex set of K_k. Denote by $J^r_{i,j}$, $r \neq i \neq j \neq r$, the subgraph of K_k induced by edges $\{u_iu_k, u_ku_j\}$. Clearly, it is isomorphic to P_3. It is not difficult to check that $\mathcal{R} = \cup_{i=1}^k \{J^{i+2}_{i,i+1}, J^{i+4}_{i,i+1}, \ldots, J^{i+2t}_{i,i+1}\}$ (indices are taken modulo k) is a P_3-decomposition of K_k and $M(K_k, \mathcal{R})$ is isomorphic to ${}^tC_k$. Thus, according to Theorem 5 (Lemma 4, if $t = 2$) and Corollary 2, K_k is balanced d-magic, a contradiction. $\square$

4. Supermagic graphs

In [15], the supermagic graphs $sK_{n,n}$ (i.e., $sK_{2[n]}$) are characterized. All supermagic graphs $mK_{k[n]}$ are described in [10]. Moreover, in the paper a general technique for constructing supermagic labellings of copies of certain kinds of regular supermagic graphs is stated. However, degree-magic labellings allow us to construct supermagic labellings for the disjoint union of some regular non-isomorphic graphs. For example, combining Propositions 1 and 4 and Theorem 7 we immediately obtain

Theorem 8. *Let $\delta > 4$ be an even integer. Let G be a δ-regular graph for which each component is a complete multipartite graph of even size. Then G is a supermagic graph. Moreover, for any δ-regular supermagic graph H, the union of disjoint graphs H and G is also a supermagic graph.*

Similarly, using Corollary 3, we have

Theorem 9. *Let $\delta \equiv 0 (\text{mod } 8)$ be a positive integer. Let G be a δ-regular graph for which each component is a circulant graph. Then G is a supermagic graph. Moreover, for any δ-regular supermagic graph H, the union of disjoint graphs H and G is also a supermagic graph.*

We conclude this paper with the following assertion.

Theorem 10. *Let $k, n_1, \dots, n_k$ be positive integers such that $k \equiv 1 (\mathrm{mod}\ 4)$ and $11 \le n_i \equiv 3 (\mathrm{mod}\ 8)$ for all $i \in \{1, \dots, k\}$. Then the complement of a union of disjoint cycles $C_{n_1} \cup \dots \cup C_{n_k}$ is supermagic.*

Proof. The complement of $C_{n_1} \cup \dots \cup C_{n_k}$ is decomposable into subgraphs $G_0, G_1, \dots, G_k$ where G_0 is isomorphic to $K_{n_1,\dots,n_k}$ and G_i, $i \in \{1, \dots, k\}$, is isomorphic to the complement of C_{n_i}. Since the complement of C_{n_i} is isomorphic to $C_{n_i}(2, 3, \dots, \lfloor n_i/2 \rfloor)$, by Corollary 3, G_i is balanced d-magic for all $i \in \{1, \dots, k\}$. According to Theorem 7, G_0 is also balanced d-magic. Thus, by Proposition 4, the complement of $C_{n_1} \cup \dots \cup C_{n_k}$ is balanced d-magic. This graph is regular and so, according to Proposition 1, it is supermagic. □

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