Lie Isomorphisms of the Skew Elements of a Simple Ring with Involution

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Let $A$ and $B$ be simple rings with involution, let $K$ and $L$ denote respectively the Lie subrings of skew elements of $A$ and $B$, and let $[K, K]$ and $[L, L]$ be the derived Lie rings of $K$ and $L$, respectively. In [2] Herstein posed the following question: can every (Lie) isomorphism of $[K, K]$ onto $[L, L]$ be lifted to an (associative) isomorphism of $A$ onto $B$? The purpose of this paper is to settle the problem in the affirmative under the following conditions: (i) the involutions are of the first kind, (ii) neither $A$ nor $B$ satisfies a generalized polynomial identity over its centroid, and (iii) $A$ is assumed to contain two nonzero orthogonal symmetric idempotents whose sum is unequal to 1. Assumption (iii) is crucial to our arguments and at present we see no way to remove this type of hypothesis. On the other hand assumption (ii) is primarily a convenient device to say that we are looking at the case where $A$ and $B$ do not satisfy any finiteness conditions. The case where $A$ or $B$ do satisfy finiteness conditions has been investigated by Jacobson (see, e.g., [5, Chap. 10]) and by Klotz [6]. At any rate we plan in a subsequent paper to treat the case where either $A$ or $B$ does satisfy a generalized polynomial identity. We mention finally that the analogous problem for rings without involution has been worked out by Howland [4]: under suitable restrictions any Lie isomorphism of $[A, A]$ onto $[B, B]$ can be lifted to an isomorphism of $A$ onto $B$.

1. Preliminaries

Throughout this section $A$ will denote a simple ring with involution of the first kind. We recall that this means that $A$ has an antiautomorphism $*$ of period 1 or 2 such that $(\lambda a)^* = \lambda a^*$, where $a \in A$ and $\lambda \in \Phi$, the centroid of $A$. $K = \{a \in A \mid a^* = -a\}$, the set of skew elements of $A$ under $*$, is a Lie ring (closed under addition and Lie multiplication $[a, b] = ab - ba$), and
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A/p = [K, K] will denote the derived Lie ring of K. The foundations for prime rings with involution given in [8] apply, of course, to simple rings with involution and we indicate the appropriate remarks here. The central closure \( \bar{A} \) of \( A \) is either \( A \) itself (if \( 1 \in A \)) or \( A \oplus \Phi \) (if \( 1 \notin A \)). In the latter case multiplication is given by \((a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda \mu)\), \( a, b \in A \), \( \lambda, \mu \in \Phi \), and the involution * of \( A \) can be extended to an involution of \( A \oplus \Phi \) by defining \((a, \lambda) \rightarrow (a^*, \lambda), a \in A, \lambda \in \Phi \). In any case \( \bar{A} \) is a closed prime algebra with 1 and with involution whose skew elements are again just K.

\( \bar{A}_\Phi \langle x \rangle \) will denote the free product over \( \Phi \) of \( \bar{A} \) and the free noncommutative algebra \( \Phi \langle x_1, x_2, \ldots, x_n, \ldots \rangle \). An additive subgroup \( R \) of \( \bar{A} \) is said to satisfy a generalized polynomial identity over \( \Phi \) if there is a nonzero element \( f(x_1, x_2, \ldots, x_n) \) of \( \Phi \langle x \rangle \) such that \( f(r_1, r_2, \ldots, r_n) = 0 \) for all \( r_i \in R \). Briefly, we shall say that \( R \) is GPI over \( \Phi \).

We now list several theorems to which we shall wish to refer in the sequel.

**Theorem A.** \( A \) is GPI over \( \Phi \) if and only if \( A \) has a minimal right ideal \( eA \), \( e \) an idempotent, such that \( [eA : \Phi] < \infty \). (see [1, p. 218, Theorem 10], or [8, p. 510, First Main Theorem]).

**Theorem B.** \( A \otimes_\Phi A^0 = A_r A_l \), where \( A^0 \) is the opposite algebra of \( A \), and \( A_r \) and \( A_l \) are, respectively, the right and left multiplications of \( A \) acting on itself. The isomorphism is given according to the rule \( a \otimes b \rightarrow a b \).

(This is essentially [7, p. 213, Lemma 1]—the proof given there works for the case that \( A \) is closed prime.)

**Theorem C.** \( M = [K, K] \) is GPI over \( \Phi \) if and only if \( A \) is GPI over \( \Phi \). (This is a corollary of [8, p. 515, Second Main Theorem]. To see this we observe that if \( f(x_1, x_2, \ldots, x_n) \) is a GPI for \( M \) then \( g = f([t_1, u_1], [v_1, w_1], [t_2, u_2], [v_2, w_2], \ldots, [t_n, u_n], [v_n, w_n]) \) is a GPI satisfied by \( S = \{ s \in A | s^* = s \} \), the symmetric elements of \( A \) under *.

**Theorem D** (Herstein). If \( [A : \Phi] > 16 \), then \( M \) is a simple Lie ring and \( A = \bar{M} \), the associative subring generated by \( M \) (see e.g., [3, Theorems 2.13 and 2.15]).

In addition to our assumption that \( A \) is a simple ring with involution of the first kind we add the assumption that \( A \) is not GPI over \( \Phi \). By Theorem A \( [A : \Phi] = \infty \) and so by Theorem D \( A = \bar{M} \).

Finally we make the assumption that \( A \) contains two nonzero orthogonal symmetric idempotents \( e_1 \) and \( e_2 \) such that \( e_1 + e_2 \neq 1 \). Whether or not \( 1 \in A \) we always have \( 1 \in \bar{A} \) and so \( e_3 = 1 - e_1 - e_2 \in \bar{A} \). We set \( A_{ij} = e_i A e_j \), \( A_i = A_{ii} \), and note that \( A \) can be written in its Pierce decomposition.
\[ \sum_{i,j=1}^{\infty} A_{ij}. \]

If \( x_{ij} \in A_{ij}, \ i \neq j \), then we let \( x_{ij} = x_{ij}^* \in A_{jk} \). Since \( e_1 \) and \( e_2 \) are symmetric \( A_1, A_2 \), and \( A_3 \) are well known to be simple rings with involution (induced by \( * \)) with centroid \( \Phi \). We let \( K_i \) denote the skew elements of \( A_i \) and set \( M_i = [K_i, K_i] \). By Theorem A it is easy to see that if \( [A_i : \Phi] < \infty \) for some \( i \) then \( A \) would have to be GPI over \( \Phi \). Therefore by our assumption that \( A \) is not GPI over \( \Phi \) we have each \( [A_i : \Phi] = \infty \), and so by Theorem D we see that \( M_i = A_i \) for each \( i \).

**Lemma 1.**

1. \( A_i = A_{ij}A_{ji} \),
2. \( A_{ij} = A_iA_{ij}A_j \),
3. \( A_{ij} = \frac{M_i A_{ij} M_j}{M_i A_{ij} M_j} = M_i A_{ij} M_j = M_i A_{ij} M_j \),
4. \( A_{ij} = A_{ik}A_{kj} \).

**Proof.**

1. is true because \( A_{ij}A_{ji} \) is a nonzero ideal of the simple ring \( A_i \).

To prove (2) we first claim that \( \sum_{i,j=1}^{\infty} A_i A_{ij} A_j \) is an ideal of \( A \). To see this it suffices to observe, using (1), that \( A_i A_{ij} A_j A_k = A_i A_{ij} A_{jk} A_{ik} A_k \subset A_i A_{ij} A_{jk} A_{ik} \). Since \( A \) is simple we then have \( A = \sum_{i,j=1}^{\infty} A_i A_{ij} A_j \) and in particular \( A_{ij} = A_i A_{ij} A_j \) for each \( i, j \). (3) follows from (2) since \( A_i = M_i \). (4) follows from (1) and (2) since \( A_{ij} = A_i A_{ij} A_j = A_{ik}A_{kj}A_{ij}A_{ik}A_{kj}A_{ij} \subset A_{ij} \).

For \( i \neq j \) we set \( M_{ij} = \{ x_{ij} - x_{ji} : x_{ij} \in A_{ij} \} \) and note that \( M_{ij} = M_{ji} \).

**Lemma 2.**

1. For \( i, j, k \) distinct, \( M_{ij} = [M_{ik}, M_{kj}] \) (and thus \( M_{ij} \subset M \)).
2. For \( i, j, k \) fixed and distinct, \( M \) is generated (as a Lie ring) by \( M_{ij} \cup M_{jk} \).
3. For \( i \neq j \), \( M_{ij} = [M_i, M_j] = [[M_i, M_j], M_i] \).

**Proof.**

1. By Lemma 1 (4) we may write \( x_{ij} = \sum_{k=1}^{\infty} x_{ik} x_{kj}^* \), \( i \neq k, j \neq k \). One then verifies that \( x_{ij} - x_{ji} = \sum_{k=1}^{\infty} [x_{ih} x_{jk}^* - x_{ki} x_{hj}^*] \in [M_{ik}, M_{kj}] \).

2. Since \( M_{ik} = [M_{ij}, M_{jk}] \) by (1) it suffices to show that \( M \) is generated by \( M_{ij} \cup M_{jk} \cup M_{ik} \). A typical element of \( K \) is of the form \( k_1 + k_2 + k_3 + x_{12} - x_{21} + x_{13} - x_{31} + x_{23} - x_{32}, k_t \in K_t, x_{ij} \in A_{ij} \). Commuting two elements of \( K \) results in a sum of elements of the form \([k_i, l_j], [k_i, a_{ij}], \) and \([a_{ij} - a_{ji}, a_{pq} - a_{qp}]\), \( i \neq j, p \neq q, k_i, l_i \in K_i \). It suffices to analyze the term \([k_i, l_j]\), say \([k_1, l_1]\). By Lemma 1 (1) \( k_1 \) can be written as a sum of terms of the form \( x_{12}y_{21} - y_{12}x_{21} \), and so we need only consider elements of the form \([x_{12}y_{21} - y_{12}x_{21}, l_1]\). But one verifies directly that

\[
[x_{12}y_{21} - y_{12}x_{21}, l_1] = [x_{12} - x_{21}, y_{21}l_1 + l_1y_{12}] - [y_{12} - y_{21}, x_{21}l_1 + l_1x_{12}].
\]

3. Using Lemma 1 (3), we see that \( x_{ij} \) is a sum of terms of the form...
Let $A$ and $B$ be simple rings with centroids $\Phi$ and $\Omega$, respectively. We assume henceforth that the following conditions are satisfied:

1. $A$ and $B$ have involutions of the first kind (both involutions will be denoted by $^*$).
2. Neither $A$ nor $B$ is GPI over its centroid.
3. $A$ contains two nonzero orthogonal symmetric idempotents $e_1$ and $e_2$ such that $e_1 + e_2 \neq 1$ ($A$ may or may not contain 1).
4. There is a (Lie) isomorphism $\phi$ of $M$ onto $N$, where $M = [K, K]$ ($K$ the skew elements of $A$) and $N = [L, L]$ ($L$ the skew elements of $B$).

Our goal is to show that there exists an isomorphism $\sigma$ of $A$ onto $B$ whose restriction to $M$ is $\phi$.

**Lemma 3.** If $m_i \neq 0 \in M_i$, $i = 1, 2, 3$ and $a = \phi(m_1)$, $b = \phi(m_2)$, $c = \phi(m_3)$, then $a, b, c$ are $\Omega$-independent.

**Proof.** $m_i$ does not lie in the center of $A_i$ since $^*$ is of the first kind. Hence, for each $i$, there exists $k_i \in M_i$ such that $[m_i, k_i] \neq 0$, since $A_i = M_i$. Now suppose $\alpha a + \beta b + \gamma c = 0$, $\alpha, \beta, \gamma \in \Omega$. Commuting this equation with $\phi(k_i)$ yields $\alpha [\phi(m_1), k_i] + \beta [\phi(m_2), k_i] + \gamma [\phi(m_3), k_i] = 0$. This reduces to $\alpha \phi(m_1) k_i = 0$, whence $\alpha = 0$. Similarly $\beta = \gamma = 0$.

The key result is the following:

**Lemma 4.** If $m_i \neq 0 \in M_i$, $i = 1, 2, 3$ and $a = \phi(m_1)$, $b = \phi(m_2)$, $c = \phi(m_3)$, then $ab = ac = bc = 0$.

**Proof.** It is straightforward to verify that $[[m, m_2], m_3] = 0$ for all $m \in M$. Applying $\phi$ to this equation and setting $n = \phi(m)$ we have $[[[n, a], b], c] = 0$ for all $n \in N$. Let $\tilde{B}$ be the central closure of $B$ and in $\tilde{B}_{\Omega}(x)$ set $f(x) = [[[x, a], b], c]$. If $f(x)$ is a nonzero element of $\tilde{B}_{\Omega}(x)$, then $N$ is GPI over $\Omega$ and by Theorem C $B$ is GPI over $\Omega$, a contradiction to assumption (2). Therefore $f(x)$ is the zero element in $\tilde{B}_{\Omega}(x)$ and in particular $[[[y, a], b], c] = 0$ for all $y \in \tilde{B}$. Equivalently,

$$
(abc)_r t_1 - (ab)_r c_1 - (ac)_r b_1 - (bc)_r a_1 \\
+ a_r (bc)_t + b_r (ac)_t + c_r (ab)_t - 1_r (abc)_t = 0
$$

(1)
in the ring $\tilde{B}, \tilde{B}_1$, where $\tilde{B}$ and $\tilde{B}_1$ are respectively the right and left multiplications determined by the elements of $\tilde{B}$. By Theorem B, $\tilde{B}, \tilde{B}_1$ is isomorphic to $\tilde{B} \otimes \tilde{B}_0$ under the rule $a \otimes b \rightarrow a_r b_1$. Therefore Eq. (1), when translated to $\tilde{B} \otimes \tilde{B}_0$, becomes (using the fact that $a, b, c$ commute with each other):

$$abc \otimes 1 - ab \otimes c =- ac \otimes b = bc \otimes a$$

$$+ a \otimes bc + b \otimes ac + c \otimes ab - 1 \otimes abc = 0.$$  

(2)

Since $\tilde{B} = L \oplus \tilde{T}$, where $L$ is the skew elements of $\tilde{B}$ and $\tilde{T}$ is the symmetric elements of $\tilde{B}$, Eq. (2) breaks up into two separate equations, one of which is

$$a \otimes bc + b \otimes ac + c \otimes ab + abc \otimes 1 = 0.$$  

(3)

By Lemma 3 we know that $a, b, c$ are $\Omega$-independent. Equation (3) then implies that

$$abc = -\alpha a - \beta b - \gamma c$$  

(4)

for some $\alpha, \beta, \gamma \in \Omega$. Substitution of (4) in (3) yields

$$a \otimes (bc - \alpha) + b \otimes (ac - \beta) + c \otimes (ab - \gamma) = 0.$$  

(5)

By the independence of $a, b, c$ we must have

$$bc = \alpha, \quad ac = \beta, \quad ab = \gamma.$$  

(6)

From (6) we easily obtain $abc = \alpha a = \beta b = \gamma c$, whence $\alpha = \beta = \gamma = 0$. Thus (6) becomes $ab = ac = bc = 0$, and the proof is complete.

We now set $N_i = \phi(M_i)$ and $N_{ij} = \phi(M_{ij}), ~ i \neq j$. From Lemma 4 it is clear that $N_i N_i = 0$ for $i \neq j$.

**Lemma 5.** (1) $N_{ij} = [N_i, N_{ij}] = [[N_i, N_{ij}], N_{ij}], ~ i \neq j$.

(2) The elements of $N_{ij}$ are sums of elements of the form

$$\phi(m_i) \phi(x_{ij} - x_{ji}) \phi(m_j) + \phi(m_j) \phi(x_{ij} - x_{ji}) \phi(m_i).$$

(3) For $i, j, k$ fixed and distinct, $B$ is generated (as a ring) by $N_{ij} \cup N_{jk}$.

**Proof.** (1) follows by applying $\phi$ to Lemma 2 (3). (2) follows from $N_{ij} = [[N_i, N_{ij}], N_{ij}]$ in conjunction with the fact that $N_i N_j = 0, ~ i \neq j$. Application of $\phi$ to Lemma 2 (2) shows that $N$ is generated (as a Lie ring) by $N_{ij} \cup N_{jk}$. Since $B$ is not GPI over $\Omega$ we have earlier remarked that in particular $[B : \Omega] = \infty$. Hence by Theorem D, $B = N$, and the proof of (3) is complete.

From Lemma 5 it is clear that $N_i N_{jk} = 0, ~ i, j, k$ distinct.
Lemma 6. For $i \neq j$, $k_i, l_i, \ldots, q_i \in M_i$, $m_j \in M_j$, $x_{ij} \in A_{ij}$, $\phi(k_i) \phi(l_i) \cdots \phi(q_i) \phi(x_{ij} - x_{ji}) \phi(m_j) = \phi(k_i l_i \cdots q_i x_{ij} + (-1)^{n+1} x_{ij} q_i \cdots l_i k_i) \phi(m_j)$, where $n$ is the "length" of the product $k_i l_i \cdots q_i$.

Proof. The proof is by induction on $n$. For $n = 1$, $\phi(k_i) \phi(x_{ij} - x_{ji}) \phi(m_j) = [\phi(k_i), \phi(x_{ij} \cdots x_{ji})] \phi(m_j) = \phi(k_i x_{ij} + x_{ji} k_i) \phi(m_j)$. We assume the lemma true for $n$ and prove it for $n + 1$. Indeed

$$
\phi(k_i) \phi(l_i) \cdots \phi(q_i) \phi(x_{ij} - x_{ji}) \phi(m_j) \quad = \quad \phi(k_i) \phi(l_i \cdots q_i x_{ij} + (-1)^{n+1} x_{ij} q_i \cdots l_i) \phi(m_j) \quad = \quad \phi(k_i l_i \cdots q_i x_{ij} + x_{ji}(-q_i) \cdots (-l_i) k_i) \phi(m_j) \quad = \quad \phi(k_i l_i \cdots q_i x_{ij} + (-1)^{n+2} x_{ij} q_i \cdots l_i k_i) \phi(m_j).
$$

Lemma 7. If $\sum k_i l_i \cdots q_i = 0$, $k_i, l_i, \ldots, q_i \in M_i$, then $\sum \phi(k_i) \phi(l_i) \cdots \phi(q_i) = 0$. (Here it is understood that the number of terms in each product may vary.)

Proof. For simplicity we may assume $i = 1$. We first note that $0 = 0^* = (\sum k_i l_i \cdots q_i)^* = \sum (-1)^n q_i l_i \cdots k_i$, where $n$ is the "length" of the product $k_i l_i \cdots q_i$. By Lemma 5 (3) $B$ is generated by $N_{12} \cup N_{23}$, where $N_{12} - [N_{12}, N_{23}]$ by Lemma 5 (1). To show $y = \sum \phi(k_i) \phi(l_i) \cdots \phi(q_i) = 0$, if suffices to show that $yB = 0$, i.e., $y(N_{12} \cup N_{23}) = 0$. But this just reduces to showing $y(x_{i12} - x_{21}) \phi(m_2) = 0$, $x_{i12} \in A_{i12}$, $m_2 \in M_2$. By Lemma 6

$$
\left\{ \sum \phi(k_i) \phi(l_i) \cdots \phi(q_i) \right\} \phi(x_{i12} - x_{21}) \phi(m_2) = \sum \phi(k_i l_i \cdots q_i x_{i12} + (-1)^{n+1} x_{i21} q_i \cdots l_i k_i) \phi(m_2) = \phi \left\{ \sum k_i l_i \cdots g_3 \right\} x_{i12} - x_{21} \left\{ \sum (-1)^n q_i l_i \cdots k_i \right\} \phi(m_2) = 0.
$$

Lemma 7 enables us to define a mapping $\mu$ from $\sum_{i=1}^r A_i$ into $B$ according to the rule:

$$
\mu(x_i) = \sum \phi(k_i) \phi(l_i) \cdots \phi(q_i),
$$

where

$$
x_i = \sum k_i l_i \cdots q_i \in A_i, \quad k_i, \; l_i, \ldots, q_i \in M_i.
$$

It is clear that $\mu$ is a well-defined ring homomorphism of $\sum_{i=1}^r A_i$ into $B$. In particular $f_1 = \mu(e_1) \in B$, $f_2 = \mu(e_2) \in B$, and $f_3 = 1 - f_1 - f_2 \in B$ are orthogonal idempotents, with $f_2 B$ and $B f_3$ contained in $B$. Evidently
Lemma 8. For \( i \neq j \), \( \phi(x_{ij} - x_{ji}) = f_i \phi(x_{ij} - x_{ji}) f_j + f_j \phi(x_{ij} - x_{ji}) f_i \).

An additive mapping \( \rho: \sum_{i=1}^{n} A_{ij} \to B \) can now be defined according to the rule

\[
\rho(x_{ij}) = f_i \phi(x_{ij} - x_{ji}) f_j, \quad x_{ij} \in A_{ij}, \quad i \neq j.
\]

We now define a mapping \( \sigma: A \to B \) according to

\[
\sigma\left( \sum_{i,j=1}^{3} x_{ij} \right) = \sum_{i=1}^{3} \mu(x_{ii}) + \sum_{i \neq j} \rho(x_{ij}), \quad x_{ij} \in A_{ij}.
\]

\( \sigma \) is clearly additive, and in view of Lemma 8 it is easily seen that \( \sigma(x_{ij}) \sigma(y_{jk}) = 0 \) if \( j \neq k \). We shall tacitly use this fact frequently in the sequel.

Lemma 9. \( \sigma(x_{ij}x_{jk}) = \sigma(x_{ij}) \sigma(x_{jk}), \quad x_{ij}, x_{jk} \in A_{ij}, \quad x_{jk} \in A_{jk}, \quad i, j, k \) distinct.

Proof. For simplicity we assume \( i = 1, j = 2, k = 3 \). Then

\[
\sigma(x_{12}x_{23}) = f_3 \phi(x_{12}x_{23} - x_{32}x_{23}) f_3 = f_1[\phi(x_{12} - x_{21}), \phi(x_{23} - x_{32})] f_3
\]

\[
= f_1 \phi(x_{12} - x_{21}) \phi(x_{23} - x_{32}) f_3
\]

\[
= f_1[f_1 \phi(x_{12} - x_{21}) f_2 + f_2 \phi(x_{12} - x_{21}) f_1] \phi(x_{23} - x_{32}) f_3
\]

\[
= \{f_1 \phi(x_{12} - x_{21}) f_2\} \{f_2 \phi(x_{23} - x_{32}) f_3\}
\]

\[
= \sigma(x_{12}) \sigma(x_{23}).
\]

Lemma 10. \( \sigma(x_{ij}x_{ij}) = \sigma(x_{ij}) \sigma(x_{ij}), \quad x_i \in A_i, \quad x_{ij} \in A_{ij}, \quad i \neq j. \)

Proof. We assume \( i = 1, j = 2, \) and first show that \( \sigma(k_1x_{12}) = \sigma(k_1) \sigma(x_{12}), \quad k_1 \in M_1 \).

Indeed,

\[
\sigma(k_1x_{12}) = f_1 \phi(k_1x_{12} + x_{21}k_1) f_3
\]

\[
= f_1[k_1, x_{12} - x_{21}] f_3
\]

\[
= f_1[\phi(k_1), \phi(x_{12} - x_{21})] f_3
\]

\[
= f_1 \phi(k_1) \phi(x_{12} - x_{21}) f_3
\]

\[
= \phi(k_1) f_1 \phi(x_{12} - x_{21}) f_3
\]

\[
= \sigma(k_1) \sigma(x_{12}).
\]

For \( k_1, l_1, ..., q_1 \in M_1 \) repeated application of the preceding argument shows
that \( o(k_1l_1 \cdots q_1x_{12}) = o(k_1) o(l_1) \cdots o(q_1) o(x_{12}) = o(k_1l_1 \cdots q_1) o(x_{12}) \). Since \( A_1 = M_1 \), the proof of the lemma is complete.

**Lemma 11.** \( o(x_{ij}y_{ij}) = o(x_{ij}) o(y_{ij}), x_{ij} \in A_{ij}, y_{ij} \in A_{ji}, i \neq j \).

**Proof.** We let \( i = 1, j = 2 \), and choose \( x_{13} \in A_{13} \). Then by repeated use of Lemmas 9 and 10 we have \( o(x_{12}y_{21}) o(x_{13}) = o(x_{12}y_{21}x_{13}) = o(x_{12}) o(y_{21}) o(x_{13}). \) Thus

\[
\{o(x_{12}y_{21}) - o(x_{12}) o(y_{21})\} o(x_{13} - x_{21}) = 0,
\]

using Lemma 8. Since \( \{o(x_{12}y_{21}) - o(x_{12}) o(y_{21})\} o(x_{23} - x_{32}) \) is clearly zero we see by Lemma 5 (3) that \( o(x_{12}y_{21}) - o(x_{12}) o(y_{21}) = 0 \).

Lemmas 9–11 complete the proof that \( o \) is a ring homomorphism of \( A \) into \( B \). \( o \) is clearly an injection since \( A \) is simple. By Lemma 8 we see that \( o(x_{12}y_{21}) - o(x_{12}) o(y_{21}) \) is clearly zero and also that \( o \) is surjective (since the \( o(x_{ij} - x_{ij}) \) generates \( B \)). This completes the proof of the

**Main Theorem.** Let \( A \) and \( B \) be simple algebras with involution of the first kind such that neither \( A \) nor \( B \) satisfies a generalized polynomial identity over its centroid. Furthermore assume that \( A \) contains two nonzero symmetric idempotents whose sum is not equal to 1. Let \( K \) and \( L \) be, respectively, the skew elements of \( A \) and \( B \). Then any (Lie) isomorphism of \( [K, K] \) onto \( [L, L] \) can be extended uniquely to an (associative) isomorphism of \( A \) onto \( B \).

**References**