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A general realization theorem for matrix-valued Herglotz–Nevanlinna functions

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Dedicated to Damir Arov on the occasion of his 70th birthday and to Yury Berezanskiĭ on the occasion of his 80th birthday

Abstract

New special types of stationary conservative impedance and scattering systems, the so-called non-canonical systems, involving triplets of Hilbert spaces and projection operators, are considered. It is established that every matrix-valued Herglotz–Nevanlinna function of the form

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t)$$

can be realized as a transfer function of such a new type of conservative impedance system. In this case it is shown that the realization can be chosen such that the main and the projection operators of the realizing system satisfy a certain commutativity condition if and only if $L = 0$. It is also shown that $V(z)$ with an additional condition (namely, L is invertible or $L = 0$), can be realized as a linear fractional transformation of the transfer function of a non-canonical scattering F_+-system. In particular, this means that every scalar Herglotz–Nevanlinna function can be realized in the above sense.

Moreover, the classical Livšic systems (Brodskiĭ–Livšic operator colligations) can be derived from F_+-systems as a special case when $F_+ = I$ and the spectral measure $\mathrm{d}\Sigma(t)$ is compactly supported. The

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realization theorems proved in this paper are strongly connected with, and complement the recent results by Ball and Staffans.

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1. Introduction

An operator-valued function $V(z)$ acting on a finite-dimensional Hilbert space $\mathfrak{E}$ belongs to the class of matrix-valued Herglotz–Nevanlinna functions if it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, if it is symmetric with respect to the real axis, i.e., $V(z)^* = V(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$, and if it satisfies the positivity condition

$$\operatorname{Im} V(z) \geqslant 0, \quad z \in \mathbb{C}_+.$$

It is well known (see e.g. [29]) that matrix-valued Herglotz–Nevanlinna functions admit the following integral representation:

$$V(z) = Q + Lz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1}$$

where $Q = Q^*$, $L \geqslant 0$, and $\Sigma(t)$ is a nondecreasing matrix-valued function on $\mathbb{R}$ with values in the class of nonnegative matrices in $\mathfrak{E}$ such that

$$\int_{\mathbb{R}} \frac{(\mathrm{d}\Sigma(t)x, x)}{1+t^2} < \infty, \quad x \in \mathfrak{E}. \tag{2}$$

The problem considered in this paper is the general operator representation of these functions with an interpretation in system theory, i.e., in terms of linear stationary conservative dynamical systems. This involves new types of stationary conservative impedance and scattering systems (non-canonical systems) involving triplets of Hilbert spaces and projection operators. The exact definition of both types of non-canonical systems is given below. It turns out that every matrix-valued Herglotz–Nevanlinna function can be realized as a matrix-valued transfer function of this new type of conservative impedance system. Moreover, assuming an additional condition on the matrix L in (1) (L is invertible or $L = 0$), it is shown that such a function is realizable as a linear fractional transformation of the transfer matrix-valued function of a conservative stationary scattering F_+-system. In this case the main operator of the impedance system is the "real part" of the main operator of the scattering F_+-system. In particular, it follows that every scalar Herglotz–Nevanlinna function can be realized in the above mentioned sense. This gives a complete solution of the realization problems announced in "*Unsolved problems in mathematical systems and control theory*" [33] in the framework of modified Brodskiĭ–Livšic operator colligations (in the scalar case via impedance and scattering systems, in the matrix-valued case via impedance systems). Furthermore, the classical canonical systems of the Livšic type (Brodskiĭ–Livšic operator colligations) can be derived from F_+-systems as a special case when $F_+ = I$ and the spectral measure $\mathrm{d}\Sigma(t)$ is compactly supported.

Realizations of different classes of holomorphic matrix-valued functions in the open right half-plane, unit circle, and upper half-plane play an important role in the spectral analysis of non-self-adjoint operators, interpolation problems, and system theory; see [1–21, 23–50]. For special classes of Herglotz–Nevanlinna functions such operator realizations are known.

Consider, for instance, a matrix-valued Herglotz–Nevanlinna function of the form

$$V(z) = \int_a^b \frac{\mathrm{d}\Sigma(t)}{t-z}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3}$$

with $\Sigma(t)$ a nondecreasing matrix-valued function on the finite interval $(a, b) \subset \mathbb{R}$. Then $V(z)$ has an operator realization of the form

$$V(z) = K^*(A - zI)^{-1}K, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{4}$$

where A is a bounded self-adjoint operator acting on a Hilbert space $\mathfrak{H}$ and K is a bounded invertible operator from the Hilbert space $\mathfrak{E}$ into $\mathfrak{H}$. Such realizations are due to Brodskiĭ and Livšic; they have been used in the theory of characteristic operator-valued functions as well as in system theory in the following sense (cf. [36–38,23,24,39]). Let J be a bounded, self-adjoint, and unitary operator in $\mathfrak{E}$ which satisfies $\operatorname{Im} A = KJK^*$. Then the aggregate

$$\Theta = \begin{pmatrix} A & K & J \\ \mathfrak{H} & & \mathfrak{E} \end{pmatrix}, \tag{5}$$

or

$$\begin{cases} (A - zI)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2\mathrm{i}K^*x, \end{cases} \tag{6}$$

is the corresponding, so-called, *canonical system* or *Brodskiĭ–Livšic operator colligation*, where $\varphi_- \in \mathfrak{E}$ is an input vector, $\varphi_+ \in \mathfrak{E}$ is an output vector, and x is a state space vector in $\mathfrak{H}$. The function $W_\Theta(z)$, defined by

$$W_\Theta(z) = I - 2\mathrm{i}K^*(A - zI)^{-1}KJ, \tag{7}$$

such that $\varphi_+ = W_\Theta(z)\varphi_-$, is the *transfer function* of the system Θ or the *characteristic function* of operator colligation. Such type of systems appear in the theory of electrical circuits and have been introduced by Livšic [37]. The relation between $V(z)$ in (4) and $W(z)$ in (7) is given by

$$V(z) = \mathrm{i}[W(z) + I]^{-1}[W(z) - I]J.$$

For an extension of the class of (compactly supported) Herglotz–Nevanlinna functions in (3) involving a linear term as in (1), see [30–32,41,42]. Obviously, general matrix-valued Herglotz–Nevanlinna functions $V(z)$ cannot be realized in the above mentioned (Brodskiĭ–Livšic) form.

The realization of a different class of Herglotz–Nevanlinna functions is provided by a linear stationary conservative dynamical system Θ of the form

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & \mathfrak{E} \end{pmatrix}. \tag{8}$$

In this system $\mathbb{A}$, the *main operator* of the system, is a bounded linear operator from $\mathfrak{H}_+$ into $\mathfrak{H}_-$ extending a symmetric (Hermitian) operator A in $\mathfrak{H}$, where $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ is a rigged Hilbert space. Moreover, K is a bounded linear operator from the finite-dimensional Hilbert space $\mathfrak{E}$ into $\mathfrak{H}_-$, while $J = J^* = J^{-1}$ is acting on $\mathfrak{E}$, $\varphi_- \in \mathfrak{E}$ is an input vector, $\varphi_+ \in \mathfrak{E}$ is an output vector, and $x \in \mathfrak{H}_+$ is a vector of the inner state of the system Θ. The system described by (8) is called a *canonical* Livšic system or Brodskiĭ–Livšic rigged operator colligation, cf., e.g. [19–21]. The operator-valued function

$$W_\Theta(z) = I - 2\mathrm{i}K^*(\mathbb{A} - zI)^{-1}KJ \tag{9}$$

is a transfer function (or characteristic function) of the system Θ. It was shown in [19] that a matrix-valued function $V(z)$ acting on a Hilbert space $\mathfrak{E}$ of the form (1) can be represented and realized in the form

$$V(z) = \mathrm{i}[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I] = K^*(\mathbb{A}_R - zI)^{-1}K, \tag{10}$$

where $W_\Theta(z)$ is a transfer function of some canonical scattering ($J = I$) system Θ, and where the "*real part*" $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ of $\mathbb{A}$ satisfies $\mathbb{A}_R \supset A$ if and only if the function $V(z)$ in (1) satisfies the following two conditions:

$$\begin{cases} L = 0, \\ Qx = \int_{\mathbb{R}} \frac{t}{1+t^2}\mathrm{d}\Sigma(t)x \quad \text{when } \int_{\mathbb{R}}(\mathrm{d}\Sigma(t)x, x)_{\mathfrak{E}} < \infty. \end{cases} \tag{11}$$

This shows that general matrix-valued Herglotz–Nevanlinna functions $V(z)$ acting on $\mathfrak{E}$ cannot be realized in the form (10) even by means of a canonical system (a Brodskiĭ–Livšic rigged operator colligation) Θ of the form (8).

The main purpose of the present paper is to solve the general realization problem for matrix-valued Herglotz–Nevanlinna functions. The case of Herglotz–Nevanlinna functions of the form (1) with a bounded measure was considered in [30–32]. In the general case, an appropriate realization for these functions will be established by introducing new types systems: so-called *non-canonical* $\varDelta_+$-systems and F_+-systems. A $\varDelta_+$-system or *impedance* system can be written as

$$\begin{cases} (\mathbb{D} - zF_+)x = K\varphi_-, \\ \varphi_+ = K^*x, \end{cases} \tag{12}$$

where $\mathbb{D}$ and F_+ are self-adjoint operators acting from $\mathfrak{H}_+$ into $\mathfrak{H}_-$ and in addition F_+ is an orthogonal projector in $\mathfrak{H}_+$. In this case the associated transfer function is given by

$$V(z) = K^*(\mathbb{D} - zF_+)^{-1}K. \tag{13}$$

It will be shown that every matrix-valued Herglotz–Nevanlinna function can be represented in the form (13).

Another type of realization problem deals with so-called non-canonical [41,42] F_+-systems,

$$\begin{cases} (\mathbb{A} - zF_+)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2\mathrm{i}K^*x, \end{cases} \tag{14}$$

also called rigged F_+-colligations. This colligation can be expressed via an array similar to the Brodskiĭ–Livšic rigged operator colligation (8):

$$\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & F_+ & K & J \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & & \mathfrak{E} \end{pmatrix}. \tag{15}$$

The additional ingredient in (15) is the operator F_+ which is an orthogonal projection in $\mathfrak{H}_+$ and $\mathfrak{H}$. The corresponding transfer function (or F_+-characteristic function) is

$$W_{\Theta,F_+}(z) = I - 2\mathrm{i}K^*(\mathbb{A} - zF_+)^{-1}KJ. \tag{16}$$

It will be shown that every matrix-valued Herglotz–Nevanlinna function with an invertible matrix L (or $L = 0$) in (1) can be represented in the form

$$V(z) = K^*(\mathbb{A}_R - zF_+)^{-1}K, \tag{17}$$

where $\mathbb{A}_R = \frac{1}{2}(\mathbb{A} + \mathbb{A}^*)$ is the “real part” of the main operator $\mathbb{A}$ in the corresponding F_+-colligation. The corresponding F_+-characteristic function $W_{\Theta,F_+}(z)$ is related to the Herglotz–Nevanlinna function $V(z)$ via

$$V(z) = \mathrm{i}[W_{\Theta,F_+}(z) + I]^{-1}[W_{\Theta,F_+}(z) - I].$$

Moreover, it will also be shown that the operators $\mathbb{D}$ and F_+ in (13) can be selected so that they satisfy a certain commutativity condition precisely when the linear term in (1) is absent, i.e., if $L = 0$. When $F_+ = I$ the constructed realization reduces to the Brodskiĭ–Livšic rigged operator colligation (canonical system) (8) as well as to the classical Brodskiĭ–Livšic operator colligation (canonical system) when the measure $\mathrm{d}\Sigma(t)$ in (1) is compactly supported; this includes all the previous results in the realization problem for matrix-valued Herglotz–Nevanlinna functions. The results in this paper depend in an essential way on the theory of extensions in rigged Hilbert spaces [49,48]; a concise exposition of this theory is provided in [49].

A different approach to realization problems is due to Ball and Staffans [16,17,44–46]. In particular, they consider canonical input-state–output systems of the type

$$\begin{cases} \dot{x} = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{18}$$

with the transfer mapping

$$T(s) = D + C(sI - A)^{-1}B. \tag{19}$$

It follows directly from [16,17,46] that for an arbitrary Herglotz–Nevanlinna function $V(z)$ of the type (1) with $L = 0$ the function $-\mathrm{i}V(\mathrm{i}z)$ can be realized in the form (19) by a canonical impedance conservative system (18) considered in [16,17,46]. However, this does not contradict the criteria for the canonical realizations (9)–(11) established by two of the authors in [19] due to the special type ($F_+ = I$) of the Livšic systems (Brodskiĭ–Livšic rigged operator colligations) under consideration. Theorem 4.1 of the present paper provides a general result for non-canonical realizations of such functions. The general realization case involving a non-zero linear term in (1) is also implicitly treated by Ball and Staffans in [16,17].

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2. Some preliminaries

Let $\mathfrak{H}$ be a Hilbert space with inner product (x, y) and let A be a closed linear operator in $\mathfrak{H}$ which is Hermitian, i.e., $(Ax, y) = (x, Ay)$, for all $x, y \in \operatorname{dom} A$. In general, A need not be densely defined. The closure of its domain in $\mathfrak{H}$ is denoted by $\mathfrak{H}_0 = \overline{\operatorname{dom} A}$. In the sequel A is often considered as an operator from $\mathfrak{H}_0$ into $\mathfrak{H}$. Then the adjoint A^* of A is a densely defined operator from $\mathfrak{H}$ into $\mathfrak{H}_0$. Associated to A are two Hilbert spaces $\mathfrak{H}_+$ and $\mathfrak{H}_-$, the spaces with a positive and a negative norm. The space $\mathfrak{H}_+$ is $\operatorname{dom} A^*$ equipped with the graph inner product:

$$(f, g)_+ = (f, g) + (A^*x, A^*y), \quad f, g \in \operatorname{dom} A^*,$$

while $\mathfrak{H}_-$ is the corresponding dual space consisting of all linear functionals on $\mathfrak{H}_+$, which are continuous with respect to $\|\cdot\|_+$. This gives rise to a triplet $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ of Hilbert spaces, which is often called a rigged Hilbert space associated to A. The norms of these spaces satisfy the inequalities

$$\|x\| \leqslant \|x\|_+, \ x \in \mathfrak{H}_+, \quad \text{and} \quad \|x\|_- \leqslant \|x\|, \ x \in \mathfrak{H}.$$

In what follows the prefixes (+)-, (·)-, and (−)- will be used to refer to corresponding metrics, norms, or inner products of rigged Hilbert spaces. For example, a $(-,\cdot)$-continuity of an operator means that it is continuous if considered as operating from $\mathfrak{H}_-$ in $\mathfrak{H}$. Recall that there is an isometric operator R, the so-called Riesz–Berezanskiĭ operator, which maps $\mathfrak{H}_-$ onto $\mathfrak{H}_+$ such that

$$\begin{aligned}(x,y)_- &= (x,Ry) = (Rx,y) = (Rx,Ry)_+, \quad x,y\in\mathfrak{H}_-,\\ (u,v)_+ &= (u,R^{-1}v) = (R^{-1}u,v) = (R^{-1}u,R^{-1}v)_-, \quad u,v\in\mathfrak{H}_+,\end{aligned} \tag{20}$$

see [22]. A closed densely defined linear operator T in $\mathfrak{H}$ is said to belong to the class Ω_A if:

(i) $A = T\cap T^*$ (i.e. A is the maximal common symmetric part of T and T^*);
(ii) $-i$ is a regular point of T.

An operator $\mathbb{A}\in[\mathfrak{H}_+,\mathfrak{H}_-]$ is called a (∗)-*extension* of $T\in\Omega_A$ if the inclusions

$$T\subset\mathbb{A} \quad\text{and}\quad T^*\subset\mathbb{A}^*$$

are satisfied. Here the adjoints are taken with respect to the underlying inner products and $[\mathfrak{H}_1,\mathfrak{H}_2]$ stands for the class of all linear bounded operators between the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$. An operator $\mathbb{A}\in[\mathfrak{H}_+,\mathfrak{H}_-]$ is a *bi-extension* of A if $\mathbb{A}\supset A$ and $\mathbb{A}^*\supset A$. Clearly, every (∗)-extension $\mathbb{A}$ of $T\in\Omega_A$ is a bi-extension of A. Taking into account again that the adjoints are taken with respect to the duality of corresponding inner products, we call a bi-extension $\mathbb{A}$ of A a *self-adjoint bi-extension* if $\mathbb{A}=\mathbb{A}^*$ and the operator $\widetilde{A}$ defined by

$$\widetilde{A} = \{\{x,\mathbb{A}x\} : x\in\mathfrak{H}_+,\ \mathbb{A}x\in\mathfrak{H}\} \tag{21}$$

is a self-adjoint extension of A in the original Hilbert space $\mathfrak{H}$. A (∗)-extension $\mathbb{A}$ of T is called *correct* if its "real part" $\mathbb{A}_R := \frac{1}{2}(\mathbb{A}+\mathbb{A}^*)$ is a self-adjoint bi-extension of A.

For two operators A and B in a Hilbert space $\mathfrak{H}$ the set of all points $z\in\mathbb{C}$ such that the operator $(A-zB)^{-1}$ exists on $\mathfrak{H}$ and is bounded will be denoted by $\rho(A,B)$ and $\rho(A)=\rho(A,I)$. For some basic facts concerning resolvent operators of the form $(A-zB)^{-1}$, see [32,41,42].

Now proper definitions for both Δ_+-systems and F_+-systems can be given.

Definition 2.1. Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_-$ be the rigged Hilbert space associated with A. The system of equations

$$\begin{cases}(\mathbb{D}-zF_+)x = K\varphi_-,\\ \varphi_+ = K^*x,\end{cases} \tag{22}$$

where $\mathfrak{E}$ is a finite-dimensional Hilbert space is called a Δ_+-system or impedance system if:

(i) $\mathbb{D}\in[\mathfrak{H}_+,\mathfrak{H}_-]$ is a self-adjoint bi-extension of A;
(ii) $K\in[\mathfrak{E},\mathfrak{H}_-]$ with $\ker K=\{0\}$ (i.e. K is invertible);
(iii) F_+ is an orthogonal projection in $\mathfrak{H}_+$ and $\mathfrak{H}$;
(iv) the set $\rho(\mathbb{D},F_+,K)$ of all points $z\in\mathbb{C}$ where $(\mathbb{D}-zF_+)^{-1}$ exists on $\mathfrak{H}\cup\operatorname{ran}K$ and $(-,\cdot)$-continuous is open.

Definition 2.2. Let A be a closed symmetric operator in a Hilbert space $\mathfrak{H}$ and let $\mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_-$ be the rigged Hilbert space associated with A. The array

$$\Theta = \Theta_{F_+} = \begin{pmatrix}\mathbb{A} & F_+ & K & J\\ \mathfrak{H}_+\subset\mathfrak{H}\subset\mathfrak{H}_- & & & \mathfrak{E}\end{pmatrix}, \tag{23}$$

where $\mathfrak{E}$ is a finite-dimensional Hilbert space is called an F_+-colligation or an F_+-system if:

(i) $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a correct $(*)$-extension of $T \in \Omega_A$;
(ii) $J = J^* = J^{-1} : \mathfrak{E} \to \mathfrak{E}$;
(iii) $\mathbb{A} - \mathbb{A}^* = 2\mathrm{i}KJK^*$, where $K \in [\mathfrak{E}, \mathfrak{H}_-]$ and $\ker K = \{0\}$ (K is invertible);
(iv) F_+ is an orthogonal projection in $\mathfrak{H}_+$ and $\mathfrak{H}$;
(v) the set $\rho(\mathbb{A}, F_+, K)$ of all points $z \in \mathbb{C}$, where $(\mathbb{A} - zF_+)^{-1}$ exists on $\mathfrak{H} \cup \operatorname{ran} K$ and $(-,\cdot)$-continuous, is open;
(vi) the set $\rho(\mathbb{A}_R, F_+, K)$ of all points $z \in \mathbb{C}$, where $(\mathbb{A}_R - zF_+)^{-1}$ exists on $\mathfrak{H} \cup \operatorname{ran} K$ and $(-,\cdot)$-continuous, and the set $\rho(\mathbb{A}, F_+, K) \cap \rho(\mathbb{A}_R, F_+, K)$ are both open;
(vii) if $z \in \rho(\mathbb{A}, F_+, K)$ then $\bar{z} \in \rho(\mathbb{A}^*, F_+, K)$; if $z \in \rho(\mathbb{A}_R, F_+, K)$ then $\bar{z} \in \rho(\mathbb{A}_R, F_+, K)$.

The system (23) is conservative in the sense that $\operatorname{Im} \mathbb{A} = KJK^*$. It is said to be a scattering system if $J = I$. In this case the main operator $\mathbb{A}$ in (23) is dissipative: $\operatorname{Im} \mathbb{A} \geqslant 0$. When $F_+ = I$ and $\mathbb{A}$ is a correct $(*)$-extension of $T \in \Omega_A$ the F_+-system in Definition 2.2 reduces to a rigged operator colligation (canonical system) of Brodskiĭ–Livšic type. It was shown in [19] that each operator T from the class Ω_A admits a correct $(*)$-extension $\mathbb{A}$, which can be included as the main operator in such a rigged operator colligation and that all the properties in Definition 2.2 are automatically fulfilled.

To each F_+-system (F_+-colligation) in Definition 2.2 one can associate a transfer function, or a characteristic function, via

$$W_\Theta(z) = I - 2\mathrm{i}K^*(\mathbb{A} - zF_+)^{-1}KJ. \tag{24}$$

Proposition 2.3. *Let Θ_{F_+} be an F_+-colligation of the form (23). Then for all $z, w \in \rho(\mathbb{A}, F_+, K)$,*

$$W_{\Theta_{F_+}}(z)JW^*_{\Theta_{F_+}}(w) - J = 2\mathrm{i}(\bar{w} - z)K^*(\mathbb{A} - zF_+)^{-1}F_+(\mathbb{A}^* - \bar{w}F_+)^{-1}K,$$
$$W^*_{\Theta_{F_+}}(w)JW_{\Theta_{F_+}}(z) - J = 2\mathrm{i}(\bar{w} - z)JK^*(\mathbb{A}^* - \bar{w}F_+)^{-1}F_+(\mathbb{A} - zF_+)^{-1}KJ.$$

Proof. By the properties (iii) and (vi) in Definition 2.2 one has for all $z, w \in \rho(\mathbb{A}, F_+, K)$

$$\begin{aligned}
&(\mathbb{A} - zF_+)^{-1} - (\mathbb{A}^* - \bar{w}F_+)^{-1} \\
&\quad = (\mathbb{A} - zF_+)^{-1}[(\mathbb{A}^* - \bar{w}F_+) - (\mathbb{A} - zF_+)](\mathbb{A}^* - \bar{w}F_+)^{-1} \\
&\quad = (z - \bar{w})(\mathbb{A} - zF_+)^{-1}F_+(\mathbb{A}^* - \bar{w}F_+)^{-1} - 2\mathrm{i}(\mathbb{A} - zF_+)^{-1}KJK^*(\mathbb{A}^* - \bar{w}F_+)^{-1}.
\end{aligned}$$

This identity together with (24) implies that

$$\begin{aligned}
&W_{\Theta_{F_+}}(z)JW^*_{\Theta_{F_+}}(w) - J \\
&\quad = [I - 2\mathrm{i}K^*(\mathbb{A} - zF_+)^{-1}KJ]J[I + 2\mathrm{i}JK^*(\mathbb{A}^* - \bar{w}F_+)^{-1}K] - J \\
&\quad = 2\mathrm{i}(\bar{w} - z)K^*(\mathbb{A} - zF_+)^{-1}F_+(\mathbb{A}^* - \bar{w}F_+)^{-1}K.
\end{aligned}$$

This proves the first equality. Likewise one proves the second identity by using

$$\begin{aligned}
&(\mathbb{A} - zF_+)^{-1} - (\mathbb{A}^* - \bar{w}F_+)^{-1} \\
&\quad = (z - \bar{w})(\mathbb{A}^* - \bar{w}F_+)^{-1}F_+(\mathbb{A}^* - zF_+)^{-1} - 2\mathrm{i}(\mathbb{A}^* - \bar{w}F_+)^{-1}KJK^*(\mathbb{A} - zF_+)^{-1}.
\end{aligned}$$

This completes the proof. □

Proposition 2.3 shows that the transfer function $W_{\Theta_{F_+}}(z)$ in (24) associated to an F_+-system of the form (23) is J-unitary on the real axis, J-expansive in the upper halfplane, and J-contractive in the lower halfplane with $z \in \rho(\mathbb{A}, F_+, K)$.

There is another function that one can associate to each F_+-system Θ_{F_+} of the form (23). It is defined via

$$V_{\Theta_{F_+}}(z) = K^*(\mathbb{A}_R - zF_+)^{-1}K, \quad z \in \rho(\mathbb{A}_R, F_+, K), \tag{25}$$

where $\rho(\mathbb{A}_R, F_+, K)$ is defined above. Clearly, $\rho(\mathbb{A}_R, F_+, K)$ is symmetric with respect to the real axis.

Theorem 2.4. *Let Θ_{F_+} be an F_+-system of the form (23) and let $W_{\Theta_{F_+}}(z)$ and $V_{\Theta_{F_+}}(z)$ be defined by (24) and (25), respectively. Then for all $z, w \in \rho(\mathbb{A}_R, F_+, K)$,*

$$V_{\Theta_{F_+}}(z) - V_{\Theta_{F_+}}(w)^* = (z - \bar{w})K^*(\mathbb{A}_R - zF_+)^{-1}F_+(\mathbb{A}_R - \bar{w}F_+)^{-1}K, \tag{26}$$

$V_{\Theta_{F_+}}(z)$ is a matrix-valued Herglotz–Nevanlinna function, and for each $z \in \rho(\mathbb{A}_R, F_+, K) \cap \rho(\mathbb{A}, F_+, K)$ the operators $I + \mathrm{i}V_{\Theta_{F_+}}(z)J$ and $I + W_{\Theta_{F_+}}(z)$ are invertible. Moreover,

$$V_{\Theta_{F_+}}(z) = \mathrm{i}[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I]J, \tag{27}$$

and

$$W_{\Theta_{F_+}}(z) = [I + \mathrm{i}V_{\Theta_{F_+}}(z)J]^{-1}[I - \mathrm{i}V_{\Theta_{F_+}}(z)J]. \tag{28}$$

Proof. For each $z, w \in \rho(\mathbb{A}_R, F_+, K)$ one has

$$(\mathbb{A}_R - zF_+)^{-1} - (\mathbb{A}_R - \bar{w}F_+)^{-1} = (z - \bar{w})(\mathbb{A}_R - zF_+)^{-1}F_+(\mathbb{A}_R - \bar{w}F_+)^{-1}. \tag{29}$$

In view of (25) this implies (26).

Clearly,

$$V_{\Theta_{F_+}}(z)^* = V_{\Theta_{F_+}}(\bar{z}).$$

Moreover, it follows from (26) and Definition 2.2 that $V_{\Theta_{F_+}}(z)$ is a matrix-valued Herglotz–Nevanlinna function.

The following identity with $z \in \rho(\mathbb{A}, F_+, K) \cap \rho(\mathbb{A}_R, F_+, K)$

$$(\mathbb{A}_R - zF_+)^{-1} - (\mathbb{A} - zF_+)^{-1} = \mathrm{i}(\mathbb{A} - zF_+)^{-1}\mathrm{Im}\,\mathbb{A}(\mathbb{A}_R - zF_+)^{-1}$$

leads to

$$K^*(\mathbb{A}_R - zF_+)^{-1}K - K^*(\mathbb{A} - zF_+)^{-1}K = \mathrm{i}K^*(\mathbb{A} - zF_+)^{-1}KJK^*(\mathbb{A}_R - zF_+)^{-1}K.$$

Now in view of (24) and (25)

$$2V_{\Theta_{F_+}}(z) + \mathrm{i}(I - W_{\Theta_{F_+}}(z))J = (I - W_{\Theta_{F_+}}(z))V_{\Theta_{F_+}}(z),$$

or equivalently, that

$$[I + W_{\Theta_{F_+}}(z)][I + \mathrm{i}V_{\Theta_{F_+}}(z)J] = 2I. \tag{30}$$

Similarly, the identity

$$(\mathbb{A}_R - zF_+)^{-1} - (\mathbb{A} - zF_+)^{-1} = \mathrm{i}(\mathbb{A}_R - zF_+)^{-1}\mathrm{Im}\,\mathbb{A}(\mathbb{A} - zF_+)^{-1}$$

with $z \in \rho(\mathbb{A}, F_+, K) \cap \rho(\mathbb{A}_R, F_+, K)$ leads to

$$[I + \mathrm{i}V_{\Theta_{F_+}}(z)J][I + W_{\Theta_{F_+}}(z)] = 2I. \tag{31}$$

The equalities (30) and (31) show that the operators are invertible and consequently one obtains (27) and (28). □

3. Impedance realizations of Herglotz–Nevanlinna functions

The realization of Herglotz–Nevanlinna functions has been obtained for various subclasses. In this section earlier realizations are combined to present a general realization of an arbitrary Herglotz–Nevanlinna function by an impedance system. The following lemma is essentially contained in [31]; for completeness a full proof is presented here.

Lemma 3.1. *Let Q be a self-adjoint operator in a finite-dimensional Hilbert space $\mathfrak{E}$. Then $V(z) = Q$ admits a representation of the form*

$$V(z) = K^*(D - zF_+)^{-1}K, \quad z \in \rho(D, F), \tag{32}$$

where K is an invertible mapping from $\mathfrak{E}$ into a Hilbert space $\mathfrak{H}$, D is a bounded self-adjoint operator in $\mathfrak{H}$, and F_+ is an orthogonal projection in $\mathfrak{H}$ whose kernel ker F_+ *is finite-dimensional.*

Proof. First assume that Q is invertible. Let $\mathfrak{H} = \mathfrak{E}$, let K be any invertible mapping from $\mathfrak{E}$ onto $\mathfrak{H}$, and let $D = KQ^{-1}K^*$. Then D is a bounded self-adjoint operator in $\mathfrak{H}$. Clearly, $V(z) = K^*(D - zF)^{-1}K$ with $F = 0$, an orthogonal projection in $\mathfrak{H}$. In the general case, Q can be written as the sum of two invertible self-adjoint operators $Q = Q^{(1)} + Q^{(2)}$ (for example, $Q^{(1)} = Q - \varepsilon I$ and $Q^{(2)} = \varepsilon I$, where ε is a real number), so that

$$Q^{(1)} = K^{(1)*}(D^{(1)} - zF^{(1)})^{-1}K^{(1)}, \quad Q^{(2)} = K^{(2)*}(D^{(2)} - zF^{(2)})^{-1}K^{(2)},$$

where $K^{(i)}$ is an invertible operator from $\mathfrak{E}$ into a Hilbert space $\mathfrak{H}^{(i)} = \mathfrak{E}$, $D^{(i)}$ is a bounded self-adjoint operator in $\mathfrak{H}^{(i)}$, and $F^{(i)} = 0$ is an orthogonal projection in $\mathfrak{H}^{(i)}$, $i = 1, 2$. (Note that since $K^{(i)}$ is an arbitrary invertible operator from $\mathfrak{E}$ into $\mathfrak{H}^{(i)} = \mathfrak{E}$ it may as well be chosen as $K^{(i)} = I_{\mathfrak{E}}$.) Define

$$\mathfrak{H} = \mathfrak{H}^{(1)} \oplus \mathfrak{H}^{(2)}, \quad K = \begin{pmatrix} K^{(1)} \\ K^{(2)} \end{pmatrix}, \quad D = \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix}, \quad F_+ = \begin{pmatrix} F^{(1)} & 0 \\ 0 & F^{(2)} \end{pmatrix}.$$

Then K is an invertible operator from $\mathfrak{E}$ into the Hilbert space $\mathfrak{H}$, D is a bounded self-adjoint operator, and $F_+ = 0$ is an orthogonal projection in $\mathfrak{H}$. Moreover,

$$\begin{aligned} Q = Q^{(1)} + Q^{(2)} &= K^{(1)^*}(D^{(1)} - zF^{(1)})^{-1}K^{(1)} + K^{(2)^*}(D^{(2)} - zF^{(2)})^{-1}K^{(2)} \\ &= K^*(D - zF_+)^{-1}K, \end{aligned}$$

which proves the lemma. □

Herglotz–Nevanlinna functions of the form (1) which satisfy the conditions in (11) can be realized by means of the theory of regularized generalized resolvents, [19,20]. By means of Lemma 3.1 these realizations can be extended to Herglotz–Nevanlinna functions of the form (1) with $L = 0$.

Theorem 3.2. *Let $V(z)$ be a Herglotz–Nevanlinna function, acting on a finite-dimensional Hilbert space $\mathfrak{E}$, with the integral representation*

$$V(z) = Q + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t), \tag{33}$$

where $Q = Q^*$ and $\Sigma(t)$ is a nondecreasing matrix-valued function on $\mathbb{R}$ satisfying (2). Then $V(z)$ admits a realization of the form

$$V(z) = K^*(\mathbb{D} - zF_+)^{-1}K, \quad z \in \mathbb{C} \setminus \mathbb{R} \subset \rho(\mathbb{D}, F_+, K), \tag{34}$$

where $\mathbb{D} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a self-adjoint bi-extension, $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ is a rigged Hilbert space, F_+ is an orthogonal projection in $\mathfrak{H}_+$ and $\mathfrak{H}$, K is an injective (invertible) operator from $\mathfrak{E}$ into $\mathfrak{H}_+$, $K^* \in [\mathfrak{H}_+, \mathfrak{E}]$. Moreover, the operators $\mathbb{D}$ and F_+ can be selected such that the following commutativity condition holds:

$$F_-\mathbb{D} = \mathbb{D}F_+, \quad F_- = R^{-1}F_+R \in [\mathfrak{H}_-, \mathfrak{H}_-], \tag{35}$$

where R is the Riesz–Berezanskiĭ operator defined in (20).

Proof. According to [19, Theorem 9] each matrix-valued Herglotz–Nevanlinna function of the form (33) admits a realization of the form

$$V(z) = K^*(\mathbb{A}_R - zI)^{-1}K = \mathrm{i}[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I], \tag{36}$$

where $W_\Theta(z)$ is the transfer function (9) of a system of the form (8) if and only if the following condition holds:

$$Qx = \int_{\mathbb{R}} \frac{t}{1+t^2} \mathrm{d}\Sigma(t)x \tag{37}$$

for every vector $x \in \mathfrak{E}$, such that

$$\int_{\mathbb{R}} (\mathrm{d}\Sigma(t)x, x)_{\mathfrak{E}} < \infty. \tag{38}$$

To prove the existence of the representation (34) for Herglotz–Nevanlinna functions $V(z)$ which do not satisfy the condition (37), the realization result in Lemma 3.1 will be used. Denote by $\mathfrak{E}_1$ the linear subspace of vectors $x \in \mathfrak{E}$ with the property (38) and let $\mathfrak{E}_2 = \mathfrak{E} \ominus \mathfrak{E}_1$, so that $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$. Rewrite Q in the block matrix form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q_{ij} = P_{\mathfrak{E}_i} Q \upharpoonright \mathfrak{E}_j, \ j = 1, 2$$

and let $\Sigma(t) = (\Sigma_{ij}(t))_{i,j=1}^2$ be decomposed accordingly. Observe, that by (2), (37), (38) the integrals

$$G_{11} := \int_{\mathbb{R}} \frac{t}{1+t^2} \mathrm{d}\Sigma_{11}(t), \quad G_{12} := \int_{\mathbb{R}} \frac{t}{1+t^2} \mathrm{d}\Sigma_{12}(t) \tag{39}$$

are convergent. Let the self-adjoint matrix G be defined by

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12}^* & C \end{pmatrix}, \tag{40}$$

where $C = C^*$ is arbitrary. Now rewrite $V(z) = V_1(z) + V_2(z)$ with

$$V_1(z) = Q - G, \quad V_2(z) = G + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t). \tag{41}$$

Clearly, for every $x \in \mathfrak{E}_1$ the equality

$$Gx = \int_{\mathbb{R}} \frac{t}{1+t^2} \mathrm{d}\Sigma(t)x$$

holds. Consequently, $V_2(z)$ admits the following representation:

$$V_2(z) = K_2^*(\mathbb{A}_R^{(2)} - zI)^{-1} K_2, \tag{42}$$

where $K_2 : \mathfrak{E} \to \mathfrak{H}_{-2}$, $K_2^* : \mathfrak{H}_{+2} \to \mathfrak{E}$ with $\mathfrak{H}_{+2} \subset \mathfrak{H}_2 \subset \mathfrak{H}_{-2}$ a rigged Hilbert space, and where $\mathbb{A}_R^{(2)} = \frac{1}{2}(\mathbb{A}^{(2)} + (\mathbb{A}^{(2)})^*)$ is a self-adjoint bi-extension of a Hermitian operator A_2. The operator K_2 is invertible and has the properties

$$\begin{aligned} &\operatorname{ran} K_2 \subset \operatorname{ran}(\mathbb{A}^{(2)} - zI), \quad \operatorname{ran} K_2 \subset \operatorname{ran}(\mathbb{A}_R^{(2)} - zI), \\ &(\mathbb{A}^{(2)} - zI)^{-1} K_2 \in [\mathfrak{E}, \mathfrak{H}_+], \quad (\mathbb{A}_R^{(2)} - zI)^{-1} K_2 \in [\mathfrak{E}, \mathfrak{H}_+] \end{aligned} \tag{43}$$

for further details, see [19]. Now, by Lemma 3.1 the function $V_1(z)$ admits the representation

$$V_1(z) = K_1^*(D_1 - zF_{+,1})^{-1} K_1,$$

where $D_1 = D_1^*$ and $F_{+,1} = 0$ are acting on a finite-dimensional Hilbert space $\mathfrak{H}_1 = \mathfrak{E} \oplus \mathfrak{E}$ and where $K_1 : \mathfrak{E} \to \mathfrak{H}_1$ is invertible. Recall from Lemma 3.1 that

$$D_1 = \begin{pmatrix} D_1^{(1)} & 0 \\ 0 & D_1^{(2)} \end{pmatrix}, \quad K_1 = \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \end{pmatrix}, \tag{44}$$

where $K_1^{(i)} : \mathfrak{E} \to \mathfrak{E}$, $i = 1, 2$, and $D_1^{(1)}$, $D_1^{(2)}$ are defined by means of the decomposition of $Q - G$ into the sum of two invertible self-adjoint operators

$$Q - G = (Q^{(1)} - G^{(1)}) + (Q^{(2)} - G^{(2)}).$$

Then

$$D_1^{(i)} = K_1^{(i)*}(Q^{(i)} - G^{(i)})^{-1} K_1^{(i)}, \quad i = 1, 2. \tag{45}$$

To obtain the realization (34) for $V(z)$ in (33), introduce the following triplet of Hilbert spaces:

$$\mathfrak{H}_+^{(1)} := \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{+2} \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_2 \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{-2} := \mathfrak{H}_-^{(1)}, \tag{46}$$

i.e., a rigged Hilbert space corresponding to the block representation of symmetric operator $D_1 \oplus A_2$ in $\mathfrak{H}^{(1)} := \mathfrak{H}_1 \oplus \mathfrak{H}_2$ (where $\mathfrak{H}_1 = \mathfrak{E} \oplus \mathfrak{E}$). Also introduce the following operators:

$$\mathbb{D} = \begin{pmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_1^{(2)} & 0 \\ 0 & 0 & \mathbb{A}_R^{(2)} \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \\ K_2 \end{pmatrix}. \tag{47}$$

It is straightforward to check that

$$\begin{aligned} V(z) = V_1(z) + V_2(z) &= K_1^{(1)*}(D_1^{(1)} - zF_{+,1})^{-1} K_1^{(1)} \\ &\quad + K_1^{(2)*}(D_1^{(2)} - zF_{+,1})^{-1} K_1^{(2)} + K_2^*(\mathbb{A}_R^{(2)} - zI)^{-1} K_2 \\ &= K^*(\mathbb{D} - zF_+)^{-1} K. \end{aligned} \tag{48}$$

By the construction, $A_2 \subset \widetilde{A}_R^{(2)} = (\widetilde{A}_R^{(2)})^* \subset \mathbb{A}_R^{(2)}$, where

$$\widetilde{A}_R^{(2)} = \{\{f, g\} \in \mathbb{A}_R^{(2)} : g \in \mathfrak{H}\},$$

and A_2 is a symmetric operator in $\mathfrak{H}_2$, cf. (21). Moreover, $\mathbb{D}$ as an operator in $[\mathfrak{H}_+^{(1)}, \mathfrak{H}_-^{(1)}]$ is self-adjoint, i.e. $\mathbb{D} = \mathbb{D}^*$, and since

$$\widehat{D} = \begin{pmatrix} D_1 & 0 \\ 0 & A^{(2)} \end{pmatrix} \subset \begin{pmatrix} D_1 & 0 \\ 0 & \mathbb{A}_R^{(2)} \end{pmatrix} = \mathbb{D} \tag{49}$$

and $A = D_1 \oplus A_2 \subset \widehat{D}$, the operator $\mathbb{D}$ is a self-adjoint bi-extension of the Hermitian operator A in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. It is easy to see that with operators in (47) one obtains the representation (34) for $V(z)$ in (33) and the system constructed with these operators satisfy the Definition 2.1 of a Δ_+-system.

Finally, from (47) one obtains $F_- \mathbb{D} = \mathbb{D} F_+$, where F_+ and F_- are connected as in (35). This completes the proof of the theorem. □

Remark 3.3. According to the recent results by Staffans [46] an operator-function $(-\mathrm{i})V(\mathrm{i}z)$, where $V(z)$ is defined by (33) can be realized by an impedance system of the form (18)–(19) (see also [16,17,44,45]). This realization is carried out by using a different approach and does not possess some of the properties contained in Theorem 3.2.

The general impedance realization result for Herglotz–Nevanlinna functions of the form (1) is now built on Theorem 3.2 and a representation for linear functions.

Lemma 3.4. *Let L be a nonnegative matrix in a finite-dimensional Hilbert space $\mathfrak{E}$. Then it admits a realization of the form*

$$zL = z\widehat{K}^* P \widehat{K} = K_3^*(D_3 - zF_3)^{-1} K_3, \tag{50}$$

where D_3 is a self-adjoint matrix in a Hilbert space $\mathfrak{H}_3$, P is the orthogonal projection onto $\overline{\mathrm{ran}}\, L$, and K_3 is an invertible operator from $\mathfrak{E}$ into $\mathfrak{H}_3$.

Proof. Since $L \geqslant 0$, there is a unique nonnegative square root $L^{1/2} \geqslant 0$ of L with

$$\ker L^{1/2} = \ker L, \quad \overline{\mathrm{ran}}\, L^{1/2} = \overline{\mathrm{ran}}\, L.$$

Define the operator $\widehat{K}$ in $\mathfrak{E}$ by

$$\widehat{K}u = \begin{cases} u, & u \in \ker L; \\ L^{1/2}u, & u \in \overline{\mathrm{ran}}\, L. \end{cases} \tag{51}$$

Then $\widehat{K}$ is invertible and $L^{1/2} = P\widehat{K}$, where P denotes the orthogonal projection onto $\overline{\mathrm{ran}}\, L$. Define

$$\mathfrak{H}_3 = \mathfrak{E} \oplus \mathfrak{E}, \quad K_3 = \begin{pmatrix} P\widehat{K} \\ \widehat{K} \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix}, \quad F_{+,3} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{52}$$

Then K_3 is an invertible operator from $\mathfrak{E}$ into $\mathfrak{H}_3$, D_3 is a bounded self-adjoint operator, and $F_{+,3}$ is an orthogonal projection in $\mathfrak{H}_3$. Moreover,

$$V_3(z) = zL = z\widehat{K}^* P \widehat{K} = K_3^*(D_3 - zF_{+,3})^{-1} K_3. \tag{53}$$

This completes the proof. □

The general realization result for Herglotz–Nevanlinna functions of the form (1) is now obtained by combining the earlier realizations.

Theorem 3.5. *Let $V(z)$ be a matrix-valued Herglotz–Nevanlinna function in a finite-dimensional Hilbert space $\mathfrak{E}$ with the integral representation*

$$V(z) = Q + zL + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \mathrm{d}\Sigma(t), \tag{54}$$

where $Q = Q^*$, $L \geqslant 0$, and $\Sigma(t)$ is a nondecreasing nonnegative matrix-valued function on $\mathbb{R}$ satisfying (2). Then $V(z)$ admits a realization of the form

$$V(z) = K^*(\mathbb{D} - zF_+)^{-1}K, \tag{55}$$

where $\mathbb{D} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ is a self-adjoint bi-extension in a rigged Hilbert space $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$, F_+ is an orthogonal projection in $\mathfrak{H}_+$ and $\mathfrak{H}$, and $K \in [\mathfrak{E}, \mathfrak{H}_-]$ is an invertible operator from $\mathfrak{E}$ into $\mathfrak{H}_-$.

Proof. Define the following matrix functions:

$$V_1(z) = Q + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t), \quad V_2(z) = zL.$$

According to Theorem 3.2 the function $V_1(z)$ has a representation

$$V_1(z) = K_1^*(\mathbb{D}_1 - zF_{+,1})^{-1}K_1,$$

where $\mathbb{D}_1$, K_1 and $F_{+,1}$ are given by the formula (47). We recall that $\mathbb{D}_1$ is a self-adjoint bi-extension in a rigged Hilbert space $\mathfrak{H}_-^{(1)} \subset \mathfrak{H}^{(1)} \subset \mathfrak{H}_+^{(1)}$ given by (46), $F_{+,1}$ is an orthogonal projection in $\mathfrak{H}_+^{(1)}$, and K_1 is an invertible mapping from $\mathfrak{E}$ into $\mathfrak{H}_-^{(1)}$.

According to Lemma 3.4 the functions $V_2(z)$ has a realization of the form (50) with components $\mathfrak{H}_3$, D_3, K_3 and $F_{+,3}$ described by (52).

Now the final result follows by introducing the rigged Hilbert space $\mathfrak{H}_3 \oplus \mathfrak{H}_+^{(1)} \subset \mathfrak{H}_3 \oplus \mathfrak{H}^{(1)} \subset \mathfrak{H}_3 \oplus \mathfrak{H}_-^{(1)}$ and the operators

$$\mathbb{D} = \begin{pmatrix} D_3 & 0 \\ 0 & \mathbb{D}_1 \end{pmatrix} \in [\mathfrak{H}_3 \oplus \mathfrak{H}_+^{(1)}, \mathfrak{H}_3 \oplus \mathfrak{H}_-^{(1)}], \quad F_+ = \begin{pmatrix} F_{+,3} & 0 \\ 0 & F_{+,1} \end{pmatrix}, \quad K = \begin{pmatrix} K_3 \\ K_1 \end{pmatrix}.$$

It is straightforward to check that with these operators one obtains the representation (55) for $V(z)$ in (54) and the system constructed with these operators satisfy the Definition 2.1 of a Δ_+-system. □

For the sake of clarity an extended version for the impedance realization in the proof of Theorem 3.5 is provided. The rigged Hilbert space used is

$$\mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{+2} \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_2 \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{-2}, \tag{56}$$

and the operators are given by

$$\mathbb{D} = \begin{pmatrix} 0 & \mathrm{i}I & 0 & 0 & 0 \\ -\mathrm{i}I & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1^{(1)} & 0 & 0 \\ 0 & 0 & 0 & D_1^{(2)} & 0 \\ 0 & 0 & 0 & 0 & \mathbb{A}_R^{(2)} \end{pmatrix},$$

$$F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} P\widehat{K} \\ \widehat{K} \\ K_1^{(1)} \\ K_1^{(2)} \\ K_2 \end{pmatrix}. \tag{57}$$

All the operators in (57) are defined above.

In conclusion of this section it is observed that the general impedance realization case involving a non-zero linear term in (54) is also implicitly treated by Ball and Staffans in [16,17].

4. F_+-system realization results

In the general impedance realization results in Theorem 3.2 and Theorem 3.5 the realizations are in terms of the operators in (34) and (55), respectively. It remains to identify the Herglotz–Nevanlinna functions as transforms of transfer functions of appropriate conservative systems.

Theorem 4.1. *Let $V(z)$ be a Herglotz–Nevanlinna function acting on a finite-dimensional Hilbert space $\mathfrak{E}$ with the integral representation*

$$V(z) = Q + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t), \tag{58}$$

where $Q = Q^*$ and $\Sigma(t)$ is a nondecreasing matrix-valued function on $\mathbb{R}$ satisfying (2). Then the function $V(z)$ can be realized in the form

$$V(z) = \mathrm{i}[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I], \tag{59}$$

where $W_{\Theta_{F_+}}(z)$ is the transfer function given by (24) of an F_+-system defined in (23). The F_+-system in (23) can be taken to be a scattering system.

Proof. By Theorem 3.2 the function $V(z)$ can be represented in the form $V(z) = K^*(\mathbb{D} - zF_+)^{-1}K$, where K, $\mathbb{D}$, and F_+ are as in (47) corresponding to the decomposition

$$V(z) = V_1(z) + V_2(z),$$

where

$$V_1(z) = Q - G, \quad V_2(z) = G + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \mathrm{d}\Sigma(t),$$

with a self-adjoint operator G of the form (40). With the notations used in the proof of Theorem 3.2 one may rewrite $V_1(z)$ and $V_2(z)$ as in (48) with

$$D_1^{(1)} = (Q - G - \varepsilon I)^{-1}, \quad D_1^{(2)} = (\varepsilon I)^{-1}, \quad K_1^{(1)} = \lambda I_{\mathfrak{E}}, \quad K_1^{(2)} = I_{\mathfrak{E}}, \tag{60}$$

$\mathbb{A}_R^{(2)} \in [\mathfrak{H}_{+2}, \mathfrak{H}_{-2}]$, $\mathbb{A}_R^{(2)} = \frac{1}{2}(\mathbb{A}^{(2)} + \mathbb{A}^{(2)*})$ is associated to a $(*)$-extension $\mathbb{A}^{(2)}$ of an operator $T_2 \in \Omega_{A_2}$ for which $(-i) \in \rho(T_2)$, cf. [19]. The remaining operators are defined in (47).

Recall that K_2 and the resolvents $(\mathbb{A}^{(2)} - zI)^{-1}$, $(\mathbb{A}_R^{(2)} - zI)^{-1}$ satisfy the properties (43). To construct an F_+-system of the form (23) introduce the operator $\mathbb{A}$ by

$$\mathbb{A} = \mathbb{D} + \mathrm{i}KK^* \in [\mathfrak{H}_+, \mathfrak{H}_-],$$

where K, $\mathbb{D}$, and F_+ are defined in (47). Then the block-matrix form of $\mathbb{A}$ is

$$\mathbb{A} = \begin{pmatrix} D_1^{(1)} + \mathrm{i}\lambda^2 I & \mathrm{i}\lambda I & \mathrm{i}\lambda K_2^* \\ \mathrm{i}\lambda I & D_1^{(2)} + \mathrm{i}I & \mathrm{i}K_2^* \\ \mathrm{i}\lambda K_2 & \mathrm{i}K_2 & \mathbb{A}^{(2)} \end{pmatrix}. \tag{61}$$

Let

$$\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & F_+ & K & I \\ \mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- & & & \mathfrak{E} \end{pmatrix}, \tag{62}$$

where the rigged Hilbert triplet $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ is defined in (46), i.e.,

$$\mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{+2} \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_2 \subset \mathfrak{E} \oplus \mathfrak{E} \oplus \mathfrak{H}_{-2}.$$

It remains to show that all the properties in Definition 2.2 are satisfied. For this purpose, consider the equation

$$(\mathbb{A} - zF_+)x = (\mathbb{D} + iKK^*)x - zF_+x = Ke,$$

or

$$\begin{pmatrix} D_1^{(1)} + \mathrm{i}\lambda^2 I & \mathrm{i}\lambda I & \mathrm{i}\lambda K_2^* \\ \mathrm{i}\lambda I & D_1^{(2)} + \mathrm{i}I & \mathrm{i}K_2^* \\ \mathrm{i}\lambda K_2 & \mathrm{i}K_2 & \mathbb{A}^{(2)} - zI \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda e \\ e \\ K_2 e \end{pmatrix}.$$

Using the decomposition of the operators and taking into account that

$$\mathbb{A}^{(2)} = \mathbb{A}_R^{(2)} + \mathrm{i}K_2K_2^*$$

this equation can be rewritten in form of the following system:

$$\begin{cases} D_1^{(1)}x_1 + \mathrm{i}\lambda^2 Ix_1 + \mathrm{i}\lambda Ix_2 + \mathrm{i}\lambda K_2^*x_3 = \lambda e, \\ D_1^{(2)}x_2 + \mathrm{i}\lambda Ix_1 + \mathrm{i}Ix_2 + \mathrm{i}K_2^*x_3 = e, \\ (\mathbb{A}^{(2)} - zI)x_3 + \mathrm{i}\lambda K_2x_1 + \mathrm{i}K_2x_2 = K_2e, \end{cases} \tag{63}$$

or

$$\begin{cases} \frac{1}{\lambda}D_1^{(1)}x_1 + \mathrm{i}\lambda Ix_1 + \mathrm{i}Ix_2 + \mathrm{i}K_2^*x_3 = e, \\ D_1^{(2)}x_2 + \mathrm{i}\lambda Ix_1 + \mathrm{i}Ix_2 + \mathrm{i}K_2^*x_3 = e, \\ (\mathbb{A}^{(2)} - zI)x_3 + \mathrm{i}\lambda K_2x_1 + \mathrm{i}K_2x_2 = K_2e. \end{cases}$$

In a neighborhood of $(-i)$ the resolvent $(\mathbb{A}^{(2)} - zI)^{-1}$ is well defined so that by (43) the third equation in (63) can be solved for x_3:

$$x_3 = (\mathbb{A}^{(2)} - zI)^{-1}K_2e - \mathrm{i}(\mathbb{A}^{(2)} - zI)^{-1}K_2(\lambda x_1 + x_2). \tag{64}$$

Substitute (64) into the first line of the system yields

$$\frac{1}{\lambda}D_1^{(1)}x_1 + \mathrm{i}I(\lambda x_1 + x_2) + K_2^*(\mathbb{A}^{(2)} - zI)^{-1}K_2(\lambda x_1 + x_2) = e - \mathrm{i}K_2^*(\mathbb{A}^{(2)} - zI)^{-1}K_2e,$$

Denoting the right hand side by C and using (24) we get

$$C = e - \mathrm{i}K_2^*(\mathbb{A}^{(2)} - zI)^{-1}K_2e = \frac{1}{2}\left[I + W_{\Theta_2}(z)\right]e.$$

Then

$$\frac{1}{\lambda}D_1^{(1)}x_1 + \mathrm{i}I(\lambda x_1 + x_2) + K_2^*(\mathbb{A}^{(2)} - zI)^{-1}K_2(\lambda x_1 + x_2) = C,$$

Multiply both sides by 2i and using (24) one more time yields

$$\frac{2\mathrm{i}}{\lambda}D_1^{(1)}x_1 - \left[I + W_{\Theta_2}(z)\right](\lambda x_1 + x_2) = 2\mathrm{i}C.$$

Denoting for further convenience $B = [I + W_{\Theta_2}(z)]$ we obtain

$$\frac{2\mathrm{i}}{\lambda}D_1^{(1)}x_1 - \lambda Bx_1 - Bx_2 = 2\mathrm{i}C,$$

or

$$\frac{2\mathrm{i}}{\lambda}D_1^{(1)}x_1 - \lambda Bx_1 - 2\mathrm{i}C = Bx_2. \tag{65}$$

Now we subtract the second equation of the system from the first and obtain

$$D_1^{(1)}x_1 = \lambda D_1^{(2)}x_2,$$

or

$$\lambda(D_1^{(1)})^{-1}D_1^{(2)}x_2 = x_1. \tag{66}$$

Applying (66) to (65) we get

$$2\mathrm{i}D_1^{(2)}x_2 - B\lambda^2(D_1^{(1)})^{-1}D_1^{(2)}x_2 - Bx_2 = 2\mathrm{i}C,$$

and using (60)

$$\frac{2\mathrm{i}}{\varepsilon}Ix_2 - B\Big[\lambda^2(Q - G - \varepsilon I)\frac{1}{\varepsilon} + I\Big]x_2 = 2\mathrm{i}C,$$

or

$$(2\mathrm{i}I - [I + W_{\Theta_2}(z)][\lambda^2(Q - G) + \varepsilon(1 - \lambda^2)I])x_2 = 2\mathrm{i}\varepsilon C. \tag{67}$$

Choosing λ and ε sufficiently small the matrix on the left hand side of (67) can be made invertible for $z = -\mathrm{i}$. Using an invertibility criteria from [23] we deduce that (67) is also invertible in a neighborhood of $(-\mathrm{i})$. Consequently, the system (63) has a unique solution and $(\mathbb{A} - zF_+)^{-1}K$ is well defined in a neighborhood of $(-\mathrm{i})$.

In order to show that the remaining properties in Definition 2.2 are satisfied we need to present an operator $T \in \Omega_A$ such that $\mathbb{A}$ is a correct $(*)$-extension of T. To construct T we note first that $(\mathbb{A} - zF_+)\mathfrak{H}_+ \supset \mathfrak{H}$ for some z in a neighborhood of $(-\mathrm{i})$. This can be confirmed by considering the equation

$$(\mathbb{A} - zF_+)x = g, \quad x \in \mathfrak{H}_+, \tag{68}$$

and showing that it has a unique solution for every $g \in \mathfrak{H}$. The procedure then is reduced to solving the system (63) with an arbitrary right hand side $g \in \mathfrak{H}$. Following the steps for solving (63) we conclude that the system (68) has a unique solution. Similarly one shows that $(\mathbb{A}^* - zF_+)\mathfrak{H}_+ \supset \mathfrak{H}$. Using the technique developed in [49] we can conclude that operators $(\mathbb{A} + \mathrm{i}F_+)^{-1}$ and $(\mathbb{A}^* - \mathrm{i}F_+)^{-1}$ are $(-,\cdot)$-continuous. Define

$$\begin{aligned} &T = \mathbb{A}, \quad \operatorname{dom} T = (\mathbb{A} + \mathrm{i}F_+)\mathfrak{H}, \\ &T_1 = \mathbb{A}^*, \quad \operatorname{dom} T = (\mathbb{A}^* - \mathrm{i}F_+)\mathfrak{H}. \end{aligned} \tag{69}$$

One can see that both $\operatorname{dom} T$ and $\operatorname{dom} T_1$ are dense in $\mathfrak{H}$ while operator T is closed in $\mathfrak{H}$. Indeed, assuming that there is a vector $\phi \in \mathfrak{H}$ that is $(\cdot)$-orthogonal to $\operatorname{dom} T$ and representing $\phi = (\mathbb{A}^* - \mathrm{i}F_+)\psi$ we can immediately get $\phi = 0$. It is also easy to see that $T_1 = T^*$. Thus, operator T defined by (84) fits the definition of correct (∗)-extension for operator $\mathbb{A}$. Property (vi) of Definition 2.2 follows from Theorem 3.5 and the fact that $\mathbb{A}_R = \mathbb{D}$.

Consequently all the properties for an F_+-system Θ in Definition 2.2 are fulfilled with the operators and spaces defined above. □

Now the principal result of the paper will be presented.

Theorem 4.2. *Let $V(z)$ be a matrix-valued Herglotz–Nevanlinna function in a finite-dimensional Hilbert space $\mathfrak{E}$ with the integral representation*

$$V(z) = Q + zL + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \mathrm{d}\Sigma(t), \tag{70}$$

where $Q = Q^*$, $L \geqslant 0$ is an invertible matrix, and $\Sigma(t)$ is a nondecreasing nonnegative matrix-valued function on $\mathbb{R}$ satisfying (2). Then $V(z)$ can be realized in the form

$$V(z) = \mathrm{i}[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I], \tag{71}$$

where $W_{\Theta_{F_+}}(z)$ is a matrix-valued transfer function of some scattering F_+-system of the form (23).

Proof. Decompose the function $V(z)$ as follows:

$$V_1(z) = Q + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \mathrm{d}\Sigma(t) \quad \text{and} \quad V_2(z) = zL,$$

and use the earlier realizations for each of these functions.

By Theorem 4.1 the function $V_1(z)$ can be represented by

$$V_1(z) = \mathrm{i}[W_{\Theta_{F_{1,+}}}(z) + I]^{-1}[W_{\Theta_{F_{1,+}}}(z) - I],$$

where $W_{\Theta_{F_{1,+}}}(z)$ is a matrix-valued transfer function of some scattering $F_{1,+}$-system,

$$W_{\Theta_{F_{1,+}}}(z) = I - 2\mathrm{i}K_1^*(\mathbb{A}_1 - zF_{1,+})^{-1}K_1, \tag{72}$$

$\mathbb{A}_1 = \mathbb{D}_1 + \mathrm{i}K_1K_1^*$ maps $\mathfrak{H}_{+1}$ continuously into $\mathfrak{H}_{-1}$, $\mathbb{D}_1$ is a self-adjoint bi-extension, and $\mathbb{D}_1 \in [\mathfrak{H}_{+1}, \mathfrak{H}_{-1}]$, $K_1 \in [\mathfrak{E}, \mathfrak{H}_{-1}]$.

Following the proof of Theorem 3.5 the function $V_2(z)$ can be represented in the form

$$V_2(z) = K_2^*(D_2 - zF_{2,+})^{-1}K_2,$$

where

$$D_2 = \begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix}, \quad F_{2,+} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad K_2 = \begin{pmatrix} P\widehat{K} \\ \widehat{K} \end{pmatrix}, \tag{73}$$

and P and $\widehat{K}$ are as in (51), so that K_2 is an invertible operator from $\mathfrak{E}$ into $\mathfrak{H}_2 = \mathfrak{E} \oplus \mathfrak{E}$. Introduce the triplet $\mathfrak{H}_{+1} \oplus \mathfrak{H}_2 \subset \mathfrak{H}_1 \oplus \mathfrak{H}_2 \subset \mathfrak{H}_{-1} \oplus \mathfrak{H}_2$, and consider the operator

$$\mathbb{A} = \mathbb{D} + \mathrm{i}KK^* \tag{74}$$

from $\mathfrak{H}_{+1} \oplus \mathfrak{H}_2$ into $\mathfrak{H}_{-1} \oplus \mathfrak{H}_2$ given by the block form

$$\mathbb{A} = \begin{pmatrix} \mathbb{D}_1 & 0 \\ 0 & D_2 \end{pmatrix} + \mathrm{i}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\begin{pmatrix} K_1^* & K_2^* \end{pmatrix} = \begin{pmatrix} \mathbb{A}_1 & \mathrm{i}K_1K_2^* \\ \mathrm{i}K_2K_1^* & \mathbb{A}_2 \end{pmatrix}. \tag{75}$$

Here $\mathbb{A}_2 = D_2 + \mathrm{i}K_2K_2^*$. It will be shown that the equation

$$(\mathbb{A} - zF_+)x = Ke, \quad e \in \mathfrak{E}, \tag{76}$$

with

$$F_+ = \begin{pmatrix} F_{1,+} & 0 \\ 0 & F_{2,+} \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \tag{77}$$

has always a unique solution $x \in \mathfrak{H}_{+1} \oplus \mathfrak{H}_2$ and

$$(\mathbb{A} - zF_+)^{-1}K \in [\mathfrak{E}, \mathfrak{H}_{+1} \oplus \mathfrak{H}_2].$$

Taking into account (75), Eq. (76) can be written as the following system:

$$\begin{cases} (\mathbb{A}_1 - zF_{1,+})x_1 + \mathrm{i}K_1K_2^*x_2 = K_1e, \\ (\mathbb{A}_2 - zF_{2,+})x_2 + \mathrm{i}K_2K_1^*x_1 = K_2e, \end{cases} \tag{78}$$

where

$$\mathbb{A}_1 = \mathbb{D}_1 + \mathrm{i}K_1K_1^*, \quad \mathbb{A}_2 = D_2 + \mathrm{i}K_2K_2^*,$$

By Theorem 4.1 it follows that

$$(\mathbb{A}_1 - zF_{1,+})^{-1}K_1 \in [\mathfrak{E}, \mathfrak{H}_{+1}].$$

Therefore, the first equation in (78) gives

$$x_1 = (\mathbb{A}_1 - zF_{1,+})^{-1}K_1e - \mathrm{i}(\mathbb{A}_1 - zF_{1,+})^{-1}K_1K_2^*x_2. \tag{79}$$

Now substituting x_1 in the second equation in (78) yields

$$(\mathbb{A}_2 - zF_{2,+})x_2 + K_2K_1^*(\mathbb{A}_1 - zF_{1,+})^{-1}K_1K_2^*x_2 = K_2e - \mathrm{i}K_2K_1^*(\mathbb{A}_1 - zF_{1,+})^{-1}K_1e. \tag{80}$$

Taking into account (72), (73), and (75) the identity (80) leads to

$$\begin{aligned} &(2\mathrm{i}I - (D_2 - zF_+^{(2)})^{-1}K_2[I + W_{\Theta_{F_{1,+}}}(z)]K_2^*)x_2 \\ &\quad = 2\mathrm{i}(D_2 - zF_+^{(2)})^{-1}(K_2e - \mathrm{i}K_2K_1^*(\mathbb{A}_1 - zF_+^{(1)})^{-1}K_1e). \end{aligned}$$

It will be shown that the matrix-function on the left-hand side, in front of x_2, is invertible. First by straightforward calculations one obtains

$$(D_2 - zF_+^{(2)})^{-1} = \begin{pmatrix} zI & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix} \in [\mathfrak{E} \oplus \mathfrak{E}, \mathfrak{E} \oplus \mathfrak{E}].$$

The matrix function $M(z)$ defined by

$$M(z) = I + W_{\Theta_{F_{1,+}}}(z) \in [\mathfrak{E}, \mathfrak{E}]$$

is invertible by Theorem 2.4. It follows from (73) that

$$K_2M(z) = \begin{pmatrix} P\widehat{K} \\ \widehat{K} \end{pmatrix} M(z) = \begin{pmatrix} L^{1/2}M(z) \\ \widehat{K}M(z) \end{pmatrix} \in [\mathfrak{E} \oplus \mathfrak{E}, \mathfrak{E} \oplus \mathfrak{E}],$$

and that

$$K_2M(z)K_2^* = \begin{pmatrix} L^{1/2}M(z)L^{1/2} & L^{1/2}M(z)\widehat{K} \\ \widehat{K}M(z)L^{1/2} & \widehat{K}M(z)\widehat{K} \end{pmatrix} \in [\mathfrak{E} \oplus \mathfrak{E}, \mathfrak{E} \oplus \mathfrak{E}].$$

For any 2×2 block-matrix

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries in $[\mathfrak{E}]$ define the matrix-function

$$N(z) = 2\mathrm{i}I - (D_2 - zF_+^{(2)})^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{i} \begin{pmatrix} 2 + za\mathrm{i} - c & zb\mathrm{i} - d \\ a & b + 2 \end{pmatrix}.$$

Since the matrix $L > 0$ is invertible it follows that $\ker L = \{0\}$ and $\widehat{K} = L^{1/2}$. Now choose

$$Z = \begin{pmatrix} L^{1/2}M(z)L^{1/2} & L^{1/2}M(z)L^{1/2} \\ L^{1/2}M(z)L^{1/2} & L^{1/2}M(z)L^{1/2} \end{pmatrix} = \begin{pmatrix} A_0 & A_0 \\ A_0 & A_0 \end{pmatrix},$$

where $A_0 = A_0(z) = L^{1/2}M(z)L^{1/2}$. Note that the matrix-function A_0 is invertible and that $A_0^{-1} = L^{-1/2}M(z)^{-1}L^{-1/2}$. With this choice of Z one obtains

$$N = N(z) = \mathrm{i} \begin{pmatrix} 2I + z\mathrm{i}A_0 - A_0 & z\mathrm{i}A_0 - A_0 \\ A_0 & A_0 + 2I \end{pmatrix}.$$

To investigate the invertibility of N consider the system

$$\begin{pmatrix} 2I + z\mathrm{i}A_0 - A_0 & z\mathrm{i}A_0 - A_0 \\ A_0 & A_0 + 2I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\begin{cases} (2I + z\mathrm{i}A_0 - A_0)x_1 + (z\mathrm{i}A_0 - A_0)x_2 = 0, \\ A_0x_1 + (A_0 + 2)x_2 = 0. \end{cases}$$

Solving the second equation for x_1 yields

$$\begin{cases} 2x_1 + z\mathrm{i}A_0x_1 - A_0x_1 + z\mathrm{i}A_0x_2 - A_0x_2 = 0, \\ x_1 = -x_2 - 2A_0^{-1}x_2. \end{cases}$$

Substituting x_1 into the first equation gives

$$(2A_0^{-1} + z\mathrm{i})x_2 = 0,$$

or equivalently,

$$A_0x_2 = \frac{2\mathrm{i}}{z}x_2. \tag{81}$$

Recall that

$$A_0 = A_0(z) = L^{1/2}M(z)L^{1/2} = L^{1/2}[I + W_{\Theta_1}(z)]L^{1/2}.$$

For every z in the lower half-plane $W_{\Theta_1}(z)$ is a contraction (see [19]) and thus $\|A_0(z)\| \leqslant 2\|L\|$. This means that for every z ($\operatorname{Im} z < 0$) the norm of the left hand side of (81) is bounded while the norm of the right side can be made unboundedly large by letting $z \to 0$ along the imaginary axis. This leads to a conclusion that $x_2 = 0$ and then also $x_1 = 0$. Hence, $N = N(z)$ is invertible.

Consequently,

$$2\mathrm{i}I - (D_2 - zF_+^{(2)})^{-1}K_2[I + W_{\Theta_1}(z)]K_2^* \tag{82}$$

is invertible and x_2 depends continuously on $e \in \mathfrak{E}$ in (80), while (79) shows that x_1 depends continuously on $e \in \mathfrak{E}$.

Now we will follow the steps taken in the proof of the Theorem 4.1 to show that the remaining properties in Definition 2.2 are satisfied. We introduce an operator $T \in \Omega_A$ such that $\mathbb{A}$ is a correct $(*)$-extension of T. To construct T we note first that $(\mathbb{A} - zF_+)\mathfrak{H}_+ \supset \mathfrak{H}$ for some z in a neighborhood of $(-\mathrm{i})$. This can be confirmed by considering the equation

$$(\mathbb{A} - zF_+)x = g, \quad x \in \mathfrak{H}_+, \tag{83}$$

and showing that it has a unique solution for every $g \in \mathfrak{H}$. The procedure then is reduced to solving the system (78) with an arbitrary right hand side $g \in \mathfrak{H}$. Inspecting the steps of solving (78) we conclude that the system (83) has a unique solution. Similarly one shows that $(\mathbb{A}^* - zF_+)\mathfrak{H}_+ \supset \mathfrak{H}$. Once again relying on [49] we can conclude that operators $(\mathbb{A} + \mathrm{i}F_+)^{-1}$ and $(\mathbb{A}^* - \mathrm{i}F_+)^{-1}$ are $(-, \cdot)$-continuous and define

$$\begin{aligned} &T = \mathbb{A}, \quad \operatorname{dom} T = (\mathbb{A} + \mathrm{i}F_+)\mathfrak{H}, \\ &T_1 = \mathbb{A}^*, \quad \operatorname{dom} T = (\mathbb{A}^* - \mathrm{i}F_+)\mathfrak{H}. \end{aligned} \tag{84}$$

Using similar to the proof of Theorem 4.1 arguments we note that both $\operatorname{dom} T$ and $\operatorname{dom} T_1$ are dense in $\mathfrak{H}$ while operator T is closed in $\mathfrak{H}$. It is also easy to see that $T_1 = T^*$. Thus, operator T defined by (84) fits the definition of correct $(*)$-extension for operator $\mathbb{A}$. Property (vi) of Definition 2.2 follows from Theorem 3.5 and the fact that $\mathbb{A}_R = \mathbb{D}$.

Therefore, the array

$$\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & K & F_+ & I \\ \mathfrak{H}_{+1} \oplus \mathfrak{H}_2 \subset \mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \mathfrak{H}_{-1} \oplus \mathfrak{H}_2 & & & \mathfrak{E} \end{pmatrix} \tag{85}$$

is an F_+-system and $V(z)$ admits the realizations

$$V(z) = K^*(\mathbb{D} - zF_+)^{-1}K = \mathrm{i}[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I].$$

This completes the proof. □

It was shown in [32] that for the case of compactly supported measure in (70) the function $V(z)$ can be realized without the restriction on the invertibility of the linear term L.

5. Minimal realization

Recall that a symmetric operator A in a Hilbert space $\mathfrak{H}$ is called a *prime operator* [49,19] if there exists no reducing invariant subspace on which it induces a self-adjoint operator. A notion of a minimal realization is now defined along the lines of the concept of prime operators. An F_+-system of the form (23) is called F_+*-minimal* if there are no nontrivial reducing invariant subspaces $\mathfrak{H}^1 = \overline{\mathfrak{H}^1_+}$, ($\mathfrak{H}^1_+$ is a $(+)$-subspace of $\operatorname{ran} F_+$) of $\mathfrak{H}$ where the symmetric operator A induces a self-adjoint operator. Here the closure is taken with respect to $(\cdot)$-metric. In the case that $F_+ = I$ this definition coincides with the one used for rigged operator colligations in [23,19].

Theorem 5.1. *Let the matrix-valued Herglotz–Nevanlinna function $V(z)$ be realized in the form*

$$V(z) = \mathrm{i}[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I], \tag{86}$$

where $W_{\Theta_{F_+}}(z)$ is the transfer function of some F_+-system (23). *Then this F_+-system can be reduced to an F_+-minimal system of the form* (23) *and its transfer function gives rise to an F_+-minimal realization of $V(z)$ via* (86).

Proof. Let the matrix-valued Herglotz–Nevanlinna function $V(z)$ be realized in the form (86) with an F_+-system of the type (23). Assume that its symmetric operator A has a reducing invariant subspace $\mathfrak{H}^1 = \overline{\mathfrak{H}^1_+}$, ($\mathfrak{H}^1_+$ is a (+)-subspace of ranF_+) on which it generates a self-adjoint operator A_1. Then there is the following $(\cdot,\cdot)$-orthogonal decomposition

$$\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1, \quad A = A_0 \oplus A_1, \tag{87}$$

where A_0 is an operator induced by A on $\mathfrak{H}^0$.

The identity (87) shows that the adjoint of A in $\mathfrak{H}$ admits the orthogonal decomposition $A^* = A_0^* \oplus A_1$. Now consider operators $T \supset A$ and $T^* \supset A$ as in the definition of the system Θ_{F_+}. It is easy to see that both T and T^* admit the $(\cdot,\cdot)$-orthogonal decompositions

$$T = T_0 \oplus A_1,$$

and

$$T^* = T_0^* \oplus A_1,$$

where $T_0 \supset A_0$ and $T_0^* \supset A_0$. Since $T \in \Omega_A$, the identity $A_0 \oplus A_1 = T \cap T^* = (T_0 \cap T_0^*) \oplus A_1$ holds and $-\mathrm{i}$ is a regular point of $T = T_0 \oplus A_1$ or, equivalently, $-\mathrm{i}$ is a regular point of T_0. This shows that $T_0 \in \Omega_{A_0}$. Clearly,

$$\mathfrak{H}_+ = \mathfrak{H}^0_+ \oplus \mathfrak{H}^1_+ = \operatorname{dom} A_0^* \oplus \operatorname{dom} A_1.$$

This decomposition remains valid in the sense of (+)-orthogonality. Indeed, if $f_0 \in \mathfrak{H}^0_+$ and $f_1 \in \mathfrak{H}^1_+ = \operatorname{dom} A_1$, then by considering the adjoint of $A : \mathfrak{H}_0 (= \overline{\operatorname{dom} A}) \to \mathfrak{H}$ as a mapping from $\mathfrak{H}$ into $\mathfrak{H}_0$ one obtains

$$(f_0, f_1)_+ = (f_0, f_1) + (A^* f_0, A^* f_1) = (f_0, f_1) + (A_0^* f_0, A_1 f_1) = 0 + 0 = 0.$$

Consequently, the inclusions $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$ can be rewritten in the following decomposed forms:

$$\mathfrak{H}^0_+ \oplus \mathfrak{H}^1_+ \subset \mathfrak{H}^0 \oplus \mathfrak{H}^1 \subset \mathfrak{H}^0_- \oplus \mathfrak{H}^1_- = \mathfrak{H}^0_+ \oplus \operatorname{dom} A_1 \subset \mathfrak{H}^0 \oplus \mathfrak{H}^1 \subset \mathfrak{H}^0_- \oplus \mathfrak{H}^1_-.$$

Now let $\mathbb{A} \in [\mathfrak{H}_+, \mathfrak{H}_-]$ be the correct $(*)$-extension of A in the definition of the system Θ_{F_+}. Then $\mathbb{A}$ admits the decomposition $\mathbb{A} = \mathbb{A}_0 \oplus A_1$ and $\mathbb{A}^* = \mathbb{A}_0^* \oplus A_1$. Since A_1 is self-adjoint in $\mathfrak{H}^1$, $\mathbb{A}_0$ is a correct $(*)$-extension of T_0, cf. (21). Moreover,

$$\frac{\mathbb{A} - \mathbb{A}^*}{2\mathrm{i}} = \frac{(\mathbb{A}_0 \oplus A_1) - (\mathbb{A}_0^* \oplus A_1)}{2\mathrm{i}} = \frac{\mathbb{A}_0 - \mathbb{A}_0^*}{2\mathrm{i}} \oplus \frac{A_1 - A_1}{2\mathrm{i}} = \frac{\mathbb{A}_0 - \mathbb{A}_0^*}{2\mathrm{i}} \oplus O, \tag{88}$$

where O stands for the zero operator. Decompose $K \in [\mathfrak{E}, \mathfrak{H}_-]$ according to $\mathfrak{H}_- = \mathfrak{H}^0_- \oplus \mathfrak{H}^1_-$ as follows $K = K_0 \oplus K_1$. Then (88) implies that

$$KJK^* = K_0 J K_0^* \oplus O. \tag{89}$$

Since $\dim \mathfrak{E} < \infty$ and $\ker K = \{0\}$, one has ran$K^* = \mathfrak{E}$ and therefore also ran$JK^* = \mathfrak{E}$. According to (89) $K_1(\operatorname{ran} JK^*) = \{0\}$ and therefore $K_1 = 0$, or equivalently, $K = K_0 \oplus O$. Let P^0_+ be the orthogonal projection operator of $\mathfrak{H}_+$ onto $\mathfrak{H}^0_+$ and let $P^1_+ = I - P^0_+$. Then $K^* = K_0^* P^0_+$, since for all $f \in \mathfrak{E}$, $g \in \mathfrak{H}_+$ one has

$$\begin{aligned}(Kf, g) &= (K_0 f, g) = (K_0 f, g_0 + g_1) = (K_0 f, g_0) + (K_0 f, g_1)\\ &= (K_0 f, g_0) = (f, K_0^* g_0) = (f, K_0^* P^0_+ g).\end{aligned}$$

Since $\mathfrak{H}^1_+$ is a closed subspace of ranF_+, $P^0_+ = I - P^1_+$ commutes with F_+ and therefore $F^0_+ := F_+ P^0_+$ defines an orthogonal projection in $\mathfrak{H}^0_+$.

Now, let $e \in \mathfrak{E}$, let $z \in \rho(\mathbb{A}, F_+, K)$, and let $x = x^0 + x^1 \in \mathfrak{H}_+ = \mathfrak{H}_+^0 \oplus \mathfrak{H}_+^1$ be such that

$$(\mathbb{A} - zF_+)x = Ke.$$

Since $K = K_0 \oplus O$ the previous identity is equivalent to

$$(\mathbb{A}_0 \oplus A_1 - zF_+)(x^0 + x^1) = (K_0 \oplus O)e.$$

Since $F_+x^1 = x^1$ and P_+^0 commutes with F_+, this yields

$$(\mathbb{A}_0 - zF_+^0)x^0 = K_0 e,$$
$$(A_1 - zI)x^1 = 0.$$

It follows from the previous equations that $z \in \rho(A_1)$ because $z \in \rho(\mathbb{A}, F_+, K)$. Thus, $\rho(\mathbb{A}, F_+, K) \subset \rho(\mathbb{A}_0, F_+^0, K_0)$ and hence $x^0 = (\mathbb{A}_0 - zF_+^0)^{-1}K_0 e$. On the other hand, $x^0 = x = (\mathbb{A} - zF_+)^{-1}Ke$ and therefore for all $e \in \mathfrak{E}$ one obtains

$$(\mathbb{A} - zF_+)^{-1}Ke = (\mathbb{A}_0 - zF_+^0)^{-1}K_0 e,$$

and

$$K^*(\mathbb{A} - zF_+)^{-1}Ke = K_0^*(\mathbb{A}_0 - zF_+^0)^{-1}K_0 e.$$

This means that the transfer functions of the system Θ_{F_+} in (23) and of the system

$$\Theta_{F_+}^0 = \begin{pmatrix} \mathbb{A}_0 & F_+^0 & K_0 & J \\ \mathfrak{H}_+^0 \subset \mathfrak{H}^0 \subset \mathfrak{H}_-^0 & & & \mathfrak{E} \end{pmatrix}$$

coincide. Therefore, the system Θ_{F_+} in (23) can be reduced to an F_+-minimal system of the same form such that the corresponding transfer functions coincide. This completes the proof of the theorem. □

The definition of minimality can be extended to Δ_+-systems in the same manner. Moreover, an F_+-system of the form (14)

$$\begin{cases} (\mathbb{A} - zF_+)x = KJ\varphi_-, \\ \varphi_+ = \varphi_- - 2\mathrm{i}K^*x, \end{cases}$$

and a Δ_+-system of the form (12)

$$\begin{cases} (\mathbb{A}_R - zF_+)x = K\varphi_-, \\ \varphi_+ = K^*x, \end{cases}$$

where $\mathbb{A}_R$ is the real part of $\mathbb{A}$, are minimal (or non-minimal) simultaneously.

For the Δ_+-systems constructed in Section 3 the minimality can be characterized as follows.

Theorem 5.2. *The realization of the matrix-valued Herglotz–Nevanlinna function $V(z)$ constructed in Theorem 3.5 is minimal if and only if the symmetric part A_2 of $\mathbb{A}_R^{(2)}$ defined by (57) is prime.*

Proof. Assume that the system constructed in Theorem 3.5 is not minimal. Let $\mathfrak{H}^1$ (with $\mathfrak{H}_+^1 \subset \operatorname{ran} F_+$) be a reducing invariant subspace from Theorem 5.1 on which A generates a self-adjoint operator A_1. Then $\mathbb{D} = \mathbb{D}_0 \oplus A_1$ and it follows from the block representations of $\mathbb{D}$ and F_+ in (57) that $\mathfrak{H}^1$ is necessarily a subspace of $\mathfrak{H}_2$ in (56) while $\mathfrak{H}_+^1$ is a subspace of $\mathfrak{H}_{+2}$. To see this let us describe $\operatorname{ran} F_+$ first. According to (56) $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_- = \mathfrak{E}^4 \oplus \mathfrak{H}_{+2} \subset \mathfrak{E}^4 \oplus \mathfrak{H}_2 \subset \mathfrak{E}^4 \oplus \mathfrak{H}_{-2}$ and hence every vector $x \in \mathfrak{H}_+$ can be written as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \text{where } x_1, x_2, x_3, x_4 \in \mathfrak{E}, x_5 \in \mathfrak{H}_{+2}.$$

By (57),

$$F_+ x = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ x_5 \end{pmatrix}, \quad \text{and} \quad \mathbb{D}(F_+ x) = \begin{pmatrix} \mathrm{i}x_2 \\ 0 \\ 0 \\ 0 \\ \mathbb{A}_R^{(2)} x_5 \end{pmatrix}.$$

This means that $x \in \mathfrak{H}_+^1 \subset \operatorname{ran} F_+$ only if $x_2 = 0$. Therefore the only possibility for a reducing invariant subspace $\mathfrak{H}^1$ is to be a subspace of $\mathfrak{H}_2$ while $\mathfrak{H}_+^1$ is a subspace of $\mathfrak{H}_{+2}$. This proves the claim $\mathfrak{H}_+^1 \subset \mathfrak{H}_{+2}$. Consequently, $\mathfrak{H}^1$ is a reducing invariant subspace for the symmetric operator A_2, in which case the operator A_2 is not prime.

Conversely, if the symmetric operator A_2 is not prime, then a reducing invariant subspace on which A_2 generates a self-adjoint operator is automatically a reducing invariant subspace for the operator A which belongs to $\operatorname{ran} F_+$. This completes the proof. □

Finally, Theorem 5.2 implies that a realization of an arbitrary matrix-valued Herglotz–Nevanlinna function in Theorem 3.5 can be provided by a minimal Δ_+-system.

6. Examples

The paper will be concluded with some simple illustrations of the main realization result.

Example 1. Consider the following Herglotz–Nevanlinna function

$$V(z) = 1 + z - \mathrm{i} \tanh\left(\frac{\mathrm{i}}{2} z l\right), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{90}$$

where $l > 0$. An explicit F_+-system Θ_{F_+} will be constructed so that $V(z) \equiv i[W_{\Theta,F_+}(z) + I]^{-1}[W_{\Theta,F_+}(z) - I] = V_{\Theta_{F_+}}(z)$. Let the differential operator T_2 in $\mathfrak{H}_2 = L^2_{[0,l]}$ be given by

$$T_2 x = \frac{1}{\mathrm{i}} \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \operatorname{dom} T_2 = \{x(t) \in \mathfrak{H}_2 : x'(t) \in \mathfrak{H}_2,\ x(0) = 0\},$$

with adjoint

$$T_2^* x = \frac{1}{\mathrm{i}} \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \operatorname{dom} T_2^* = \left\{x(t) \in \mathfrak{H}_2 : x'(t) \in \mathfrak{H}_2,\ x(l) = 0\right\}.$$

Let A_2 be the symmetric operator defined by

$$A_2 x = \frac{1}{\mathrm{i}} \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \operatorname{dom} A_2 = \{x(t) \in \mathfrak{H}_2 : x'(t) \in \mathfrak{H}_2,\ x(0) = x(l) = 0\}, \tag{91}$$

with adjoint

$$A_2^* x = \frac{1}{\mathrm{i}} \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \operatorname{dom} A_2^* = \left\{x(t) \in \mathfrak{H}_2 :\ x'(t) \in \mathfrak{H}_2\right\}.$$

Then $\mathfrak{H}_+ = \operatorname{dom} A_2^* = W_2^1$ is a Sobolev space with the scalar product

$$(x, y)_+ = \int_0^l x(t)\overline{y(t)}\,\mathrm{d}t + \int_0^l x'(t)\overline{y'(t)}\,\mathrm{d}t.$$

Now consider the rigged Hilbert space

$$W_2^1 \subset L^2_{[0,l]} \subset (W_2^1)_-,$$

and the operators

$$\mathbb{A}_2 x = \frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{i}x(0)[\delta(x-l) - \delta(x)],$$
$$\mathbb{A}_2^* x = \frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{i}x(l)[\delta(x-l) - \delta(x)],$$

where $x(t) \in W_2^1$ and $\delta(x)$, $\delta(x-l)$ are delta-functions in $(W_2^1)_-$. Define the operator K_2 by

$$K_2 c = c \cdot \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)], \quad c \in \mathbb{C}^1,$$

so that

$$K_2^* x = \left(x, \frac{1}{\sqrt{2}}[\delta(x-l) - \delta(x)]\right) = \frac{1}{\sqrt{2}}[x(l) - x(0)],$$

for $x(t) \in W_2^1$.

Let $D_1 = K_1 Q_1^{-1} K_1^* = 1$, where $Q = 1$, and $K_1 = 1$, $K_1 : \mathbb{C} \to \mathbb{C}$. Following (52) define

$$\mathfrak{H}_3 = \mathbb{C} \oplus \mathbb{C}, \quad K_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix}, \quad F_{+,3} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Now the corresponding F_+-system can be constructed. According to (57) one has

$$\mathbb{D} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbb{A}_{2,R} \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} 1 \\ 1 \\ 1 \\ K_2 \end{pmatrix},$$

and it follows from (74) and (75) that

$$\mathbb{A} = \mathbb{D} + \mathrm{i}KK^* = \begin{pmatrix} \mathrm{i} & 2\mathrm{i} & \mathrm{i} & \mathrm{i}K_2^* \\ 0 & \mathrm{i} & \mathrm{i} & \mathrm{i}K_2^* \\ \mathrm{i} & \mathrm{i} & 1+\mathrm{i} & \mathrm{i}K_2^* \\ \mathrm{i}K_2 & \mathrm{i}K_2 & \mathrm{i}K_2 & \mathbb{A}_2 \end{pmatrix}.$$

Consequently, the corresponding F_+-system is given by

$$\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & K & F_+ & I \\ \mathbb{C}^3 \oplus W_2^1 \subset \mathbb{C}^3 \oplus L^2_{[0,l]} \subset \mathbb{C}^3 \oplus (W_2^1)_- & & & \mathbb{C} \end{pmatrix}, \tag{92}$$

where $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and all the operators are described above. It is well known (see for example [23]) that the symmetric operator A_2 defined in (91) does not have nontrivial invariant subspaces on which it induces self-adjoint operators. Thus, the F_+-system in (92) is an F_+-minimal realization of the function $V(z)$ in (90), cf. Section 5. The transfer function of this system is

$$W_{\Theta_{F_+}}(z) = \frac{2 - \mathrm{i}(1 + \mathrm{e}^{\mathrm{i}zl})(z+1)}{2\mathrm{e}^{\mathrm{i}zl} + \mathrm{i}(1 + \mathrm{e}^{\mathrm{i}zl})(z+1)} = \frac{1 - \mathrm{i} - z\mathrm{i} - \tanh\left(\frac{\mathrm{i}}{2}zl\right)}{1 + \mathrm{i} + z\mathrm{i} + \tanh\left(\frac{i}{2}zl\right)}.$$

Example 2. Consider the following Herglotz–Nevanlinna function

$$V(z)=\begin{pmatrix}1&0\\0&1\end{pmatrix}+z\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}-\mathrm{i}\tanh(\pi\mathrm{i}z)&0\\0&\frac{1-z}{z^2-z-1}\end{pmatrix}. \tag{93}$$

An explicit F_+-system Θ_{F_+} will be constructed so that $V(z)\equiv\mathrm{i}[W_{\Theta,F_+}(z)+I]^{-1}[W_{\Theta,F_+}(z)-I]=V_{\Theta_{F_+}}(z)$. Let T_{21} be a differential operator $\mathfrak{H}_2=L^2_{[0,2\pi]}$ given by

$$T_{21}x=\frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t},\quad \operatorname{dom}T_{21}=\left\{x(t)\in\mathfrak{H}_2: x'(t)\in\mathfrak{H}_2,\quad x(0)=0\right\},$$

with adjoint

$$T^*_{21}x=\frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t},\quad \operatorname{dom}T^*_{21}=\left\{x(t)\in\mathfrak{H}_2: x'(t)\in\mathfrak{H}_2,\ x(2\pi)=0\right\}.$$

Let A_{21} be the symmetric operator defined by

$$A_{21}x=\frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t},\quad \operatorname{dom}A_{21}=\left\{x(t)\in\mathfrak{H}_2: x'(t)\in\mathfrak{H}_2,\ x(0)=x(2\pi)=0\right\}, \tag{94}$$

with adjoint

$$A^*_{21}x=\frac{1}{\mathrm{i}}\frac{\mathrm{d}x}{\mathrm{d}t},\quad \operatorname{dom}A^*_{21}=\left\{x(t)\in\mathfrak{H}_2: x'(t)\in\mathfrak{H}_2\right\}.$$

Then $\mathfrak{H}_+=\operatorname{dom}A^*_{21}=W^1_2$ is a Sobolev space with the scalar product

$$(x,y)_+=\int_0^{2\pi}x(t)\overline{y(t)}\,\mathrm{d}t+\int_0^{2\pi}x'(t)\overline{y'(t)}\,\mathrm{d}t.$$

Consider the rigged Hilbert space

$$W^1_2\subset L^2_{[0,2\pi]}\subset(W^1_2)_-,$$

and the operators

$$\mathbb{A}_{21}x=\tfrac{1}{\mathrm{i}}\tfrac{\mathrm{d}x}{\mathrm{d}t}+\mathrm{i}x(0)\left[\delta(x-2\pi)-\delta(x)\right],$$
$$\mathbb{A}^*_{21}x=\tfrac{1}{\mathrm{i}}\tfrac{\mathrm{d}x}{\mathrm{d}t}+\mathrm{i}x(2\pi)\left[\delta(x-2\pi)-\delta(x)\right],$$

where $x(t)\in W^1_2$ and $\delta(x)$, $\delta(x-2\pi)$ are delta-functions in $(W^1_2)_-$. Define the operator K_{21} by

$$K_{21}c=c\cdot\frac{1}{\sqrt{2}}[\delta(x-2\pi)-\delta(x)],\quad c\in\mathbb{C}^1,$$

so that

$$K^*_{21}x=\left(x,\frac{1}{\sqrt{2}}[\delta(x-2\pi)-\delta(x)]\right)=\frac{1}{\sqrt{2}}[x(2\pi)-x(0)],$$

for $x(t)\in W^1_2$. Define

$$T_{22}=\begin{pmatrix}\mathrm{i}&\mathrm{i}\\-\mathrm{i}&1\end{pmatrix}\quad\text{and}\quad K_{22}=\begin{pmatrix}1\\0\end{pmatrix}, \tag{95}$$

and set

$$\mathbb{A}_2=\begin{pmatrix}\mathbb{A}_{21}&0\\0&T_{22}\end{pmatrix}\quad\text{and}\quad K_2=\begin{pmatrix}K_{21}&0\\0&K_{22}\end{pmatrix}. \tag{96}$$

Now let $D_1 = K_1 Q^{-1} K_1^* = I_2$, where $Q = I_2$, and $K_1 = I_2$, $K_1 : \mathbb{C}^2 \to \mathbb{C}^2$. Following (52) define

$$\mathfrak{H}_3 = \mathbb{C}^4, \quad K_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad F_{+,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now the corresponding F_+-system will be constructed. According to (57) one has

$$\mathbb{D} = \begin{pmatrix} D_3 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & D_1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \mathbb{A}_{2,R} \end{pmatrix}, \quad F_+ = \begin{pmatrix} F_{+,3} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & I \end{pmatrix}, \quad K = \begin{pmatrix} K_3 \\ \cdots \\ K_1 \\ \cdots \\ K_2 \end{pmatrix}, \tag{97}$$

and it follows from (74) and (75) that

$$\mathbb{A} = \mathbb{D} + \mathrm{i} K K^*. \tag{98}$$

Consequently, the corresponding F_+-system is given by

$$\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & & K & F_+ & I \\ \mathbb{C}^6 \oplus W_2^1 \subset \mathbb{C}^6 \oplus L^2_{[0,2\pi]} \subset \mathbb{C}^6 \oplus (W_2^1)_- & & & & \mathbb{C}^2 \end{pmatrix}, \tag{99}$$

where all the operators are described above. The transfer function of this system is given by

$$W_{\Theta_{F_+}}(z) = \begin{pmatrix} \frac{1-\mathrm{i}-z\mathrm{i}-\tanh(\pi \mathrm{i} z)}{1+\mathrm{i}+z\mathrm{i}+\tanh(\pi \mathrm{i} z)} & 0 \\ 0 & \frac{z^3+\mathrm{i}z^2-(3+i)z-\mathrm{i}}{-z^3+\mathrm{i}z^2+(3-\mathrm{i})z-\mathrm{i}} \end{pmatrix}.$$

It is easy to see that the maximal symmetric part of the operator T_{22} in (95) is a non-densely defined operator

$$A_{22} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}, \quad \operatorname{dom} A_{22} = \left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} : c \in \mathbb{C} \right\}. \tag{100}$$

Consequently, the symmetric operator A_2 defined by $\mathbb{A}_2$ in (96), $\mathbb{D}$ in (97), and $\mathbb{A}$ in (98) is given by

$$\begin{aligned} A_2 &= \begin{pmatrix} \frac{1}{\mathrm{i}} \frac{\mathrm{d}x}{\mathrm{d}t} & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{pmatrix}, \\ \operatorname{dom} A_2 &= \left\{ \begin{pmatrix} x(t) \\ 0 \\ c \end{pmatrix} : x(t), x'(t) \in \mathfrak{H}_2, x(0) = x(2\pi) = 0, c \in \mathbb{C} \right\}. \end{aligned} \tag{101}$$

Hence, this operator A_2 does not have nontrivial invariant subspaces on which it induces self-adjoint operators. Thus, F_+-system in (99) is an F_+-minimal realization of the function $V(z)$ in (93), cf. Section 5.

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