ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

Discrete Mathematics 294 (2005) 291–296

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

The number of Moore families on $n = 6$

M. Habib[a], L. Nourine[b]

[a] *LIRMM-Université Montpellier II, 161 rue Ada, F34392 Montpellier Cedex 5, France*

[b] *LIMOS-Université Blaise Pascal, Campus des cézeaux F63173 AUBIERE, France*

Received 16 May 2000; received in revised form 27 October 2004; accepted 1 November 2004

Abstract

This paper studies the generating problem for Moore families on an n-set (i.e. families closed under intersection containing the n-set) or closure operators. We show a bijection between Moore families and ideal color sets of the colored poset based on $n.2^{n-1}$, where $n.2^{n-1}$ is the sum of n Boolean lattices with $n-1$ atoms. By applying an algorithm to generate ideal color sets, we can determine that the number of Moore families on 6 elements is exactly 75 973 751 474.

Keywords: Moore family; Upper locally distributive lattices; Closure system; Colored posets

Enumerating combinatorial objects has been always an attractive research area in combinatorics. Advances in computation have helped combinatorists to determine various values even in cases where they fail to obtain explicit formulas. This yields new motivation to search for efficient generation algorithms.

The number of Moore families on n elements is known for $n \leqslant 5$ by Higuchi [3] (see Table 1). These values were computed using a lexicographic depth first search of the covering graph of the lattice of Moore families. Up to now the major drawback of this method is the time needed for computation of the next Moore family in a given lexicographic order.

In the following, we will present a new method based on a bijection between Moore families and ideal color sets of a colored poset [5], that leads us to compute the number of Moore families for $n = 6$.

E-mail addresses: habib@lirmm.fr (M. Habib), nourine@isima.fr (L. Nourine).

0012-365X/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2004.11.010

Table 1
Known values of the number of Moore families on an *n*-set

n	$\|\mathscr{M}_n\|$	
1	2	
2	7	
3	61	
4	2480	
5	1385552	Higuchi [3]
6	75 973 751 474	This paper

For definitions on lattices and ordered sets not given here, see Davey and Priestley's book [2].

1. Some definitions and properties for Moore families

Let X be an n-set and 2^X its power set. A Moore family on X is a family of subsets of X closed by set-intersection and containing the set X. A Moore family is also known as a closure system; i.e. the set of all closed sets of a closure operator. The set $\mathscr{M}_n$ of all possible Moore families on an n-set, ordered by set-inclusion is a lattice, called the lattice of all Moore families, denoted by $\mathscr{M}_n = (\mathscr{M}_n, \subseteq)$. The intersection of two Moore families is also a Moore family, therefore the whole family $\mathscr{M}_n$ is also a closure system.

Clearly, the set of all Moore families on a set X is exactly the set of all meet-sublattices of the Boolean lattice 2^X containing the set X (Fig. 1). So, the number of Moore families on a set X is equal to the number of meet-sublattices containing the set X and equal to the number of join-sublattices of 2^X containing the empty set.

Our first strategy to count the number of Moore families on a set X is based on counting the number of join-sublattices of 2^X. Consider the Boolean lattice $2^X = (2^X, \subseteq)$ with $|X| = n$. Let S_n denote the set of all join-sublattices of 2^X containing the empty set. The following simple lemma gives us a method to count the size of S_n.

Lemma 1. *Let L be a join-sublattice of 2^X and $A \in L, A \neq \emptyset$. Then $L \backslash A$ is a join-sublattice of L iff A is a join-irreducible element in the lattice L.*

Proof. Obvious since a join-irreducible element cannot be the join of two other elements. □

Let $A \in 2^X$ be a minimal join-irreducible of the lattice 2^X. Then, the set S_n can be decomposed into two sets as follows:

- $S_n/A = \{L \in S_n \mid A \in L\}$: the join-sublattices containing A.
- $S_n \backslash A = \{L \in S_n \mid A \notin L\}$: the join-sublattices not containing A.

This decomposition induces a recursive method to generate all join-sublattices S_n, and therefore to count them.

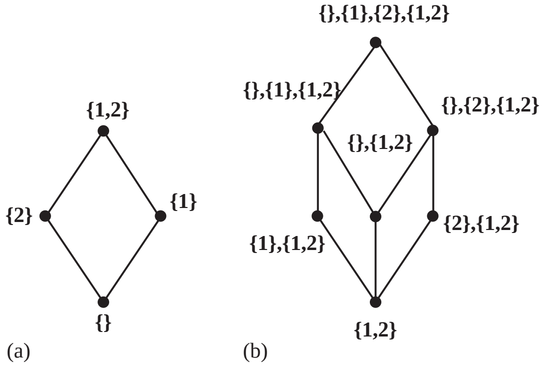

Fig. 1. (a) The Boolean lattice $2^{\{1,2\}}$, (b) $\mathscr{M}_2$: the lattice of Moore families on $X = \{1, 2\}$ (i.e. each element of $\mathscr{M}_2$ is a meet-sublattice of $2^{\{1,2\}}$).

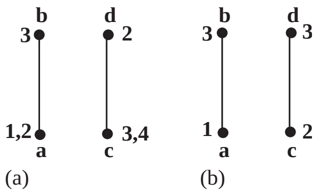

Fig. 2. (a) A colored poset, (b) a simple colored poset.

We first implemented this method (based on Lemma 1) to generate join-sublattices of the Boolean lattice 2^6, but we had to stop the computation after several days without any result. This was due to the time needed to update the covering graph of L when a join-irreducible has been deleted from the covering graph of L.

To avoid this brute force technique, we use the ideal color sets of a colored poset introduced in [5] as a representation of any lattice. In the following, we show how to count efficiently the number of join-sublattices of the Boolean lattice 2^X. This leads us to compute the number of Moore families for $|X| = 6$.

Definition 1. A colored poset, denoted by $P = (X, <, \gamma)$, is the poset $P = (X, <)$ equipped with a coloring $\gamma : X \rightarrow 2^M$ such that $x < y$ implies $\gamma(x) \cap \gamma(y) = \emptyset$, where M is a set of colors. In other words, γ is a set-coloring of the comparability graph of P.

A colored poset is said to be simple if the color set of any element is a singleton (i.e., the coloring γ is from X to M), see Fig. 2b.

Let $P = (X, <, \gamma)$ be a colored poset and I a subset of X. Let us recall that an order ideal I of P satisfies $x \in I$ and $y < x$ implies $y \in I$. We define the *ideal color set* as the set of colors of elements in I ideal of P, i.e $C(I) = \bigcup_{x \in I} \gamma(x)$. In Fig. 2a, if $I = \{a, c\}$ then $C(I) = \{1, 2, 3, 4\}$. Notice that two different ideals can have the same color sets, i.e. if $J = \{a, b, c\}$ then $C(I) = C(J)$ in Fig. 2a.

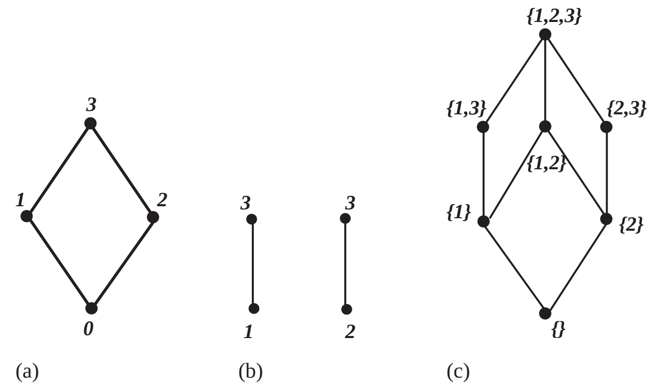

Fig. 3. (a) 2^2 and its atomistic coloring; i.e. numbers from 0 to 3, (b) colored poset P_2, (c) the lattice of ideal color sets of P_2 which is dually isomorphic to $\mathscr{M}_2$ in Fig. 1.

The set of all ideal color sets of P, denoted by $\mathscr{C}(P)$, is a closed under union and containing the empty set, and hence a lattice, called the lattice of all ideal color sets of P.

Let $\mathscr{I}(P)$ be the set of all order ideals of P. We define the mapping gen : $\mathscr{C}(P) \to \mathscr{I}(P)$ by $\text{gen}(C) = \{x \in X \mid \gamma(\downarrow x) \subseteq C\}$ where $\downarrow x = \{y \in X \mid y \leqslant x\}$ (i.e. gen(C) is the unique largest ideal I of P with $\gamma(I) = C$). In Fig. 2a, gen$(\{1, 2, 3, 4\}) = \{a, b, c, d\}$. Note that the mapping gen is an order embedding of $\mathscr{C}(P)$ into $\mathscr{I}(P)$.

In the following, we construct a simple colored poset in which its ideal color sets are in bijection with the join-sublattices of the Boolean lattice 2^X (Fig. 3).

Let Q be a Boolean lattice on n atoms, say $a_0, a_1, \ldots, a_{n-1}$. We consider the mapping $\gamma : Q \to [0, 2^n - 1]$ as follows:

$$\gamma(x) = \begin{cases} 0 & \text{if x is the bottom element,} \\ 2^i & \text{if } x = a_i \text{ for some } i \in [0, n-1], \\ \sum_{a \in J(x)} \gamma(a) & \text{otherwise,} \end{cases} \tag{1}$$

where $J(x)$ is the set of all atoms below x in Q.

The application γ is a simple coloring since each element of 2^X has only one color. Moreover $x <_Q y$ implies $J(x) \subset J(y)$ and therefore $\gamma(x) \neq \gamma(y)$ (i.e. $\gamma(x) \cap \gamma(y) = \emptyset$).

We define the colored poset P_n as the disjoint sum of all intervals $[a, \top]$ of Q where a is an atom of Q and $\top$ the top element of Q. The color of an element in P_n is equal to its color in Q. Since two elements of Q do not have the same color, we can identify an element in Q with its color.

Theorem 1. *Let L be a subset of 2^X and $C = \{c \in 2^X \mid c \notin L\}$. L is a join-sublattice of 2^X containing the color 0 iff the set C is an ideal color set of P_n.*

Proof. Suppose that L is a join-sublattice of 2^X containing the color 0. If C is empty, we are done so we may assume that C is not empty. Let c be a color in C. Since c is not in L, there must be an atom a in 2^X with the property that $a \leqslant c$ and for all b in 2^X such that $a \leqslant b \leqslant c$, b is also not in L. If this was not the case, we could express c as the join of elements in L, and since L is a join-sublattice, c would have to be a member of L. It is now clear that the entire set $\downarrow c$ in the component $[a, \top]$ consists of elements not in L and is an order ideal. Taking the union of all such order ideals that can be constructed using the colors in C yields an order ideal whose color set is exactly C.

Now, suppose that C is an ideal color set of P_{n}. We want to show that L is a join-sublattice containing 0. Notice that L and C are complements. Clearly 0 is in L since it cannot be in C by the definition of P_{n}. Now suppose that d and e are in L, but their join c is not in L. Since c is in C, there must be some atom a, such that $\downarrow c$ is a subset of C and also a subset of $[a, \top]$. The coloring has been set up so that the numbers corresponding to elements of 2^X can be thought of a bit vectors, with join corresponding to bitwise *OR*. In addition, the atoms of 2^X are exactly the bit vectors having just a single 1. Since $c = d + e$, either d or e must be in $[a, \top]$, and hence in C. This contradicts the fact that both d and e are in L. Thus, we see that L is closed under joins. □

Given a number n, let us sketch the algorithm that generates all ideal color sets of P_{n}. Clearly, we can use any algorithm for generating order ideals of *a* poset; but as one can easily remark, the number of order ideals is greater than the number of ideal color sets of P_{n}. For P_2, we have 9 order ideals and only 7 ideal color sets.

To avoid these extra order ideals, we use the divide and conquer algorithm for generating order ideals of the poset P_{n} [4]. Indeed, for each ideal color set C of P_{n}, we first generate the ideal color sets containing C and then the ideal color sets not containing C. Let C be the empty ideal color set corresponding to the empty order ideal I. At each step we do the following:

- Choose a minimal element x of P_{n}.
- Compute $R = \mathrm{gen}(C \cup \gamma(x)) \backslash I$. Increase the number of ideal color sets by 1.
- Compute the ideal color sets of $P_{\mathrm{n}} \backslash R$ with $C = C \cup \gamma(x)$ and $I = I \cup R$. Namely, the color of any element in R is a color of an element in $I \cup \{x\}$, thus adding these elements to $I \cup \{x\}$ do not change the ideal color set corresponding to $I \cup \{x\}$. Note that these ideal color sets contain $C \cup \gamma(x)$.
- Compute the ideal color sets of $P_{\mathrm{n}} \backslash \{\uparrow y \mid y \in P_{\mathrm{n}} \text{ and } \gamma(y) \in C \cup \gamma(x)\}$ with C and I, where $\uparrow y = \{z \in P_{\mathrm{n}} \mid y \leqslant z\}$. These ideal color sets do not contain $C \cup \gamma(x)$.

We have written a program to count the number of ideal color sets of the colored poset P_6. The whole computation for generating $\mathscr{M}_6$, used about 24 h on Pentium III 600 MHz.

The C-program can be found in "www.lirmm.fr/~habib/PROGRAM/Moore.cc".

Since the enumeration problem of Moore families remains open, we are now studying structural aspects of colored posets. It could be possible to find an explicit formula or a better upper bound [1].

Acknowledgements

The authors are very grateful to the referees for very helpful comments and the simplification of the proof of Theorem 1.

References

[1] G. Burosch, J. Demetrovics, G. Katona, D.J. Kleitman, A. Sapozhenko, On the number of closure operators, Combinatorics, Paul Erdos is Eighty, vol. 1, Bolyai Soc. Math., 1993, pp. 91–95.
[2] B.A. Davey, H.A. Priestley, Introduction to Lattices and Orders, Cambridge University Press, Cambridge, 1991.
[3] A. Higuchi, Note: lattices of closure operators, Discrete Math. 179 (1998) 267–272.
[4] R. Medina, L. Nourine, Algorithme efficace de Génération des idéaux d'un ensemble ordonné, C.R. Acad. Sci. Paris 319 (I) (1994) 1115–1120.
[5] L. Nourine, Une Structuration Algorithmique de la Théorie des Treillis, Habilitation, Université de Montpellier II, France, July 2000.