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Fast Construction of a Symmetric Nonnegative Matrix with a Prescribed Spectrum

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Abstract—In this paper, for a prescribed real spectrum, using properties of the circulant matrices and of the symmetric persymmetric matrices, we derive a fast and stable algorithm to construct a symmetric nonnegative matrix which realizes the spectrum. The algorithm is based on the fast Fourier transform.

Keywords—Inverse eigenvalue problem, Nonnegative matrix, Circulant matrix, Fast Fourier transform, Persymmetric matrix.

1. INTRODUCTION

There are many applications involving nonnegative matrices. We mention areas like game theory, Markov chains (stochastic matrices), theory of probability, probabilistic algorithms, numerical analysis, discrete distribution, group theory, matrix scaling, theory of small oscillations of elastic systems (oscillation matrices), and economics.

Many references concerning properties of nonnegative matrices are available. Some fundamental results are collected into the following main theorem of Perron and Frobenius [1].

THEOREM 1. *Let* $\mathbf{A}$ *be a nonnegative irreducible* $n \times n$ *matrix. Then,*

1. $\mathbf{A}$ *has a positive real eigenvalue* $r(\mathbf{A})$ *equal to its spectral radius* $\rho(\mathbf{A})$,
2. *to* $r(\mathbf{A})$ *there corresponds a positive eigenvector,*
3. $r(\mathbf{A})$ *increases when any entry of* $\mathbf{A}$ *increases,*
4. $r(\mathbf{A})$ *is a simple eigenvalue of* $\mathbf{A}$.

The eigenvalue $r(\mathbf{A})$ is called the Perron root of $\mathbf{A}$ and the corresponding positive eigenvector is called the Perron vector of $\mathbf{A}$.

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The general inverse eigenvalue problem for nonnegative matrices is the following.

PROBLEM 1. Given a set $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ of complex numbers, find necessary and sufficient conditions for σ to be the spectrum of a nonnegative matrix.

An important particular case of Problem 1 follows.

PROBLEM 2. Given a set $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ of real numbers, find necessary and sufficient conditions for σ to be the spectrum of a nonnegative matrix.

These problems have been studied by many authors and very important results have been obtained by Suleimanova [2], Perfect [3], Salzman [4], Ciarlet [5], Kellogg [6], Fiedler [7], Soules [8], Borobia [9], Radwan [10], and Wumen [11].

However, "very few of these results are ready for implementation to actually compute this matrix" [12, p. 18].

We recall the first and perhaps the most important sufficient condition, due to Suleimanova [2], for the existence of a symmetric nonnegative matrix with a prescribed real spectrum.

THEOREM 2. *Let $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a real set satisfying*

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n \geq 0, & \\ \lambda_j < 0, \qquad & j = 2, 3, \dots n. \end{aligned} \tag{1}$$

Then there exists a nonnegative $n \times n$ matrix with spectrum σ.

Perfect [3] proved this theorem showing that the companion matrix of the polynomial $p(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)$ is nonnegative. The construction of the companion matrix of the polynomial p requires to evaluate the elementary symmetric functions at $\lambda_1, \lambda_2, \dots, \lambda_n$. That is,

$$\begin{aligned} c_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_n, \\ c_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n, \\ c_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \cdots + \lambda_{n-2}\lambda_{n-1}\lambda_n, \\ &\ \vdots \\ c_n &= \lambda_1\lambda_2\lambda_3 \dots \lambda_n. \end{aligned}$$

This computation requires a number of arithmetic operations which increases exponentially with n. In addition, it is well known that the zeros of a polynomial are very sensitive to changes in its coefficients.

The most constructive result is the sufficient condition studied by Soules. The construction of the matrix depends on the specification of the Perron vector; in particular, the components of the Perron vector need to satisfy some inequalities in order for the construction to work.

In this paper, we relax the sufficient condition of Suleimanova and we obtain a fast and stable procedure based on the fast Fourier transform [13] to construct a symmetric nonnegative matrix with a prescribed real spectrum. The procedure does not require to know the Perron vector.

The fast Fourier transform is a very fast and a very stable algorithm to compute the discrete Fourier transform. In particular, this algorithm can be used to compute very efficiently the eigenvalues of a circulant matrix. Thus, for the purpose of this paper, we begin considering the case of the inverse eigenvalue problem for a circulant nonnegative matrix. This is done in Section 2. In Section 3, we consider a prescribed real spectrum. This spectrum is used to define an inverse eigenvalue problem for a real circulant $(2n) \times (2n)$ matrix. We arrive to a real symmetric persymmetric nonnegative matrix. Then, using properties of this class of matrices, a real symmetric nonnegative $n \times n$ matrix which realizes the prescribed real spectrum is easily obtained. In Section 4, some experimental results are included.

2. CIRCULANT NONNEGATIVE MATRICES

In this section, we recall basic facts of the circulant matrices and we study the inverse eigenvalue problem for circulant nonnegative matrices.

An $n \times n$ matrix $\mathbf{C} = (c_{ij})$ is a circulant matrix if $c_{ij} = c_{i+1,j+1}$ and the subscripts are taken modulo n, that is,

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdot & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdot & \cdot & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_2 \\ c_2 & \cdot & \cdot & \cdot & \cdot & c_1 \\ c_1 & c_2 & \cdot & c_{n-2} & c_{n-1} & c_0 \end{bmatrix}. \tag{2}$$

The circulant matrix given in (2) is denoted by

$$\mathbf{C} = \operatorname{circ}(c_0, c_1, \dots, c_{n-1}).$$

The circulant matrices have very nice properties. We recall some of them.

1. If $\omega = \exp(2\pi i/n)$, $i^2 = -1$, then the vectors
$$\mathbf{1}_n = [1, 1, \dots, 1]^\top, \qquad \mathbf{v}_j = \left[1, \omega^{j-1}, \omega^{2(j-1)}, \dots, \omega^{(n-1)(j-1)}\right]^\top, \tag{3}$$
$j = 2, 3, \dots, n$, form an orthogonal basis of eigenvectors of any complex circulant $n \times n$ matrix.
2. Moreover,
$$\mathbf{v}_{n-j+2} = \overline{\mathbf{v}}_j, \qquad j = 2, 3, \dots, \left[\frac{n+1}{2}\right], \tag{4}$$
where $\overline{\mathbf{v}}_j$ denotes the vector whose components are the complex conjugates of the components of $\mathbf{v}_j$ and $[(n+1)/2]$ is the greatest integer not exceeding $(n+1)/2$.
3. If $\mathbf{C}$ is the circulant $n \times n$ matrix given in (2) then its eigenvalues are
$$\lambda_1 = c_0 + c_1 + c_2 + \cdots + c_{n-1}, \qquad \lambda_j = c_0 + c_1\omega^{j-1} + c_2\omega^{2(j-1)} + \cdots + c_{n-1}\omega^{(n-1)(j-1)}, \tag{5}$$
$j = 2, \dots, n$, with eigenvectors $\mathbf{1}_n, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$, respectively. If, in addition, $\mathbf{C}$ is a real matrix, then
$$\lambda_{n-j+2} = \overline{\lambda}_j, \qquad j = 2, 3, \dots, \left[\frac{n+1}{2}\right]. \tag{6}$$
4. Let
$$\mathbf{F} = (f_{kj}) = [\mathbf{1}_n, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n]. \tag{7}$$
Then,
$$f_{kj} = \omega^{(k-1)(j-1)}, \quad 1 \le k, \quad j \le n, \qquad \text{and} \qquad \mathbf{F}\overline{\mathbf{F}} = \overline{\mathbf{F}}\mathbf{F} = n\mathbf{I}_n, \tag{8}$$
where $\mathbf{I}_n$ denotes the identity matrix of order n. In the literature, $(1/\sqrt{n})\,\mathbf{F}$ is called the Fourier matrix.

We consider the following subset of $\mathbb{C}^{(n-1)}$,

$$\mathcal{S}^{(n-1)} = \left\{(\lambda_2, \lambda_3, \dots, \lambda_n) \mid \lambda_{n-j+2} = \overline{\lambda}_j,\ j = 2, 3, \dots, \left[\frac{n+1}{2}\right]\right\}.$$

EXAMPLE 1. We have

$$\left(5+3i, \sqrt{2}-2i, 6-4i, -1+i, -1-i, 6+4i, \sqrt{2}+2i, 5-3i\right) \in \mathcal{S}^8,$$
$$(-2+i, 3-i, 5i, -3, -3, -5i, 3+i, -2-i) \in \mathcal{S}^8,$$
$$(5+3i, 6-4i, -1+i, -4-2i, 5, -4+2i, -1-i, 6+4i, 5-3i) \in \mathcal{S}^9,$$
$$(5+3i, 6-4i, -1, -4-2i, 5, -4+2i, -1, 6+4i, 5-3i) \in \mathcal{S}^9,$$
$$(-2, -3, -5, -7, 0, -7, -5, -3, -2) \in \mathcal{S}^9.$$

The following lemma is clear.

LEMMA 3. *If* $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$, *then* $\{\lambda_2, \lambda_3, \dots, \lambda_n\}$ *is closed under complex conjugation and* $\lambda_{m+2} \in \mathbb{R}$ *if* $n = 2m+2$.

REMARK 1. If $\{\lambda_2, \lambda_3, \dots, \lambda_n\}$ is closed under complex conjugation and $\operatorname{Im} \lambda_j \neq 0$, $2 \leq j \leq n$, then the elements of the set can be reindexed such that $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$.

LEMMA 4. *If* $\mu \in \mathbb{R}$ *and* $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$, *then there exists a real circulant* $n \times n$ *matrix* $\mathbf{C} = \operatorname{circ}(c_0, c_1, \dots, c_{n-1})$ *such that*

$$\sigma(\mathbf{C}) = \{\mu, \lambda_2, \lambda_3, \dots, \lambda_n\}.$$

PROOF. Let

$$\mathbf{q} = [\mu, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n]^\top.$$

Since $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$,

$$\mathbf{q} = \left[\mu, \lambda_2, \lambda_3, \dots, \overline{\lambda}_3, \overline{\lambda}_2\right]^\top.$$

Let $\mathbf{P}$ be the permutation matrix $\mathbf{P} = [\mathbf{e}_1, \mathbf{e}_n, \mathbf{e}_{n-1}, \dots, \mathbf{e}_3, \mathbf{e}_2]$, where $\mathbf{e}_j$ denotes the j^{th} column of $\mathbf{I}_n$. Then,

$$\mathbf{P}\overline{\mathbf{q}} = \begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_n^\top \\ \mathbf{e}_{n-1}^\top \\ \vdots \\ \mathbf{e}_3^\top \\ \mathbf{e}_2^\top \end{bmatrix} \begin{bmatrix} \mu \\ \overline{\lambda}_2 \\ \overline{\lambda}_3 \\ \vdots \\ \lambda_3 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \overline{\lambda}_3 \\ \overline{\lambda}_2 \end{bmatrix} = \mathbf{q}.$$

Also, using the property in (4), one can see that $\mathbf{P}\overline{\mathbf{F}} = \overline{\mathbf{F}}\mathbf{P} = \mathbf{F}$. Let

$$\mathbf{C} = \operatorname{circ}(c_0, c_1, c_2, \dots, c_{n-1}).$$

Taking in consideration that the eigenvalues of $\mathbf{C}$ are given by the expressions in (5), we impose

$$\begin{aligned} c_0 + c_1 + c_2 + \cdots + c_{n-1} &= \mu, \\ c_0 + \omega c_1 + \omega^2 c_2 + \cdots + \omega^{n-1} c_{n-1} &= \lambda_2, \\ c_0 + \omega^2 c_1 + \omega^4 c_2 + \cdots + \omega^{2(n-1)} c_{n-1} &= \lambda_3, \\ c_0 + \omega^3 c_1 + \omega^3 c_2 + \cdots + \omega^{3(n-1)} c_{n-1} &= \lambda_4, \\ &\ \,\vdots \\ c_0 + \omega^{n-3} c_1 + \omega^{2(n-3)} c_2 + \cdots + \omega^{(n-1)(n-3)} c_{n-1} &= \overline{\lambda}_4, \\ c_0 + \omega^{n-2} c_1 + \omega^{2(n-2)} c_2 + \cdots + \omega^{(n-1)(n-2)} c_{n-1} &= \overline{\lambda}_3, \\ c_0 + \omega^{n-1} c_1 + \omega^{2(n-1)} c_2 + \cdots + \omega^{(n-1)(n-1)} c_{n-1} &= \overline{\lambda}_2. \end{aligned}$$

This system of linear equation in the unknowns $c_0, c_1, c_2, \dots, c_{n-1}$ can be written as follows

$$\mathbf{Fc} = \mathbf{q}, \tag{9}$$

where

$$\mathbf{c} = [c_0, c_1, c_2, \dots\dots, c_{n-1}]^\top .$$

After complex conjugation, from (9), we obtain $\overline{\mathbf{F}}\overline{\mathbf{c}} = \overline{\mathbf{q}}$. Then, $\mathbf{P}\overline{\mathbf{F}}\overline{\mathbf{c}} = \mathbf{P}\overline{\mathbf{q}}$. Since $\mathbf{P}\overline{\mathbf{F}} = \mathbf{F}$ and $\mathbf{P}\overline{\mathbf{q}} = \mathbf{q}$, we have $\mathbf{F}\overline{\mathbf{c}} = \mathbf{q}$. Therefore, $\mathbf{Fc} = \mathbf{F}\overline{\mathbf{c}}$. Finally, using the fact that $\mathbf{F}$ is an invertible matrix, we conclude that $\mathbf{c} = \overline{\mathbf{c}}$. Thus, system (9) has a real solution $\mathbf{c} = \mathbf{F}^{-1}\mathbf{q} = (1/n)\,\overline{\mathbf{F}}\mathbf{q}$ and $\mu, \lambda_2, \lambda_3, \dots, \lambda_n$ are eigenvalues of the real circulant $n \times n$ matrix $\mathbf{C}$. The proof is complete. ∎

REMARK 2. The matrix $\mathbf{C}$ in Lemma 4 can be computed as follows.

1. Compute $\mathbf{c} = (1/n)\,\overline{\mathbf{F}}\mathbf{q}$.
2. Define $\mathbf{C} = \operatorname{circ}(c_0, c_1, c_2, \dots, c_{n-1})$.

The product $\overline{\mathbf{F}}\mathbf{q}$ may be accomplished by the fast Fourier transform. A direct calculation of the product requires $(n-1)^2$ multiplications and $n(n-1)$ additions. For large n this would require too much computer time. The advantage of the fast Fourier transform consists of the tremendous savings that it gives in the number of multiplications and additions; in particular, if n is a power of 2, the number of operations to compute the product $\overline{\mathbf{F}}\mathbf{q}$ is $(1/2)\,n\log_2 n$ multiplications and $n\log_2 n$ additions. For instance, if $n = 2^{10}$, then the fast Fourier transform requires 5,120 multiplications as opposed to 1,046,529 multiplications for a direct calculation. Moreover, since the singular values of $\mathbf{F}$ are the nonnegative square roots of the eigenvalues of $\mathbf{F}\overline{\mathbf{F}} = n\mathbf{I}$, its spectral condition number is equal to 1. Therefore, $\mathbf{Fc} = \mathbf{q}$ is a very well conditioned system of linear equations.

REMARK 3. For

$$\mu = -\sum_{j=2}^{n} \lambda_j + \sum_{j=2}^{n} |\operatorname{Im} \lambda_j|$$

the explicit formulas for the entries of the circulant matrix $\mathbf{C}$ of Lemma 4 follow.

1. If $n = 2m+1$, then

$$c_k = \frac{1}{n}\begin{pmatrix} 2\displaystyle\sum_{j=2}^{m+1}\left(\cos\frac{2k(j-1)\pi}{2m+1} - 1\right)\operatorname{Re}\lambda_j \\ +2\displaystyle\sum_{j=2}^{m+1}\left(\operatorname{Im}\lambda_j \sin\frac{2k(j-1)\pi}{2m+1} + |\operatorname{Im}\lambda_j|\right)\end{pmatrix},$$

   for $k = 0, 1, \dots, 2m$.

2. If $n = 2m+2$, then

$$c_k = \frac{1}{n}\begin{pmatrix} 2\displaystyle\sum_{j=2}^{m+1}\left(\cos\frac{k(j-1)\pi}{m+1} - 1\right)\operatorname{Re}\lambda_j + \left((-1)^k - 1\right)\lambda_{m+2} \\ +2\displaystyle\sum_{j=2}^{m+1}\left(\operatorname{Im}\lambda_j \sin\frac{k(j-1)\pi}{m+1} + |\operatorname{Im}\lambda_j|\right)\end{pmatrix},$$

   for $0, 1, \dots, 2m+1$.

For the rest of this section, let

$$\mu = -\sum_{j=2}^{n} \lambda_j + \sum_{j=2}^{n} |\operatorname{Im} \lambda_j|. \tag{10}$$

and let $\mathbf{C} = \operatorname{circ}(c_0, c_1, c_2, \dots, c_{n-1})$ be the real circulant of Lemma 4 and

$$c = \min\{c_0, c_1, c_2, \dots, c_{n-1}\}. \tag{11}$$

THEOREM 5. *If* $(\lambda_2, \lambda_3, \ldots, \lambda_n) \in \mathcal{S}^{(n-1)}$ *and*

$$\lambda_0 = \max\left\{\mu, \max_{2\le k\le n} |\lambda_k|\right\}, \qquad \textit{wherever } c \ge 0,$$

or

$$\lambda_0 = \max\left\{\mu - nc, \max_{2\le k\le n} |\lambda_k|\right\}, \qquad \textit{wherever } c < 0,$$

then there exists a circulant nonnegative matrix $\mathbf{A}_0$ *such that* $r(\mathbf{A}_0) = \lambda_0$ *and* $\sigma(\mathbf{A}_0) = \{\lambda_0, \lambda_2, \ldots, \lambda_n\}$. *And, for* $\lambda > \lambda_0$, *there exists a circulant positive matrix* $\mathbf{A}_\lambda$ *such that* $r(\mathbf{A}_\lambda) = \lambda$ *and* $\sigma(\mathbf{A}_\lambda) = \{\lambda, \lambda_2, \ldots, \lambda_n\}$.

PROOF. From Lemma 4, there exists a real circulant $n \times n$ matrix

$$\mathbf{C} = \operatorname{circ}(c_0, c_1, \ldots, c_{n-1}),$$

such that $\{\lambda_2, \lambda_3, \ldots, \lambda_n\} \subset \sigma(\mathbf{C})$. The other eigenvalue of $\mathbf{C}$ is $\mu = -\sum_{j=2}^{n} \lambda_j + \sum_{j=2}^{n} |\operatorname{Im} \lambda_j| = \sum_{j=0}^{n-1} c_j$. We recall that $\mathbf{1}_n, \mathbf{v}_2, \ldots, \mathbf{v}_n$, given in (3), are orthogonal eigenvectors corresponding to $\mu, \lambda_2, \lambda_3, \ldots, \lambda_n$, respectively. Let

$$\mathbf{A}_\lambda = \mathbf{C} + \frac{\lambda - \mu}{n} \mathbf{1}_n \mathbf{1}_n^\top.$$

Then,

$$\mathbf{A}_\lambda = \operatorname{circ}\left(c_0 + \frac{\lambda-\mu}{n}, c_1 + \frac{\lambda-\mu}{n}, c_2 + \frac{\lambda-\mu}{n}, \ldots, c_{n-1} + \frac{\lambda-\mu}{n}\right)$$

and

$$\mathbf{A}_\lambda \mathbf{1}_n = \mathbf{C}\mathbf{1}_n + (\lambda-\mu)\mathbf{1}_n = \mu\mathbf{1}_n + (\lambda-\mu)\mathbf{1}_n = \lambda\mathbf{1}_n.$$

Moreover, since $\mathbf{1}_n^\top \mathbf{v}_j = 0$, we have

$$\mathbf{A}_\lambda \mathbf{v}_j = \mathbf{C}\mathbf{v}_j = \lambda_j \mathbf{v}_j, \qquad \text{for } j = 2, 3, \ldots, n-1.$$

We have proved that

$$\sigma(\mathbf{A}_\lambda) = \{\lambda, \lambda_2, \lambda_3, \ldots, \lambda_n\}.$$

Let c be as defined in (11).

CASE 1. Suppose $c \ge 0$. Then, $\mu \ge 0$. Let

$$\lambda_0 = \max\left\{\mu, \max_{2\le k\le n} |\lambda_k|\right\},$$

and $\mathbf{A}_0 = \mathbf{A}_{\lambda_0}$ is a circulant nonnegative matrix.

CASE 2. Now, suppose $c < 0$. Let

$$\lambda_0 = \max\left\{\mu - nc, \max_{2\le k\le n} |\lambda_k|\right\}.$$

Then, $\lambda_0 > \mu$ and

$$c_j + \frac{\lambda_0 - \mu}{n} \ge c + \frac{\lambda_0 - \mu}{n} \ge 0, \qquad \text{for } j = 0, 1, 2, \ldots, n-1.$$

Therefore, $\mathbf{A}_0 = \mathbf{A}_{\lambda_0}$ is a circulant nonnegative matrix.

In both cases, $\sigma(\mathbf{A}_0) = \{\lambda_0, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ and $r(\mathbf{A}_0) = \lambda_0$. Clearly, if $\lambda > \lambda_0$, then $\mathbf{A}_\lambda$ is a circulant positive matrix and, $\sigma(\mathbf{A}_\lambda) = \{\lambda, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ and $r(\mathbf{A}_\lambda) = \lambda$. ∎

EXAMPLE 2. Let $\lambda_2 = 5+3i$, $\lambda_3 = -2-11i$, $\lambda_4 = 7+8i$, $\lambda_5 = 7-8i$, $\lambda_6 = -2+11i$, $\lambda_7 = 5-3i$. For these given numbers, the use of Theorem 5 gives $\lambda_0 = 50.0214$ and

$$\mathbf{A}_0 = \operatorname{circ}(10.0031,\ 7.7640,\ 8.1776,\ 0.0000,\ 11.1150,\ 9.0021,\ 4.9596),$$

to four decimal places.

COROLLARY 6. *If $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is such that $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$ and $\lambda_1 = \lambda_0$, (respectively, $\lambda_1 > \lambda_0$), where λ_0 is as in Theorem 5, then there exists a circulant nonnegative, (respectively, positive) $n \times n$ matrix with spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_n)\}$.*

EXAMPLE 3. Consider λ_j, $j = 2, 3, 4, 5, 6, 7$, of Example 2, and let $\lambda_1 = 51$. Since $\lambda_1 > \lambda_0$, there exists a circulant positive matrix $\mathbf{A}$ with spectrum $\{\lambda_j\}_{j=1}^{7}$,

$$\begin{aligned}\mathbf{A} &= \mathbf{A}_0 + \frac{51 - \lambda_0}{7} \mathbf{1}_7 \mathbf{1}_7^\top \\ &= \operatorname{circ}(10.1429,\ 7.9038,\ 8.3174,\ 0.1398,\ 10.2548,\ 9.1419,\ 5.0994).\end{aligned}$$

COROLLARY 7. *If $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{R}$ is such that $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$ and*

$$\begin{aligned}\lambda_1 + \lambda_2 + \dots + \lambda_n &= 0, \qquad \text{(respectively, } > 0), \\ \lambda_j &< 0, \qquad j = 2, 3, \dots n.\end{aligned}$$

Then there exists a circulant nonnegative, (respectively, positive) $n \times n$ matrix with spectrum σ.

PROOF. Since $\operatorname{Im} \lambda_j = 0$ for all j, it follows that $\mu = -\sum_{j=2}^{n} \lambda_j$. From the proof of Theorem 5 there exists a circulant matrix $\mathbf{C} = \operatorname{circ}(c_0, c_1, \dots, c_n)$ such that

$$\sigma(\mathbf{C}) = \left\{ -\sum_{j=2}^{n} \lambda_j, \lambda_2, \lambda_3, \dots, \lambda_n \right\}.$$

Now, from the formulas given in Remark 3 and from the hypothesis $\lambda_j < 0$, $j = 2, 3, \dots n$, we have that $c_0 = 0$ and $c_k > 0$ for $k = 1, \dots, n-1$. Then, $c \geq 0$ and thus, $\lambda_0 = \max\{\mu, \max_{2 \leq k \leq n} |\lambda_k|\} = \mu$. Finally, since $\lambda_1 \geq \mu$, from Corollary 6, we have that λ_1 and $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are the Perron root and the spectrum, respectively, of the circulant nonnegative matrix

$$\mathbf{C} + \frac{\lambda_1 + \sum_{j=2}^{n} \lambda_j}{n} \mathbf{1}_n \mathbf{1}_n^\top.$$

Clearly, if $\lambda_1 > \mu$, then the matrix is positive. The proof is complete. ∎

EXAMPLE 4. Let $\lambda_1 = 13, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3, \lambda_5 = -3, \lambda_6 = -2, \lambda_7 = -1$. This prescribed spectrum satisfies the hypothesis of the previous lemma and it is realized by the circulant positive matrix

$$\mathbf{A} = \operatorname{circ}(0.1429,\ 2.5784,\ 1.9011,\ 1.9490,\ 1.9490,\ 1.9011,\ 0.5784).$$

COROLLARY 8. *If $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{C}$ is such that $(\lambda_2, \lambda_3, \dots, \lambda_n) \in \mathcal{S}^{(n-1)}$ and if*

$$\begin{aligned}\lambda_1 + \sum_{j=2}^{n} \lambda_j - \sum_{j=2}^{n} |\operatorname{Im} \lambda_j| &\geq 0, \\ \operatorname{Re}(\lambda_j) &\leq 0, \qquad j = 2, 3, \dots n,\end{aligned}$$

then there exists a circulant nonnegative $n \times n$ matrix with spectrum σ.

PROOF. We have

$$\mu = -\sum_{j=2}^{n} \lambda_j + \sum_{j=2}^{n} |\operatorname{Im} \lambda_j|.$$

From the proof of Theorem 5 there exists a circulant matrix $\mathbf{C} = \operatorname{circ}(c_0, c_1, \dots, c_n)$ such that

$$\sigma(\mathbf{C}) = \{\mu, \lambda_2, \lambda_3, \dots, \lambda_n\}.$$

Without loss of generality, we may suppose $\operatorname{Im}\lambda_j \le 0$ for $j = 2, 3, \ldots, [(n+1)/2]$. We assume $n = 2m + 2$. The proof for $n = 2m + 1$ is similar. From the formulas given in Remark 3, we have

$$c_k = \frac{1}{n}\left(\begin{array}{c} 2\sum_{j=2}^{m+1}\left(\cos\frac{k(j-1)\pi}{m+1} - 1\right)\operatorname{Re}\lambda_j + \left((-1)^k - 1\right)\lambda_{m+2} \\ +2\sum_{j=2}^{m+1}\left(\sin\frac{k(j-1)\pi}{m+1} - 1\right)\operatorname{Im}\lambda_j \end{array}\right).$$

Now, using the hypothesis $\operatorname{Re}\lambda_j \le 0$, $j = 2, 3, \ldots n$, we have that $c_k \ge 0$ for $k = 1, \ldots, n-1$. Then, $c \ge 0$, and thus, $\lambda_0 = \max\{\mu, \max_{2\le k\le n}|\lambda_k|\} = \mu$. Finally, since $\lambda_1 \ge \mu$, from Corollary 6, we have that $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of the circulant nonnegative matrix $\mathbf{C} + ((\lambda_1 - \mu)/n)\mathbf{I}$. ∎

EXAMPLE 5. Let $\lambda_1 = 48$, $\lambda_2 = -7 - i$, $\lambda_3 = -6 - 5i$, $\lambda_4 = -5$, $\lambda_5 = -5$, $\lambda_6 = -6 + 5i$, and $\lambda_7 = -7 + i$. This set satisfies the conditions of the previous corollary and it is the spectrum of the matrix

$$\mathbf{A} = \operatorname{circ}(1.7143,\ 8.8949,\ 7.6147,\ 6.9152,\ 8.9011,\ 8.2973,\ 5.6626).$$

3. ALGORITHM TO CONSTRUCT A SYMMETRIC NONNEGATIVE MATRIX

In this section, we consider a prescribed real spectrum $\{\lambda_i\}_{i=1}^n$.

LEMMA 9. *If*

$$\lambda_j < 0, \qquad \text{for } j = 2, 3, \ldots, n, \tag{12}$$

then there exists a symmetric circulant nonnegative $(2n) \times (2n)$ *matrix* $\mathbf{C}$ *with zeroes on its diagonal such that*

$$\sigma(\mathbf{C}) = \left\{-2\sum_{j=2}^{n}\lambda_j, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, \lambda_n, 0, \lambda_n, \lambda_{n-1}, \ldots, \lambda_3, \lambda_2\right\}.$$

PROOF. Clearly,

$$(\lambda_2, \lambda_3, \ldots, \lambda_{n-1}, \lambda_n, 0, \lambda_n, \lambda_{n-1}, \ldots, \lambda_3, \lambda_2) \in \mathcal{S}^{(2n-1)}.$$

Let

$$\mu = -2\sum_{j=2}^{n}\lambda_j.$$

Then, by Lemma 4, there exists a real circulant $(2n) \times (2n)$ matrix $\mathbf{C}$,

$$\mathbf{C} = \operatorname{circ}\left(c_0, c_1, c_2, \ldots, c_{n-1}, c_n, c_{n+1}, \ldots, c_{2n-1}\right),$$

such that

$$\sigma(\mathbf{C}) = \{\mu, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, \lambda_n, 0, \lambda_n, \lambda_{n-1}, \ldots, \lambda_3, \lambda_2\}.$$

Since $\operatorname{tr}(\mathbf{C}) = \mu + 2\sum_{j=2}^{n}\lambda_j = 0$, it follows that $c_0 = 0$. The real vector

$$\mathbf{c} = [0, c_1, c_2, \ldots, c_{2n-2}, c_{2n-1}]^\top$$

is given by the equation

$$\mathbf{Fc} = \mathbf{q}, \tag{13}$$

where

$$\mathbf{q} = [\mu, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}, \lambda_n, 0, \lambda_n, \lambda_{n-1}, \ldots, \lambda_3, \lambda_2]^{\top} \tag{14}$$

and

$$\begin{aligned} \mathbf{F} &= (f_{kj}), & f_{kj} &= \omega^{(k-1)(j-1)}, \quad 1 \le k, \quad j \le 2n, \\ \omega &= \exp\left(\frac{2\pi i}{2n}\right) = \exp\left(\frac{\pi i}{n}\right), & i^2 &= -1. \end{aligned} \tag{15}$$

From equation (13), we have $\mathbf{Fc} = \mathbf{q}$. After complex conjugation and using the fact that $\mathbf{c}$ is a real vector, we obtain $\overline{\mathbf{F}}\mathbf{c} = \mathbf{q}$. Now, we observe that $\overline{\mathbf{F}} = \mathbf{FP}$, where $\mathbf{P}$ is the permutation matrix $\mathbf{P} = [\mathbf{e}_1, \mathbf{e}_{2n}, \mathbf{e}_{2n-1}, \ldots, \mathbf{e}_3, \mathbf{e}_2]$. Here $\mathbf{e}_k$ denotes the k^{th} column of $\mathbf{I}_{2n}$. Then, $\mathbf{FPc} = \mathbf{q}$. Therefore, $\mathbf{FPc} = \mathbf{Fc}$. Thus, $\mathbf{Pc} = \mathbf{c}$. This implies

$$c_{2n-1} = c_1, \; c_{2n-2} = c_2, \; \ldots, c_{n+1} = c_{n-1}.$$

Hence,

$$\mathbf{C} = \operatorname{circ}\,(0, c_1, c_2, c_3, \ldots, c_{n-2}, c_{n-1}, c_n, c_{n-1}, c_{n-2}, \ldots, c_3, c_2, c_1)$$

is a real symmetric circulant matrix. Finally, we prove that $\mathbf{C}$ is a nonnegative matrix. From Remark 3, for $k = 1, 2, \ldots, n$, we have

$$c_k = \frac{1}{n} \sum_{j=2}^{n} \left(\cos \frac{k(j-1)\pi}{n} - 1\right) \lambda_j.$$

Since $\lambda_j < 0$ for $j = 2, 3, \ldots, n$, we conclude that $c_k > 0$ for $k = 1, 2, \ldots, n$. Hence, $\mathbf{C}$ is a nonnegative matrix. ∎

The matrix $\mathbf{C}$ in Lemma 9 is symmetric with respect to the main diagonal and also it is symmetric to the secondary diagonal. Therefore, it is a symmetric persymmetric matrix. This class of matrices have special properties [14]. In particular, the matrix $\mathbf{C}$ can be partitioned as follows:

$$\mathbf{C} = \begin{bmatrix} \mathbf{U} & \mathbf{VJ} \\ \mathbf{JV} & \mathbf{U} \end{bmatrix},$$

where

$$\mathbf{U} = \begin{bmatrix} 0 & c_1 & c_2 & c_3 & \cdot & c_{n-3} & c_{n-2} & c_{n-1} \\ c_1 & 0 & c_1 & c_2 & \cdot & \cdot & \cdot & c_{n-2} \\ c_2 & c_1 & 0 & c_1 & \cdot & \cdot & \cdot & c_{n-3} \\ c_3 & c_2 & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_3 \\ c_{n-3} & \cdot & \cdot & \cdot & \cdot & \cdot & c_1 & c_2 \\ c_{n-2} & \cdot & \cdot & \cdot & \cdot & c_1 & 0 & c_1 \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdot & c_3 & c_2 & c_1 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & \cdot & c_{n-2} & c_{n-1} & c_n \\ c_2 & c_3 & c_4 & c_5 & \cdot & c_{n-1} & c_n & c_{n-1} \\ c_3 & c_4 & c_5 & \cdot & \cdot & c_n & c_{n-1} & c_{n-2} \\ c_4 & c_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_4 \\ c_{n-2} & c_{n-1} & c_n & \cdot & \cdot & \cdot & c_4 & c_3 \\ c_{n-1} & c_n & c_{n-1} & \cdot & \cdot & c_4 & c_3 & c_2 \\ c_n & c_{n-1} & c_{n-2} & \cdot & c_4 & c_3 & c_2 & c_1 \end{bmatrix},$$

and

$$\mathbf{J} = \begin{bmatrix} 0 & . & . & . & . & . & 0 & 1 \\ . & . & . & . & . & . & 1 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . & . \\ 1 & 0 & . & . & . & . & . & 0 \end{bmatrix}.$$

Matrices $\mathbf{U}$ and $\mathbf{V}$ are both symmetric matrices, each of them of order $n \times n$. We see that $\mathbf{U}$ is a Toeplitz matrix. The entries of $\mathbf{U} = (u_{ij})$ and $\mathbf{V} = (v_{ij})$ are given by

$$u_{ii} = 0, \quad \text{for all } i \qquad \text{and} \qquad u_{ij} = c_{|j-i|}, \quad \text{for } i \neq j, \tag{16}$$

$$\begin{aligned} v_{ij} &= c_{i+j-1}, && \text{if } i + j \leq n + 1, \qquad \text{and} \\ v_{ij} &= c_{2n-i-j+1}, && \text{if } i + j > n + 1. \end{aligned} \tag{17}$$

One can easily check that

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{J} & -\mathbf{J} \end{bmatrix}$$

is an orthogonal matrix and that

$$\mathbf{Q}^\top \mathbf{C} \mathbf{Q} = \begin{bmatrix} \mathbf{U} + \mathbf{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} - \mathbf{V} \end{bmatrix}.$$

Therefore,

$$\sigma(\mathbf{C}) = \sigma(\mathbf{U} + \mathbf{V}) \cup \sigma(\mathbf{U} - \mathbf{V}).$$

Even more, it is easy to prove that

$$\begin{aligned} \sigma(\mathbf{U} + \mathbf{V}) &= \{\mu, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n\}, \\ \sigma(\mathbf{U} - \mathbf{V}) &= \{0, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n\}, \end{aligned}$$

and that the n-dimensional vector

$$\mathbf{1}_n = [1, 1, \dots, 1, 1]^\top$$

is an eigenvector for the eigenvalue μ.

Let

$$\mathbf{B} = \mathbf{U} + \mathbf{V}.$$

Since $\mathbf{U}$ is a symmetric nonnegative matrix and $\mathbf{V}$ is a symmetric positive matrix, it follows that $\mathbf{B}$ is a symmetric positive matrix. The entries of $\mathbf{B} = (b_{ij})$ can be easily obtained from (16) and (17),

$$b_{ij} = \begin{cases} c_{2i-1}, & \text{if } j = i \text{ and } 2i \leq n + 1, \\ c_{2n-2i+1}, & \text{if } j = i \text{ and } 2i > n + 1, \\ c_{|j-i|} + c_{i+j-1}, & \text{if } j \neq i \text{ and } i + j \leq n + 1, \\ c_{|j-i|} + c_{2n-i-j+1}, & \text{if } j \neq i \text{ and } i + j > n + 1. \end{cases} \tag{18}$$

We summarize the above results into the following theorem.

THEOREM 10. *If*

$$\lambda_j < 0, \qquad \text{for } j = 2, 3, \dots, n \qquad \text{and} \qquad \mu = -2 \sum_{j=2}^{n} \lambda_j,$$

then a symmetric positive matrix $\mathbf{B}$ *such that*

$$\sigma(\mathbf{B}) = \{\mu, \lambda_2, \lambda_3, \dots, \lambda_n\}$$

can be constructed by a procedure based on the fast Fourier transform.

COROLLARY 11. *Let* $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ *be a real prescribed spectrum. If*

$$\lambda_j < 0, \qquad \text{for } j = 2, 3, \dots, n \qquad \text{and} \qquad \lambda_1 \geq -2 \sum_{j=2}^{n} \lambda_j, \tag{19}$$

then a symmetric positive $\mathbf{A}$*, having the prescribed spectrum, can be constructed by a procedure based on the fast Fourier transform.*

PROOF. From Theorem 10, a symmetric positive matrix $\mathbf{B}$ such that

$$\sigma(\mathbf{B}) = \{\mu, \lambda_2, \lambda_3, \dots, \lambda_n\}$$

can be constructed by a procedure based on the fast Fourier transform, where $\mu = -2 \sum_{k=2}^{n} \lambda_k$. Let

$$\mathbf{A} = \mathbf{B} + \frac{\lambda_1 - \mu}{n} \mathbf{1}_n \mathbf{1}_n^\top.$$

Since $\lambda_1 \geq \mu$, $\mathbf{A}$ is a symmetric positive matrix. Moreover,

$$\mathbf{A}\mathbf{1}_n = \mathbf{B}\mathbf{1}_n + \frac{\lambda - \mu}{n} \mathbf{1}_n \mathbf{1}_n^\top \mathbf{1}_n = \mu \mathbf{1}_n + \frac{\lambda - \mu}{n} \left(\mathbf{1}_n^\top \mathbf{1}_n\right) \mathbf{1}_n = \lambda \mathbf{1}_n.$$

Let $\mathbf{v}_k$ be an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ_k, $k = 2, 3, \dots, n$. Since $\mathbf{B}$ is a real symmetric matrix, the eigenvectors $\mathbf{v}_k$ can be chosen orthogonal with the eigenvector $\mathbf{1}_n$. Then,

$$\mathbf{A}\mathbf{v}_k = \mathbf{B}\mathbf{v}_k + \frac{\lambda - \mu}{n} \mathbf{1}_n \mathbf{1}_n^\top \mathbf{v}_k = \lambda_k \mathbf{v}_k, \qquad \text{for } k = 2, 3, \dots, n.$$

Therefore, $\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$. This finishes the proof. ∎

ALGORITHM 1. *The algorithm to construct the matrix* $\mathbf{A}$ *in Corollary 11 follows.*

1. *Compute*

$$\mu = -2 \sum_{j=2}^{n} \lambda_j.$$

2. *Compute*

$$\mathbf{c} = \frac{1}{2n} \overline{\mathbf{F}} \mathbf{q},$$

where $\mathbf{F}$ *is the matrix given in (15) and* $\mathbf{q}$ *is the vector in (14), via the fast Fourier transform.*

3. *Construct the matrix* $\mathbf{B} = \mathbf{U} + \mathbf{V} = (b_{ij})$ *given by (18).*
4. *Construct the matrix* $\mathbf{A} = (a_{ij})$, $a_{ij} = b_{ij} + \alpha$ *where*

$$\alpha = \frac{\lambda_1 + 2 \sum\limits_{j=2}^{n} \lambda_j}{n}.$$

Next, we improve the condition given in (19). Let

$$b = \begin{cases} \min \{c_1, c_3, c_5 \dots, c_{n-1}\}, & \text{if } n \text{ is even}, \\ \min, \{c_1, c_3, c_5 \dots, c_n\}, & \text{if } n \text{ is odd}. \end{cases} \tag{20}$$

COROLLARY 12. *Let $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$ be a prescribed real spectrum. Let b be given by (20). If*

$$\lambda_j < 0, \qquad \text{for } j = 2, 3, \dots, n \qquad \text{and} \qquad \lambda_1 \geq -2 \sum_{j=2}^{n} \lambda_j - bn, \tag{21}$$

then a symmetric nonnegative matrix **A**, *having the prescribed spectrum, can be constructed by Algorithm 1.*

PROOF. Let $\mu = -2 \sum_{j=2}^{n} \lambda_j$. Clearly, the smallest entry of the matrix **B** of Theorem 10 occurs on its diagonal and it is the number b defined in (20). Let $\lambda_0 = \mu - bn$. Then, since $\lambda_1 \geq \lambda_0$, we have

$$b_{ij} + \frac{\lambda_1 - \mu}{n} \geq b + \frac{\lambda_0 - \mu}{n} = 0.$$

Therefore, the matrix $\mathbf{A} = \mathbf{B} + ((\lambda_1 - \mu)/n)\,\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$ is nonnegative and it has the prescribed spectrum. ∎

Since $-2 \sum_{j=2}^{n} \lambda_j - bn < -2 \sum_{j=2}^{n} \lambda_j$, the condition in (21) strictly improves the condition in (19).

EXAMPLE 6. Let $\lambda_2 = -1$, $\lambda_3 = -1.5$, $\lambda_4 = -3$, $\lambda_5 = -3.8$. Then, $\mu = -2 \sum_{j=2}^{5} = 18.6$. For these eigenvalues the matrix **B** of Theorem 10 is

$$\begin{bmatrix} 2.4058 & 4.6972 & 3.7357 & 3.8028 & 3.9585 \\ 4.6972 & 1.4442 & 4.7643 & 3.8915 & 3.8028 \\ 3.7357 & 4.7643 & 1.6000 & 4.7643 & 3.7357 \\ 3.8028 & 3.8915 & 4.7643 & 1.4442 & 4.6972 \\ 3.9585 & 3.8028 & 3.7357 & 4.6972 & 2.4058 \end{bmatrix},$$

rounded the entries to four decimal places, and its smallest entry is $b = 1.4442$. From Corollary 11, for $\lambda_1 \geq 18.6$ there exists a symmetric positive matrix with spectrum $\{\lambda_j\}_{j=1}^{5}$. For $\lambda_1 = 19$, the matrix is

$$\begin{bmatrix} 2.4858 & 4.7772 & 3.8157 & 3.8828 & 4.0385 \\ 4.7772 & 1.5242 & 4.8443 & 3.9715 & 3.8828 \\ 3.8157 & 4.8443 & 1.6800 & 4.8443 & 3.8157 \\ 3.8828 & 3.9715 & 4.8443 & 1.5242 & 4.7772 \\ 4.0385 & 3.8828 & 3.8157 & 4.7772 & 2.4858 \end{bmatrix}.$$

Now, from Corollary 12, for $\lambda_1 \geq \mu - 5b = 11.3788$ there exists a symmetric nonnegative matrix which realizes the spectrum. For $\lambda_1 = 11.3788$, the matrix with spectrum $\{11.3788, -1, -1.5, -3, -3.8\}$ is

$$\begin{bmatrix} 0.9615 & 3.2530 & 2.2915 & 2.3585 & 2.5148 \\ 3.2530 & 0 & 3.3201 & 2.4472 & 2.3585 \\ 2.2915 & 3.3201 & 0.1558 & 3.3201 & 2.2915 \\ 2.3585 & 2.4472 & 3.3201 & 0 & 3.2530 \\ 2.5143 & 2.3585 & 2.2915 & 3.2530 & 0.9615 \end{bmatrix}.$$

4. COMPUTATIONAL RESULTS

All the computations were performed on a personal computer equipped with Intel Pentium chips. Because of memory limitations we took $n \leq 450$ in all the experiments. We used MATLAB for Windows, Version 4.2c.1.

Table 1 shows the average CPU time in seconds required to construct the symmetric nonnegative $n \times n$ matrix **A** of Corollary 12, with randomly generated prescribed eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_n$ from the range $[-a, 0)$, $a = 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2}, 1, 10^2, 10^4, 10^6, 10^8$. These eigenvalues were generated using the function *rand* of MATLAB. The other MATLAB functions that we used were

ifft to compute $\mathbf{c} = (1/2n)\overline{\mathbf{F}}\mathbf{q}$,
toeplitz to construct the matrix $\mathbf{U}$ and the matrix $\mathbf{VJ}$,
fliplr to obtain the matrix $\mathbf{V}$,
cputime to see the CPU time in seconds required in each construction of the matrix $\mathbf{A}$.

Table 1. CPU time in seconds.

$\frac{a}{n}$	64	100	128	200	256	300	350	400	450
10^{-8}	0.06	0.11	0.20	0.48	0.82	1.16	1.53	1.97	2.63
10^{-6}	0.06	0.12	0.21	0.49	0.88	1.16	1.55	2.02	2.61
10^{-4}	0.06	0.13	0.21	0.51	0.85	1.18	1.56	2.04	2.65
10^{-2}	0.05	0.13	0.20	0.50	0.87	1.18	1.46	2.01	2.67
1	0.06	0.13	0.20	0.49	0.86	1.20	1.52	2.02	2.68
10^{2}	0.06	0.12	0.21	0.51	0.86	1.18	1.55	2.02	2.69
10^{4}	0.06	0.12	0.19	0.49	0.86	1.18	1.49	2.09	2.62
10^{6}	0.06	0.15	0.22	0.49	0.86	1.19	1.48	2.01	2.63
10^{8}	0.06	0.14	0.22	0.50	0.88	1.20	1.51	2.01	2.69

For each n and for each a, we ran our MATLAB program ten times for a same prescribed spectrum and then we calculated the corresponding average of the CPU times. These averages are shown in Table 1.

These experimental results confirm that the algorithm that we propose to construct a symmetric nonnegative matrix $\mathbf{A}$ with a prescribed spectrum is a very fast procedure. In addition, as we mentioned in Remark 2, the procedure is very stable.

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