Generic degrees are complemented

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Introduction

The notions of forcing and generic set were introduced by Cohen in 1963 to prove the independence of the Axiom of Choice and the Continuum Hypothesis in set theory. Let \( \omega \) be the set of natural numbers, i.e., \( \{0, 1, 2, 3, \ldots \} \). A string is a mapping from an initial segment of \( \omega \) into \( \{0, 1\} \). We identify a set \( A \subseteq \omega \) with its characteristic function.

We now consider a set generic over the arithmetic sets. A set \( A \subseteq \omega \) is called \( n \)-generic if it is Cohen-generic for \( n \)-quantifier arithmetic. This is equivalent to saying that for every \( \Sigma^0_n \)-set of strings \( S \), there is a \( \sigma \in A \) such that \( \sigma \in S \) or \( (\forall \nu \geq \sigma)(\nu \notin S) \). By degree we mean Turing degree (of unsolvability). We call a degree \( n \)-generic if it has an \( n \)-generic representative. For a degree \( a \), let \( D(=a) \) denote the set of degrees which are recursive in \( a \).

Before Cohen’s work, there was a precursor of the notion of forcing in recursion theory. Friedberg showed that for every degree \( b \) above the complete degree \( 0' \), i.e., the degree of a complete r.e. set, there is a degree \( a \) such that \( a' = a \cup 0' = b \). He actually proved this result by using the notion of forcing for \( \Sigma^0_1 \) statements.

In the construction of a real which satisfies some recursion-theoretic property, the notion of forcing makes the situation clear and it has become quite popular, see Lerman [8]. There is another important method in recursion theory, namely the priority method. Friedberg and Muchnik first independently invented the priority method to prove the existence of incomparable recursively enumerable degrees. The finite injury argument used there was improved by the infinite argument by Sacks, see [9]. Further, the \( 0'' \)-priority argument was introduced by

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Lachlan, see for example Soare [11]. The priority argument is now an important method in recursion theory. But these two methods, the forcing and the priority methods, are not independent. They are related to each other.

Now consider an \( n \)-generic set \( G \). \( G \) has several recursion-theoretic properties. For example, the odd and even parts of \( G \), \( G_0 \) and \( G_1 \), are Turing incomparable. This is proved as follows; given a reduction procedure \( \Phi \) and a condition \( \sigma \) on \( G \), there is a condition \( \nu \) on \( G \) extending \( \sigma \) such that \( \varphi(\nu_0) \neq \nu_1 \), where \( \nu_0 \) is the odd part of \( \nu \) and \( \nu_1 \) is the even part of \( \nu \); in other words the computation \( \Phi \) with oracle \( \nu_0 \) is not equal to \( \nu_1 \). This is a \( \sum^0_1 \) outcome. So for all \( n \geq 1 \), if \( G \) is \( n \)-generic then \( G_0 \) and \( G_1 \) are Turing incomparable. Likewise when we consider some more complicated property about \( G \), if it is \( \Sigma^0_n \) then either all \( n \)-generic sets satisfy that property or all \( n \)-generic satisfy its negation.

But when \( G \) has a lower genericity than \( n \)-genericity then we cannot decide easily whether \( G \) satisfies a \( \Sigma^0_n \)-property or not. This is where the priority argument comes in. Assume \( G \) is \( m \)-generic and \( m < n \). To satisfy some requirements, it is suffices to produce a set of witness for each requirement which is \( \Sigma_m \) and dense by using the dynamic technique of the priority method. Then the \( m \)-genericity of \( G \) guarantees that \( G \) satisfies those requirements.

There are several situations similar to the above case. A nonrecursive degree \( a \) is called minimal if for no nonrecursive degree \( b \), \( b < a \). Spector's minimal degree construction below \( 0' \) uses the straightforward notion of forcing; given a condition \( \sigma \), find an extension \( \nu \) such that either \( \nu \) forces \( \Phi(G) \) recursive or Turing equivalent to \( G \). Sacks showed the existence of a minimal degree below \( 0' \) by using the priority argument to handle the requirements which \( 0' \)-oracle cannot decide.

For a degree \( a \), we say \( D(\preceq a) \) is complemented if for every \( b < a \) there is a \( c \) such that \( b \cap c = 0 \) and \( b \cup c = a \). We prove that \( D(\preceq a) \) is complemented for any \( 2 \)-generic degree \( a \). Posner [11] showed \( D(\preceq 0') \) is complemented by nonuniform method. Given \( a < 0' \) we construct a \( b < 0' \) such that \( a \cup b = 0' \) and \( a \cap b = 0 \) by the different methods depending on whether \( a \) satisfies \( a'' = 0'' \) or not. Slaman and Steel [14] showed by the uniform method that \( D(\preceq 0') \) is complemented. We show a stronger result in the sense that for each \( n \geq 2 \), any \( n \)-generic degree \( a \), and any nonrecursive degree \( b < a \), there are \( n \)-generic degree \( c < a \) and \( n \)-generic degree \( d < b \) such that for any nonrecursive degree \( e \preceq c \) and any degree \( f \) such that \( d \leq f < a \), \( e \cup f = a \) and \( e \cap f = 0 \). This gives an affirmative answer to a question in Jockusch [6].

Our notation is standard. Let \( A \oplus B = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in B\} \) for any set \( A \) and \( B \). Lower case Greek letters other than \( \omega \) denote strings. A string is a mapping from \( \omega \) into \( \{0, 1\} \). Fix a recursive enumeration of all strings. For strings \( \sigma \) and \( \nu \), \( \sigma \geq \nu \) denotes that \( \sigma \) extends \( \nu \), and in this case we say that \( \nu \) is a substring of \( \sigma \). Further \( \sigma \) and \( \nu \) are said to be compatible if either extends the other. If \( \sigma \) and \( \nu \) are incomparable we denote this by \( \sigma \upharpoonright \nu \). We identify a set \( A \subseteq \omega \) with its characteristic function. So \( \sigma \preceq A \) means that the characteristic
function of $A$ extends the string $\sigma$ and in this case we say that $\sigma$ is a beginning of $A$ or initial segment of $A$. We write $\sigma \ast \nu$ for the usual concatenation of $\sigma$ and $\nu$. We identify $0, 1$ with the corresponding strings $0, 1$ of length 1. We use $i$ only for $0$ or $1$ and let $[i] = 1 - i$. $\emptyset$ denotes the empty string. For each $n$, $i^{(n)}$ denotes a string $\sigma$ of length $n$ such that $\sigma(m) = i$ for all $m < n$. For a string $\sigma$, $|\sigma|$ denotes the length of $\sigma$, and $\sigma^-$ is the substring of $\sigma$ such that $|\sigma^-| = |\sigma| - 1$. Let $\sigma^*$ be an extension of $\sigma^-$ such that $|\sigma^*| = |\sigma|$ and $\sigma^*(|\sigma^-|) = [\sigma(|\sigma^-|)]$. Further for $\sigma$ such that $|\sigma| \geq 2$, let $\sigma^{**}$ be the extension of $\sigma^-$ such that $|\sigma^{**}| = |\sigma|$, $\sigma^{**}(|\sigma^-|) = \sigma(|\sigma^-|)$ and $\sigma^{**}((|\sigma^-|)) = [\sigma(|\sigma^-|)]$. For two strings $\sigma$ and $\nu$, $\sigma \cap \nu$ is the substring $\lambda$ of $\sigma$ such that $\sigma(m) = \nu(m)$ for all $m < |\lambda|$, and $\sigma(|\lambda|) \neq \nu(|\lambda|)$ or at least one of them is not defined. For a string $\sigma$ and a natural number $n$ such that $n < |\sigma|$, let $\sigma[n]$ be the substring of $\sigma$ of length $n$. Let $\langle \cdot, \cdot \rangle$ be a recursive bijection from all pairs $(\sigma, k)$ of strings and natural numbers to natural numbers such that for all $k$, if $|\nu| < |\sigma|$ then $\langle \nu, k \rangle < \langle \sigma, k \rangle$. Define $(n)_0, (n)_1$ by $n = \langle (n)_0, (n)_1 \rangle$. Let $\langle \cdot, \cdot, \cdot \rangle$ be a recursive bijection from all the triples $\langle \sigma, m, s \rangle$ to natural numbers such that $\langle \sigma, m, s \rangle < \langle \sigma, m, s - 1 \rangle$. In this bijection we can assume the following

(1) If $\langle a, m, s \rangle < \langle a', m', s' \rangle < \langle a, m, s + 1 \rangle$, then $\langle a, m, s + 1 \rangle < (a', m', s' + 1) < \langle a, m, s + 2 \rangle$.

Define $(n)_j$ for $0 \leq j \leq 2$ by $n = \langle (n)_0, (n)_1, (n)_2 \rangle$. Let $n[+1] = \langle (n)_0, (n)_1, (n)_2 + 1 \rangle$ and $n[-1] = \langle (n)_0, (n)_1, (n)_2 - 1 \rangle$. For convenience, if $n < 0$ let $n[k] = 0$ for all integers $k$. We use the same notation $(n)$ for both $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot, \cdot \rangle$. But it will be clear from the context to know which is the case. Let $\Phi_n$ be the $n$th partial (reduction) operator for some fixed recursive enumeration of all such operators. In this enumeration we may assume that for all $m$, there is an even number $e$ and an odd number $o$ such that $\Phi_n = \Phi_e = \Phi_o$. Let $\Phi_n(\sigma)(x) = y$ mean that the $n$th reduction operator with oracle $\sigma$ and input $x < |\sigma|$ yields output $y$ in at most $|\sigma|$ steps and further that $\Phi_n(\sigma)(u)$ is defined for all $u < x$. Of course $B$ is recursive in $A$ iff for some $e$, $\Phi_e(A) = B$. For two reduction operators $\Phi$ and $\Psi$, $\Psi \geq \Phi$ denotes that for every number $n$ and every string $\sigma$, if $\phi(\sigma)(n)$ is defined then $\Psi(\sigma(n)) = \Phi(\sigma)(n)$. Strings $\sigma$ and $\nu$ are called $\Phi$- (or $n$-)split if $\Phi(\sigma)$ (or $\Phi_n(\sigma)$, respectively) and $\Phi(\nu)$ (or $\Phi_n(\nu)$, respectively) are incomparable, and $\sigma$ and $\nu$ are called $\Phi$- (or $n$-)compatible if $\Phi(\sigma)$ (or $\Phi_n(\sigma)$, respectively) and $\Phi(\nu)$ (or $\Phi_n(\nu)$, respectively) are compatible.

Let $S_n$ be the $n$th r.e. set of strings in some fixed recursive enumeration of all r.e. sets of strings. For each $n$, we fix a recursive enumeration of all elements in $S_n$ with the property that any string enumerated at stage $m$ has length less than $m$. Let $S_{n,m}$ be the finite subset of $S_n$ enumerated by the end of stage $m$.

A set $S$ of strings is called dense if every string has an extension in $S$. A set $P$ of strings is called dense along $A$ if there are infinitely many initial segments of $A$ which have extensions in $P$. $\sigma$ is called $\Psi$-good if for every $\lambda \geq \Psi(\sigma)$ there is a
\( \tau \geq \sigma \) such that \( \Psi(\tau) \geq \lambda \). \( \sigma \) is called \( \Psi \)-good above \( \nu \) if for any \( \tau \geq \nu \) there is \( \sigma' \geq \sigma \) such that \( \Psi(\sigma') \geq \tau \).

Results

Theorem. For each \( n \geq 2 \), any \( n \)-generic degree \( a \), and any nonrecursive degree \( b < a \), there are \( n \)-generic degree \( c < a \) and \( n \)-generic degree \( d < b \) such that \( e \cup f = a \) and \( e \cap f = 0 \) for any nonrecursive degree \( e \leq c \) and any degree \( f \) such that \( d \leq f < a \).

Given 1-generic \( A \), by Theorem 4.1 in Jockusch [6], let \( \Theta \) be such that \( \deg(\Theta(A)) \leq b \) and \( \Theta(A) \) is \( n \)-generic.

We construct a reduction procedure \( \Psi \) so that \( \Psi(A) \) is \( n \)-generic and if \( \Phi(\Psi(A)) \) is total and nonrecursive then \( \Theta(A) \oplus \Phi(\Psi(A)) \geq \tau A \). (So for each \( m_0, m_1 \) if \( \Phi_m(\Psi(A)) = \Phi_m(\Theta(A)) \) then it is recursive.) For this, given a reduction procedure \( \Phi \) and condition \( \alpha \), we make a \( \beta > \alpha \) such that there is no string \( \gamma \) such that \( \gamma \mid \beta, \Theta(\beta) = \Theta(\gamma) \) and \( \Phi(\Psi(\beta)) = \Phi(\Psi(\gamma)) \). So from \( \Theta(\beta) \) and \( \Phi(\Psi(\beta)) \) we can compute \( \beta \). For each different \( \Phi_0, \Phi_1 \) we take different such \( \beta_0, \beta_1 \) with \( \Theta(\beta_0) \mid \Theta(\beta_1) \). The density of such \( \beta \) and the \( n \)-genericity of \( A \) guarantee the complementation of \( \Psi(A) \) with \( \Theta(A) \).

We also maintain the property that \( \alpha \) has infinitely many extensions \( \gamma \) such that \( \Theta(\gamma) \mid \Theta(\beta) \) and for which there are no axioms in \( \Psi \) other than those already apply to \( \alpha \). Then we can use these \( \gamma \) to make a \( \Psi \)-good, i.e., whenever we need an axiom of the form \( \Psi(\delta) = \mu \) such that \( \mu \geq \Psi(\alpha) \), we use such \( \gamma \) for \( \delta \). This strategy is compatible with the previous one because \( \Theta(\gamma) \mid \Theta(\beta) \).

We begin with a definition and a lemma which play an important role throughout the proof of the theorem.

Definition. For reduction operators \( \Phi \) and \( \Psi \), \( \Phi \) is called totally \( \Psi \)-splittable from \( \alpha \) if

1. for every string \( \delta \geq \Psi(\alpha) \) there are two strings \( \delta_0, \delta_1 \) such that \( \delta_i > \delta \) for each \( i \) and \( \delta_0 \) and \( \delta_1 \) are \( \Phi \)-split, and

2. for every string \( \beta \geq \alpha \) and \( x \in \omega \), there exists \( \gamma \geq \beta \) such that \( \Phi(\Psi(\gamma))(x) \) is defined.

If \( \alpha = \emptyset \) then we just say \( \Phi \) is totally \( \Psi \)-splittable. If \( \Psi \) is identity then we say \( \Phi \) is totally splittable from \( \alpha \).

Lemma 1. (1) If \( A \) and \( B \) are 2-generic and \( \Psi(A) \neq B \) then \( \alpha \) is almost \( \Psi \)-good for any \( \alpha < A \), i.e., \( \alpha \) is \( \Psi \)-good above \( \sigma \) for some \( \sigma < B \) (so \( \alpha \) is \( \Psi \)-good above \( \Theta(\alpha_0) \) for some \( \alpha_0 < A \)).
(2) Suppose that $A$ and $B$ are 2-generic, $\emptyset \leq_T C \leq_T B \leq_T A$, $\Psi(A) = B$, and $\Phi'(B) = C$ for some reduction operators $\Psi$ and $\Phi'$. Then for some $\Phi$, $\Phi(B) = \Phi'(B)$, and $\Phi$ is totally $\Psi$-splittable.

(3) For all $n \geq 1$, assume $\Psi$ is a partial recursive operator and there is a dense $\Sigma^0_2$ (or dense along $A$) set $P$ of almost $\Psi$-good string. If $A$ is $n$-generic than $\Psi(A)$ is total and $n$-generic.

**Proof.** (1) Given $\alpha < A$ let $R$ be the set of strings $r$ such that $\forall \beta \geq \alpha (\Psi(\beta) \neq r)$. Then $R$ is $\Pi^0_1$. So there is a $\beta < B$ such that $\alpha \in R$ or no extension of $\sigma$ is in $R$. As $\alpha < A$ and $\Psi(A) = B$, no extension of $\sigma$ is in $R$. This means $\alpha$ is $\Psi$-good above $\sigma$.

(2) Let $S$ be the set of strings $\alpha$ such that either (i) $(\exists x)(\forall \beta \geq \alpha)(\Phi'(\Psi(\beta))(x)$ is undefined), or (ii) for any $\beta_0, \beta_1 \geq \alpha$, $\Phi'(\Psi(\beta_0))$ and $\Phi'(\Psi(\beta_1))$ are compatible. Then $S$ is a $\Sigma^0_2$ set of strings, and $A$ extends no string in $S$ because $\Phi'(\Psi(A))$ is total and nonrecursive. Since $A$ is 2-generic we may choose a string $\beta < A$ such that no extension of $\beta$ is in $S$. By (1), let $\gamma < A$ be such that $\beta$ is $\Psi$-good above $\Psi(\gamma)$. Then given any $\nu \geq \Psi(\gamma)$ there is a $\beta' \geq \beta$ such that $\Psi(\beta') \geq \nu$. As no extension of $\beta$ is in $S$, there are $\beta_0, \beta_1 \geq \beta'$ such that $\Psi(\beta_0) \geq \nu$, and $\Psi(\beta_0)$ and $\Psi(\beta_1)$ are $\Phi'$-split. Hence $\Phi'$ it totally $\Psi$-splittable from $\gamma$. Let $\sigma = \Psi(\gamma)$. Define $\Phi$ by $\Phi(\nu) = \Phi'((\sigma \cap \nu) \ast \nu$ for $\nu \mid \sigma$, and $\Phi(\nu) = \Phi'(\nu)$ for each $\nu$ which is compatible with $\sigma$. Then clearly $\Phi(\Psi(A)) = \Phi'(\Psi(A))$ and $\Phi$ is totally $\Psi$-splittable.

(3) To show that $\Psi(A)$ is total, let for each $n$, $S_n = \{ \sigma \mid \Psi(\sigma)(n) \text{ is defined} \}$. Then $S_n$ is a dense recursive set of string. (In fact for any $\sigma$ let $\nu$ be such that $\nu \in P$ and $\nu \geq \sigma$, and let $\nu' \geq \nu$ be such that $|\Psi(\nu')| > n$.) Then by the 1-genericity of $A$, for each $n$ there is a $\sigma < A$ such that $\sigma \in S_n$. So $\Psi(A)$ is total. Next let $S$ be an arbitrary $\Sigma^0_n$ set of strings. Let $T$ be the set of strings $\nu$ such that $\Psi(\nu) \geq \lambda$ for some $\lambda \in S$. Then $T$ is a $\Sigma^0_n$ set of strings. As $A$ is $n$-generic, there is a $\nu < A$ such that $\nu \in T$ or not extension of $\nu$ is in $T$. If there is a $\nu < A$ such that $\nu \in T$ then $\Psi(A)$ extends some string $\lambda$ in $S$. If there is a $\nu \leq A$ such that no extension of $\nu$ is in $T$ then let $\delta \in P$ be a string such that $\nu < \delta \in A$. (Such a $\delta$ exists because $P$ is a dense (or dense along $A$) $\Sigma^0_n$ set.) Since $\delta$ is almost $\Psi$-good, let $\lambda$ be such that $\delta \leq \lambda < A$ and $\delta$ is $\Psi$-good above $\Psi(\lambda)$. As for any $\xi \geq \Psi(\lambda)$ there is a $\mu \geq \delta$ such that $\Psi(\mu) \geq \xi$, it follows that no extension of $\Psi(\lambda)$ is in $S$. Since $S$ was an arbitrary $\Sigma^0_n$ set of strings it follows that $\Psi(A)$ is $n$-generic. □

By Jockusch [6], there is an $n$-generic $\theta < \beta$. Take any such $\theta$ and let $D$ be an $n$-generic representative of the degree $\theta$. Let $\Theta$ be a reduction procedure such that $\Theta(A) = D$. By letting $\Psi$ be identity in Lemma 1(2), we may assume $\Theta$ is totally splittable. We construct a reduction procedure $\Psi_n$ at stage $n$ such that $\Psi_n \geq \Psi_{n-1}$ and $\lim_n \Psi_n = \Psi$ satisfies that $\Psi(A)$ is a set of the desired degree $\tau$. Before we construct $\Psi$, we briefly give the motivation of the construction. Within
the motivation, we use letters $\alpha$, $\beta$, $\gamma$ to refer to conditions on $A$, and $\sigma$, $\tau$, $\delta$ to refer to conditions on $\Theta(A)$ or $\Psi(A)$.

To prove the theorem, it suffices to show that

(1) $\Psi(A)$ is total and $n$-generic.

(2) For each $m$, if $\Phi_m(\Psi(A))$ is nonrecursive then $\Theta(A) \oplus \Psi_m(\Psi(A)) =_r A$.

(3) For each $m_0, m_1$, if $\Phi_{m_0}(\Psi(A)) = \Phi_{m_1}(\Theta(A))$ then it is recursive.

Clearly (3) is derived from (2). To satisfy (1), we execute a construction so that every initial segment of $A$ is $\Psi$-good. Then by Lemma 1(3) $\Psi$ preserves the $n$-genericity. If $\Phi_{m_0}(\Psi(A))$ is nonrecursive then by Lemma 1(2), we may assume $\Phi_{m_0}$ is totally $\Psi$-splitable.

During the course of the construction $\alpha$ may be $m$-satisfied, and put some string $\beta \geq \alpha$ into $T_m$. Here $m$ is the index of the reduction procedure $\Phi_m$ and $\beta$ is used to compute $A$ from $\Theta(A)$ and $\Phi_m(\Psi(A))$. Let $T_m \cap n$ be the set of strings enumerated into $T_m$ by the end of stage $n$. So $T_m = \bigcup_{n=0}^m A_m$. Clearly $T_m$ is r.e. Further we claim that

(4) If $\alpha \in T_m \cap n$, $\Theta(A) > \Theta(\alpha)$, and $\Phi_n(\Psi(A)) > \Phi_n(\Psi_n(\alpha))$ then $A <_n \alpha$.

(5) $T_m$ is dense along $A$ if $\Phi_m$ is totally $\Psi$-splitable.

(4) shows the procedure to compute $A$ from $\Theta(A)$ and $\Phi_m(\Psi(A))$. By (5) $A$ extends infinitely many elements in $T_m$. As $T_m$ is r.e and dense along $A$, the 1-genericity of $A$ satisfies the condition (2): let $\alpha_0 = \emptyset$. Given $\alpha, \beta < A$ find a string $\alpha_{s+1} > \alpha_s$ and $n$ such that $\alpha_{s+1} \in T_m \cap n$, $\Theta(A) > \Theta(\alpha_{s+1})$, and $\Phi_n(\Psi(A)) > \Phi_n(\Psi_n(\alpha_{s+1}))$. Such an $\alpha_{s+1}$ exists by the 1-genericity of $A$. Then by the condition (4), $A > \alpha_{s+1}$. Let $E$ be such a reduction procedure defined as above, i.e. $E(\Theta(A), \Phi_m(\Psi(A))) = A$.

The construction is organized in terms of strategies. During the course of executing a strategy we may take one of the following actions.

(a) Enumerate axioms into $\Psi$.
(b) Prohibit such enumeration. We restrain $\Psi$ away from $\sigma$ above $\alpha$ by prohibiting the enumeration of any axioms $\Psi(\beta) = \tau$ such that $\tau \geq \sigma$ and $\beta \geq \alpha$.

Note that restraint above $\alpha$ implies restraint above any extension of $\alpha$. The crux of the problem is, for each $\alpha$, to understand what axioms enumerated so far imply about the values of $\Psi$ on $A$ when $A$ extends $\alpha$. In other words, given the axioms so far, what is the forcing relation for $\Psi$? The analysis can be made very manageable by the following.

(I). For each stage and each condition $\alpha$ maintain the property that $\Theta(\alpha)$ has infinitely many extensions $\sigma$ such that there are no axioms in $\Psi$ with input $\beta$ such that $\Theta(\beta) \geq \sigma$ other than those already apply to $\beta$.

This property implies that at each state $s$ the axioms enumerated into $\Psi$ do no more than the following: If it does not follow that $\Psi$ is restrained from $\nu$ above $\alpha$ for any $\nu \mid \sigma$, then

$$(\alpha \models \Psi(A) \text{ extends } \sigma) \leftrightarrow \Psi(\alpha) \text{ extends } \sigma.$$
(II). To satisfy (4) impose: if \( \alpha \) is enumerated in \( T_m \) then for any \( \beta \upharpoonright \alpha \),

(II-i) if \( \Theta(\beta) = \Theta(\alpha) \) then \( \Phi_m(\Psi(\beta)) \upharpoonright \Phi_m(\Psi(\alpha)) \), and

(II-ii) if \( \Theta(\beta) < \Theta(\alpha) \) then restrain \( \Phi_m(\Psi) \) away from \( \Phi_m(\Psi(\alpha)) \) above \( \beta \).

This shows that there is no \( \beta \upharpoonright \alpha \) whose values on \( \Psi_m(\Psi) \) and \( \Theta \) are the same as the values \( \Phi_m(\Psi(\alpha)) \) and \( \Theta(\alpha) \), respectively. To do this, when \( \alpha \) is enumerated in \( T_m \), for each \( \beta \upharpoonright \alpha \) such that \( \Theta(\beta) \) is compatible with \( \Theta(\alpha) \), any extension \( \gamma \) of \( \beta \) such that \( \Theta(\gamma) = \Theta(\alpha) \) satisfies that its image on \( \Phi_m(\Psi) \) is incompatible with the new value \( \Phi_m(\Psi(\alpha)) \).

Assuming that the construction respect the conditions (4) and (I), for any stage of the construction and any \( \alpha \), we are free to extend \( \Psi \) and \( \Xi \) so that there is an extension \( \beta \) of \( \alpha \) such that \( \Xi(\Theta(\beta) \oplus \Phi_m(\Psi(\beta))) = \beta \). We can enumerate relational axioms and respect (I) by choosing \( \beta \) and \( \Theta(\beta) \) to be sufficiently long length. Combining (I) and (II) and the possibility of global restraint we obtain the following analysis of the forcing relation.

\[ \alpha \Vdash \Psi(\lambda) \] does not extend \( \sigma \Leftrightarrow \) one of

(a) \( \Psi(\alpha) \) is incompatible with \( \sigma \),

(b) \( \Psi \) is restrained away from \( \sigma \) above \( \alpha \).

(III). The third strategy is used to make \( T_m \) dense along \( A \). Given \( \alpha < A \) and stage \( k \), let \( S = \{\lambda_1, \ldots, \lambda_l\} \) be the set of all possible values of \( \Psi \) at stage \( k \) with input \( \gamma \) whose value on \( \Theta \) is \( \Theta(\alpha) \). Assume \( \lambda_i \) and \( \lambda_j \) are incompatible if \( i \neq j \). We keep an increasing sequence of strings \( \nu_n, \Theta(\alpha) \) at stage \( n \geq k \) such that there is an \( \beta_n \upharpoonright \alpha \) such that \( \Theta(\beta_n) = \nu_n \) and there is no axiom in \( \Psi \) with input \( \gamma \) whose value on \( \Theta \) extends \( \nu_n \) other than those already apply to \( \gamma \). During this action, we also look for the extensions \( \lambda_{p,i}, \) of each element \( \lambda_p \) of \( S \) such that \( \Phi_m(\lambda_{p,i}) \upharpoonright \Phi_m(\lambda_{q,j}) \) for each \( \langle p, i \rangle \neq \langle q, j \rangle, 0 \leq p, q \leq l \), and \( 0 \leq i, j \leq 1 \). If \( \Phi_m \) is totally \( \Psi \)-splittable then we can find such \( \lambda_{p,i} \) at some stage \( n \) and take \( \alpha' \upharpoonright \alpha \) such that \( \Theta(\alpha') = \nu_n \). Let \( \Phi_{n-1}(\alpha') = \lambda_p \). Define \( \Psi_n(\alpha') = \lambda_{p,1} \), and for \( \beta \upharpoonright \alpha' \) such that \( \Theta(\beta) = \nu_n \), let \( \Psi_{n-1}(\beta) = \lambda_q \) \( \Rightarrow \Psi_n(\beta) = \lambda_{q,0} \) for all \( q \) such that \( 0 \leq q \leq l \). (So for \( \beta \upharpoonright \alpha' \) such that \( \Theta(\beta) < \nu_n \) and \( \Psi_n(\alpha') = \lambda_q, \Psi \) is restrained away from \( \lambda_{p,1} \) above \( \beta \).) This sequence \( \beta_n \) can be regraded as a function \( p \) from \( \langle \alpha, m, s \rangle \) to strings \( \gamma > \alpha \) such that \( \Theta(\gamma) > \Theta(\alpha) \) and

(III-i) \( p(\langle \alpha, m, s \rangle) > p(\langle \alpha, m, s - 1 \rangle) \);

(III-ii) \( p(\langle \alpha, m, s \rangle) \geq p(\langle \alpha', m', s' \rangle) \geq p(\langle \alpha, m, s - 1 \rangle) \) implies

\[ \langle \alpha', m', s' \rangle = \langle \alpha, m, s \rangle \text{ or } \langle \alpha', m', s' \rangle = \langle \alpha, m, s - 1 \rangle. \]

As we described above, given \( \langle \alpha, m \rangle \), we define \( S \) as the set of all possible values of \( \Psi \) with input \( \gamma \) whose value on \( \Theta \) is \( \Theta(\alpha) \). And we look for extensions \( \lambda_{p,i} \) of \( \lambda_p \) for each \( \lambda_p \) \( \in S \). Assume \( \langle \alpha', m' \rangle \neq \langle \alpha, m \rangle \) and \( p(\langle \alpha', m', n' \rangle) \leq p(\langle \alpha, m, n \rangle) \). We take an action as above at stage \( n \) for the requirement \( \langle \alpha', m' \rangle \), and put new axioms of the form \( \Psi_n(\beta) = \lambda_{q,i} \) for each \( \beta \) such that \( \Theta(\beta) = \Theta(p(\langle \alpha', m', n' \rangle)) \). Then \( \Theta(\beta) \leq \Theta(p(\langle \alpha, m, n \rangle)) \), and \( \Psi_n(\beta) \) may be different from all value in \( S \). Then we have to find other extensions for these new
values $\Psi_\alpha(\beta)$ so that we are able to take an action for the requirement $\langle \alpha, m \rangle$. This makes the satisfaction of the requirement $\langle \alpha, m \rangle$ impossible. So (III-ii) is necessary to make each different strategy with different index independent.

(IV). The final strategy is to make a $\Psi$-good condition extending $\alpha < A$. Given a string $\tau \geq \Psi(\alpha)$, we have to have an extension $\alpha' > \alpha$ such that $\Psi(\alpha') > \tau$. For this purpose let $m$ be such that

(IV-i) $\Phi_m$ is totally splittable above any $\delta$ such that $\delta = \tau$ or $\delta \mid \Psi(\alpha)$;

(IV-ii) for any $\sigma \geq \Psi(\alpha)$, $|\Phi_m(\sigma)| > 0$ implies $\sigma \geq \tau$ (i.e., there is no $\nu \geq \Psi(\alpha)$ such that $\Phi_m(\nu) \mid \Phi_m(\tau) \text{ and } \nu \mid \tau$).

By using the same notation as (III), we assume that for $\lambda_i, \lambda_j \in S$, $\lambda_i$ and $\lambda_j$ are incompatible if $i \neq j$. So there is no $\lambda \in S$ such that $\Psi(\alpha) \leq \lambda \mid \tau$ as $\Psi(\alpha) \in S$. Then we can find such $\lambda_p, i's$ at some stage $n$ by (IV-i). For at least one $p$, $\lambda_p,0$ and $\lambda_{p,1}$ extend $\tau$ by (IV-ii) and because $S$ contains $\Psi(\alpha)$. So for some $\alpha' \geq \alpha$ such that $\Theta(\alpha') = \nu_n$, the new axiom $\Psi_n(\alpha')$ extends $\tau$ by the definition of $m$. So $\alpha$ is $\Psi$-good.

To satisfy (III-i) and (III-ii) above together we define $p$ by induction on $e = \langle \alpha, m, s \rangle$ as follows. Let $\pi-1(0) = (0)_0$. Assume by induction hypothesis that

(1) $p(e^0)$ is defined for all $e^0 < e$, and

(2) for any $e^0 < e$ and any $e^1$ such that $e^0 < e^1 < e^0[+1]$, $\pi_{e-1}(e^1)$ is defined, and $\pi_{e-1}(e^1)$ and $p(e^0)$ are $\Theta$-split. (So if $p(e^1) \geq \pi_{e-1}(e^1)$ is defined then $p(e^1)$ and $p(e^0)$ are $\Theta$-split. We use $\pi_{e-1}(e^1)$ to define $p(e^1)$ later.) Let $p(e)$ be the least extension $\pi$ of $\pi_{e-1}(e)$ such that $\Theta(\pi(e)) = \Theta(p(e'))$ for all $e^1 < e$,

(3-i) for any $e^1$ such that $e < e^1 \leq e - 1[+1]$, $\pi_{e-1}(e^1)$ has an extension $\pi_e(e^1)$ such that $\pi$ and $\pi_e(e^1)$ are $\Theta$-split, and

(3-ii) for each $e^1$ such that $e - 1[+1] < e^1 < e[+1]$, $(e^1)_0$ has an extension $\pi_e(e^1)$ such that $\pi$ and $\pi_e(e^1)$ are $\Theta$-split.

(4) $Let \pi_e(e[+1]) = p(e)$.

This completes the definition of $p(e)$. Note for any $\gamma < p(e), \Theta(\gamma) < \Theta(p(e))$. As $\Theta$ is totally splittable, there is an extension $\pi$ of $\pi_{e-1}(e)$ which satisfies (3-i) and (3-ii). Now we check the induction hypothesis (1) and (2). (1) is clear. For (2), by the induction hypothesis, it is enough to check the case $e^0 = e$. But it is clear by (3-i) and (3-ii).

Lemma 2. Let $e = \langle \alpha, m, s \rangle$ and $e' = \langle \alpha', m', s', \rangle$.

(1) $p(e) > \alpha$

(2) $p(\langle \alpha, m, s \rangle) > p(\langle \alpha, m, s - 1 \rangle)$ if $s > 0$.

(3) $p(\langle \alpha, m, s \rangle) \neq p(\langle \alpha', m', s', \rangle)$ iff $\langle \alpha, m, s \rangle > \langle \alpha', m', s', \rangle$.

(4) $\langle \alpha, m, s \rangle \neq \langle \alpha', m', s' \rangle$ and $p(\langle \alpha, m, s \rangle) > p(\langle \alpha', m', s' \rangle)$ iff $p(\langle \alpha', m', s' \rangle) < p(\langle \alpha, m, 0 \rangle)$.

(5) $p(\langle \alpha, m, s \rangle) \geq p(\langle \alpha', m', s' \rangle)$ implies $\langle \alpha, m, s \rangle > \langle \alpha, m, s - 1 \rangle$.

(6) $p(\langle \alpha, m, s \rangle) \leq p(\langle \alpha', m', s' \rangle)$ implies $\langle \alpha, m, s \rangle < \langle \alpha, m, s - 1 \rangle$.
Proof. (1) (2) \( p(e) \geq \pi_{e-1}(e) \geq \pi_{e-2}(e) \) by the construction of \( p(e) \). (2) is clear by (4) of the construction of \( p(e) \). If \( e = 0 \) then \( p(e) = \pi_{e-1}(0) = (e)_0 \) by the construction. Assume \( e > 0 \). If \( s = 0 \) then for some \( e^1 \), \( e^1 - 1 + 1 < e < e^1 + 1 \). So by (3-ii) of the construction of \( p(e) \), \( \pi_e(e) \geq (e)_0 \). By using (2), (1) holds.

(3) \((\Rightarrow)\) Assume \( p((\alpha, m, s)) \neq p((\alpha', m', s')) \). Further assume for a contradiction that \( \langle \alpha, m, s \rangle \leq \langle \alpha', m', s' \rangle \). Clearly \( \langle \alpha, m, s \rangle \neq \langle \alpha', m', s' \rangle \). Let \( t \geq s \) be such that \( \langle \alpha, m, t \rangle < \langle \alpha', m', s' \rangle < \langle \alpha, m, t + 1 \rangle \). Let \( e^2 = \langle \alpha, m, t + 1 \rangle \). Then by (3) of the definition of \( p(\langle \alpha', m', s' \rangle), p(\langle \alpha', m', s' \rangle) \) and \( \pi_e(e^2) \) are \( \Theta \)-split. As \( p(e^2) \geq \pi_e(e^2) \), \( p(\langle \alpha', m', s' \rangle) \) and \( p(e^2) \) are \( \Theta \)-split. By (2), \( p(\langle \alpha', m', s' \rangle) \) is a contradiction.

\((\Leftarrow)\) Assume \( \langle \alpha, m, s \rangle > \langle \alpha', m', s' \rangle \). Let \( t' \geq s' \) be such that \( \langle \alpha', m', t' \rangle < \langle \alpha, m, s \rangle < \langle \alpha', m', t' + 1 \rangle \). Let \( e^3 = \langle \alpha', m', t' + 1 \rangle \). Then by (3) of the definition of \( p(\langle \alpha', m', s' \rangle), p(\langle \alpha, m, s \rangle) \) and \( \pi_e(e^3) \) are \( \Theta \)-split. As \( p(e^3) \geq \pi_e(e^3) \), \( p(\langle \alpha', m', s' \rangle) \) and \( p(\langle \alpha', m', t' + 1 \rangle) \) are \( \Theta \)-split. By (2), \( p(\langle \alpha', m', s' \rangle) \) is a contradiction.

(4) \((\Leftarrow)\) Clear by (2) and the construction of \( p \).

\((\Rightarrow)\) Assume for a contradiction that \( p(\langle \alpha', m', s' \rangle) \neq p(\langle \alpha, m, 0 \rangle) \). If \( p(e') \parallel p(\langle \alpha, m, 0 \rangle) \) then by (2) \( p(e') \parallel p(e) \). If \( p(\langle \alpha, m, 0 \rangle) = p(e') \) then \( \langle \alpha, m, 0 \rangle = e' \). So assume \( p(\langle \alpha, m, 0 \rangle) = p(e') \). Then \( e' = \langle \alpha, m, t + 1 \rangle \). By (3) of the construction of \( p(e') \), \( \pi_e(e') \) and \( \pi_e(\langle \alpha, m, t + 1 \rangle) \) are \( \Theta \)-split. So \( p(e') \) and \( p(\langle \alpha, m, t + 1 \rangle) \) are \( \Theta \)-split. By (2), \( p(\langle \alpha', m', s' \rangle) \geq p(\langle \alpha', m', t' + 1 \rangle) \). So (3) holds.

(5) By (4), if \( \langle \alpha, m \rangle \neq \langle \alpha', m' \rangle \) then \( p(\langle \alpha', m', s' \rangle) < p(\langle \alpha, m, 0 \rangle) \). As \( p(\langle \alpha', m', s' \rangle) \geq p(\langle \alpha, m, s - 1 \rangle) \) if \( s > 0 \), by (2) \( p(\langle \alpha', m', s' \rangle) > p(\langle \alpha, m, 0 \rangle) \). This is a contradiction. So \( \langle \alpha, m \rangle = \langle \alpha', m' \rangle \). Hence, by (2), \( s' = s \), or \( s' = s - 1 \) if \( s > 0 \).

We now give the construction.

Construction

Stage 0. \( S(0) = \{\emptyset\} \). \( \Psi_0 = \emptyset \) and \( f(0) = 0 \).

Stage \( \eta \). Let \( f(e) \) be the greatest number \( e' \leq e \) such that \( p(e') \leq p(e) \) and \( S(e') \) is defined at the end of stage \( \eta \).

I. For \( e = \langle \alpha, m, n \rangle \) such that \( \langle \alpha, m \rangle \leq n \), we say \( e \) needs attention at stage \( \eta \) if:

(I-i) \( \alpha \) is not \( m \)-satisfied by the end of stage \( n - 1 \).

(I-ii) For each \( e_0 < e, (e_0) \leq n \) or \( p(e_0) \parallel p(e) \).

(I-iii) For each \( \gamma < p(e) \) if \( \Theta(\gamma) = \Theta(p(f(e))) \) then (I) or (II) in the construction is applied for \( \gamma \) and \( p(f(e)) \) at some earlier stage \( \eta \).

(I-iv) Let \( S(f(e)) = \{\lambda_k \mid 0 < k < l \} \). Then there are \( \lambda_{k, l} > \lambda_k \) for each \( k, 0 \leq k \leq l \), such that \( |\lambda_{k, l}| \leq n \), and \( \lambda_{k_0, i_0} \) and \( \lambda_{k_1, i_1} \) are \( m \)-split for each \( 0 \leq k_0, k_1 \leq l, 0 \leq i_0, i_1 \leq 1, \) and \( \langle k_0, i_0 \rangle \neq \langle k_1, i_1 \rangle \).
If \( e \) needs attention at stage \( n \) then let \( e_n = (\alpha_n, m_n, n) \) be the least such number. Let \( \lambda_{k,0}, \lambda_{k,1} \) be the unique pair such that \( \Psi_{n-1}(p(e_n)) = \lambda_{k,0} \cap \lambda_{k,1} \). Let \( \Psi_n(p(e_n)) = \lambda_{k,1} \). We say \( \alpha_n \) is \( m_n \)-satisfied at stage \( n \). For all \( k \) and \( \lambda \), we say \( \lambda \) is \( k \)-satisfied at stage \( n \) if \( \lambda \) is \( k \)-satisfied at some stage \( n' \leq n \). Let \( S(e_n) = \{ \lambda_{k,i} \mid 0 \leq k \leq l \text{ and } 0 \leq i \leq 1 \} \).

II. For each \( e = (\alpha, m, s) \) and \( \beta \) such that \( |\beta| \leq n \) and \( \Theta(\beta) = \Theta(p(e)) \) if

(II-i) \( \alpha \) is \( m \)-satisfied at stage \( n \),

(II-ii) \( \beta \mid p(e) \),

(II-iii) there is a unique pair \( \lambda_{k,0}, \lambda_{k,1} \) in \( S(e) \) such that \( \Psi_{n-1}(\beta) = \lambda_{k,0} \cap \lambda_{k,1} \).

Let \( \Psi_n(\beta) = \lambda_{k,0} \).

III. For any \( \lambda \) let

\[
\Psi_n(\lambda) = \bigcup \{ \Psi_m(\lambda') \mid (\exists m \leq n)(\lambda' \leq \lambda \text{ and } \Psi_m(\lambda') \text{ is explicitly defined at stage } m) \},
\]

\[
\Psi(\lambda) = \bigcup \{ \Psi_m(\lambda') \mid \exists m (\lambda' \leq \lambda \text{ and } \Psi_n(\lambda') \text{ is explicitly defined at stage } m) \}.
\]

Let \( T_{m,n} \) be the set of strings \( p(e) \) such that \( e = (\alpha, m, n') \), \( n' \leq n \), and \( \alpha \) is \( m \)-satisfied by the end of stage \( n \). Let \( T_m = \bigcup_{n=0}^n T_{m,n} \).

This completes the construction.

**Lemma 3.** Let \( e = (\alpha, m, n) \).

(1) \( S(e) \) is defined at stage \( n \) for the first time iff \( \alpha \) is \( m \)-satisfied at stage \( n \) for the first time and \( (\alpha, m) \leq n \). (So by \( S(e) \downarrow \) we mean \( S(e) \) is defined at stage \( e \).)

(2) If \( \alpha \) is \( m \)-satisfied at stage \( n \) then for all \( e' < e \), \( (e')_2 < n \) or \( p(e') \mid p(e) \).

(3) If \( \tau, \mu \in S(e) \) and \( \tau \neq \mu \) then \( \tau \) and \( \mu \) are \( m \)-split.

(4) If \( S(e) \downarrow \) then for all \( e' \) such that \( p(e') \leq p(e) \) and \( f(e) < e' < e, S(e') \downarrow \).

(5) If \( \Theta(\beta) = \Theta(\beta') = \Theta(p(e)) \) and \( \beta \) and \( \beta' \) are compatible then \( \Psi_s(\beta) = \Psi_s(\beta') \) for all \( s \).

(6) If \( S(e) \downarrow \), \( S(e') \downarrow \), and \( p(e) \neq p(e') \) (so \( e \geq e' \) by Lemma 2(3)) then every element of \( S(e') \) is extensible to some element in \( S(e) \).

(7) If \( \Theta(\beta) = \Theta(p(e)) \) and \( \Psi_s(\beta) \neq \Psi_{s+1}(\beta) \) then for some \( \beta' \leq \beta \) and \( e' \leq e \), (I) or (II) in the construction is applied at stage \( s + 1 \), i.e., \( \Psi_{s+1}(\beta') \) is explicitly defined at stage \( s + 1 \). Further, if \( \beta' = p(e') \) then \( s + 1 = (e')_2 \) and (I) in the construction is applied. If \( \beta' \mid p(e') \) then \( (e')_2 \leq s + 1 \) and (II) in the construction is applied. Also \( \Theta(\beta') = \Theta(p(e')) \leq \Theta(\beta) = \Theta(p(e)) \).

(8) If \( \Psi_{s+1}(\beta) \) is explicitly defined at stage \( s + 1 \) then:

(8-i) \( \Psi_s(\beta) < \Psi_{s+1}(\beta) = \Psi(\beta) \).

(8-ii) For \( \beta' < \beta \) and \( t \geq s \), if \( \Psi_{s+1}(\beta') \) is explicitly defined at stage \( t + 1 \), then \( t = s \) and \( \Theta(\beta) = \Theta(\beta') \) (so \( \Psi_{s+1}(\beta') = \Psi_{s+1}(\beta) \) by (5)).

(8-iii) \( \Psi_{s+1}(\beta') = \Psi_s(\beta') \) for all \( \beta' > \beta \), and for all \( \beta' > \beta \) and \( s' \geq s + 1 \) such that \( \Theta(\beta') = \Theta(\beta) \), \( \Psi_{s'}(\beta') = \Psi_{s+1}(\beta) \).

(9) Assume \( \alpha \) is \( m \)-satisfied at stage \( n \). Then \( f(\langle \alpha, m, s \rangle) = f(\langle \alpha, m, n \rangle) \) for all \( s = n \).

(10) Assume that \( e_0 < e \) implies \( (e_0)_2 \leq n \) or \( p(e_0) \mid p(e) \), and that \( \alpha \) is not \( m \)-satisfied at stage \( s = n \). Then

(10-i) \( f(\langle \alpha, m, s \rangle) = f(\langle \alpha, m, n \rangle) \).
(10-ii) If \( \beta \geq \alpha, \; \Theta(\beta) = \Theta(p(\langle \alpha, m, n \rangle)) \) and \( |\beta| \leq s \) then \( \Psi_s(\beta) \in S(f(\langle \alpha, m, n \rangle)) = S(f(\langle \alpha, m, s \rangle)) \).

(11) Assume \( \alpha \) is \( m \)-satisfied at stage \( n \) for the first time. Let \( \beta \) be such that \( \Theta(\beta) = \Theta(p(e)) \), and \( \Theta(\beta_0) < \Theta(\beta) \) for any \( \beta_0 < \beta \). Let \( u = \max\{n, |\beta|\} \) if \( \beta \neq p(e) \). If \( \beta = p(e) \), let \( u = n \). Then

(i) There is a stage \( s' \) such that \( u > s' > n \) and there is unique pair \( \lambda_{k,0}, \lambda_{k,1} \) in \( S(e) \) such that \( \Psi_s(\beta) \leq \lambda_{k,0} \cap \lambda_{k,1} \) for each \( t \) such that \( s' < t < \mu \).

(ii) (I) or (II) in the construction is applied for \( \beta \) and \( e \) at stage \( u \), and \( \Psi_s(\beta) \in S(e) \) for all \( t_1 \geq u \)

**Proof.** (1) (2) (3) Clear by the construction.

(4) Clear by (1), (2) and the definition of \( f \).

(5) We prove this by induction on \( 2^{|\beta|} |\beta| \). Given \( \beta \) and \( \beta' \) which satisfy the assumption, assume by the induction hypothesis that (5) holds for all \( \gamma, \gamma' \) such that \( 2^{\gamma} |\gamma| < 2^{\gamma'} |\gamma'| \). We prove (5) for \( 2^{\gamma} |\gamma| \) by induction on stage \( s \). If \( \Psi_{s+1}(\beta) \) is defined by (I) in the construction at stage \( s + 1 \), then \( \beta = p(e_n) \). By the definition of \( p(e_n) \), \( \beta' \geq \beta \). So by (III) in the construction \( \Psi_{s+1}(\beta') - \Psi_{s+1}(\beta) \). If \( \Psi_{s+1}(\beta) \) is defined by (II) in the construction at stage \( s + 1 \), then \( \beta \mid p(e) \). By the definition of \( p(e) \), \( \beta' \neq p(e) \). So \( \beta' \mid p(e) \). If \( \beta' \geq \beta \) then \( \Psi_{s+1}(\beta') \) is also explicitly defined by (II) in the construction. By induction hypothesis, \( \Psi_s(\beta') = \Psi_s(\beta) \). So \( \Psi_{s+1}(\beta') = \Psi_{s+1}(\beta) \) by the construction. If \( \beta' > \beta \) then \( \Psi_s(\beta') = \Psi_s(\beta) \) by (III) in the construction. Finally, if \( \Psi_{s+1}(\beta) \) is defined by (III) in the construction at stage \( s + 1 \), then there is a \( \beta_0 < \beta \) such that \( \Psi_{s+1}(\beta_0) \) is explicitly defined at stage \( s + 1 \) and \( \Psi_{s+1}(\beta) = \Psi_{s+1}(\beta_0) \). Let \( \beta_1 \leq \beta' \) be such that \( \Theta(\beta_1) = \Theta(\beta_0) \). Then by the induction hypothesis, \( \Psi_{s+1}(\beta_0) = \Psi_{s+1}(\beta) \). So \( \Psi_{s+1}(\beta') = \Psi_{s+1}(\beta) \) by (III) in the construction.

(6) By induction on \( e \). By (I-iii) in the construction, every element of \( S(f(e)) \) is extended by some element in \( S(e) \) at stage \( n \). By the definition of \( f(e) \), \( e > f(e) \equiv e' \). By the induction hypothesis, every element of \( S(e') \) is extended by some element of \( S(f(e)) \). So (6) holds.

(7) By the construction if \( \Psi_s(\beta) \neq \Psi_{s+1}(\beta) \) then for some \( \beta' \leq \beta \) and \( e' \) such that \( \Theta(\beta') = \Theta(p(e')) \), \( (e')_n \) is \( (e')_s \)-satisfied at stage \( (e')_2 \), and (I) or (II) in the construction is applied at stage \( s + 1 \). Further if \( \beta' = p(e') \) then \( s + 1 = (e')_2 \) and (I) is applied. Otherwise, \( \max\{(e')_2, |\beta|\} = s + 1 \) and (II) is applied. As \( \Theta(\beta) = \Theta(p(e)) \equiv \Theta(\beta') = \Theta(p(e')) \), \( e' \equiv e' \) by Lemma 2(3).

(8) Assume \( \Psi_{s+1}(\beta) \) is explicitly defined at stage \( s + 1 \). Then for some \( e \) and \( k \) such that \( \lambda_k \in S(f(e)) \) and \( \Theta(\beta) = \Theta(p(e)) \), \( \Psi_s(\beta) \leq \lambda_{k,0} \cap \lambda_{k,1} \) by the construction at stage \( s + 1 \). By (3), \( \lambda_{k,0} \mid \lambda_{k,1} \). As \( \Psi_{s+1}(\beta) \leq \lambda_{k,i} \) for some \( i \), \( \Psi_s(\beta) < \Psi_{s+1}(\beta) \). Note by (7) that if (I) in the construction is applied at stage \( s + 1 \) then \( \beta = p(e) \) and \( s + 1 = (e')_2 \). If (II) in the construction is applied then \( \max\{(e)_2, |\beta|\} = s + 1 \). Also \( \alpha \) is \( m \)-satisfied at stage \( n \leq s \). Assume that \( \Psi_s(\beta') \neq \Psi_{s+1}(\beta') \) for some \( s' \geq s \) and \( \beta' \leq \beta \). Then by (7), for some \( e' \leq e \) and \( \beta' \leq \beta' \), \( \Theta(\beta') = \Theta(p(e')) \equiv \Theta(\beta) = \Theta(p(e)) \), and (I) or (II) in the construction is applied at stage \( s' + 1 > s \).
If \( \Theta(\beta') < \Theta(\beta) \) then \( |\beta'| < |\beta| \). As \( p(e') < p(e) \), \( e' < e \) by Lemma 2(3), so \( (e')_2 < (e)_2 \leq s + 1 \) by (2). If \( \beta = p(e) \) then by (I-iii) in the construction at stage \( s + 1 \), (I) or (II) in the construction is applied for \( \beta'' \) and \( e' \) at some earlier stage \( \leq s \). This is a contradiction. If \( \beta \neq p(e) \) then (II) in the construction is applied at stage \( s + 1 \). If \( \beta'' = p(e') \) then (I) in the construction is applied at stage \( (e')_2 \). But \( (e')_2 < s + 1 \) by (2). This is a contradiction. If \( \beta'' \neq p(e') \) then (II) in the construction is applied at stage \( s' + 1 \). Note by (II) in the construction at stage \( s + 1 \), \( |\beta'| \leq s + 1 \). As \( |\beta'| < |\beta| \leq s + 1 \), and \( (e')_2 \), \( (e')_1 \leq (e')_2 \leq s + 1 \). So (II) in the construction is applied at some earlier stage \( < s + 1 \). This is a contradiction. Hence \( \Theta(\beta'') = \Theta(\beta) \) and \( e' = e \). And \( \Psi_{s+1}(\beta') = \Psi_{s+1}(\beta) \) by (5). Hence we proved \( \Psi_s(\beta) = \Psi(\beta) \) in (8-i) and (8-ii).

For (8-iii) if \( \Psi_{s+1}(\beta') \neq \Psi_{s+1}(\beta) \) for some \( \beta' > \beta \), then by the construction there are \( \beta'' < \beta' \) and \( t \leq s + 1 \) such that \( \beta < \beta'' < \beta' \), \( \Psi_t(\beta') \) is explicitly defined at stage \( t \), and \( \Psi_t(\beta') = \Psi_{s+1}(\beta') \). Then by (8-ii) \( \Theta(\beta'') = \Theta(\beta) \), so by (5) \( \Psi_{s+1}(\beta') = \Psi_{s+1}(\beta) \), a contradiction. The other half of (8-iii) is clear by (5).

(9) First note by (1), \( S(e) \). For all \( e' \) such that \( p(\langle \alpha, m, n \rangle) < p(e') < p(\langle \alpha, m, s \rangle) \), by Lemma 2(5), \( e' \) is of the form \( \langle \alpha, m, s' \rangle \) for some \( s' \) such that \( n < s' \leq s \). By (I-ii) in the construction at stage \( s' \), \( e' \) does not need attention at stage \( (e')_2 \). So \( S(e') \). Hence by the definition of \( f \), \( f(\langle \alpha, m, s \rangle) = f(\langle \alpha, m, n \rangle) \).

(10) (11) We prove these by induction on \( e \). For (10) assume that \( e_0 < e \) implies \( (e_0)_2 \leq n \) or \( p(e_0) \leq p(e) \) and that \( \alpha \) is not \( m \)-satisfied at stage \( s \). Then \( S(\langle \alpha, m, s' \rangle) \) for all \( s' < s \) by (1). Further by (1) and Lemma 2(4)(5), for each \( e_0 < e \) such that \( p(e_0) \leq p(e) \), \( (e_0)_1 \)-satisfied at stage \( (e_0)_2 \) iff \( (e_0)_2 \)-satisfied at stage \( s \).

Hence (10-i) holds. Next we show that \( \beta \geq \alpha \), \( \Theta(\beta) = p(\langle \alpha, m, n \rangle) \) and \( |\beta| \leq s \) imply \( \Psi_s(\beta) \in S(f(e)) \). Let \( \beta' < \beta \) be such that \( \Theta(\beta') = \Theta(p(f(e))) \). As \( e > f(e) \), \( s \geq \max((f(e))_2 + 1, |\beta'|) \). By (5) and the induction hypothesis of (11-ii), \( \Psi_s(\beta') \in S(f(e)) \). We show \( \Psi_s(\beta') = \Psi_s(\beta) \). If not then for some \( \beta'' \) and \( e_0 \) such that \( \beta \geq \beta'' > \beta' \), \( \Theta(\beta'') = \Theta(p(e_0)) \), \( f(\langle \alpha, m, s \rangle) < e_0 \leq f(\langle \alpha, m, n \rangle) \) (for (5) and (I) or (II) in the construction is applied at some stage \( s' \leq s \). So \( (e_0)_2 \)-satisfied at stage \( (e_0)_2 \). And by the assumption of (10), \( (e_0)_2 < n \). So \( f(e) \geq e_0 \), a contradiction. This completes the proof of (10).

Next we prove (11). First note \( e > f(e) \). Let \( \beta' < \beta \) be such that \( \Theta(\beta') = p(f(e)) \) and for all \( \beta'' < \beta' \), \( \Theta(\beta'') < \Theta(\beta') \). Note that \( \Theta(\gamma) < \Theta(p(e)) \) for all \( \gamma < p(e) \). If \( \beta' = p(f(e)) \) then (I) in the construction is applied for \( \beta' \) at stage \( (f(e))_2 \). So let \( s' = (f(e))_2 \). If \( \beta' \neq p(f(e)) \) then let \( s' = \max((f(e))_2, |\beta'|) \). As \( f(e) < e \) and \( \beta' < \beta \), by (2), \( s' < u \). By the induction hypothesis of (11-ii), \( \Psi_s(\beta') \in S(f(e)) \). Note by (3) that \( \lambda_{k_0}, \lambda_{k_1} \in S(f(e)) \) and \( \lambda_{k_0} \neq \lambda_{k_1} \) imply \( \lambda_{k_0} \neq \lambda_{k_1} \). So by (I-iii) in the construction at stage \( (e)_2 \), any element \( \lambda_k \) of \( S(f(e)) \) is extended by some unique pair of elements \( \lambda_{k_0}, \lambda_{k_1} \) in \( S(e) \). In (8-iii) let \( s-1 = s' \). Then \( \Psi_s(\beta') = \Psi_s(\beta') \). So to prove (11-i), it is enough to show \( \Psi_t(\beta) = \Psi_t(\beta) \) for all \( t \) such that \( s' < t < u \).
Assume for a contradiction that there are such $t$, $\Psi_t(\beta) \neq \Psi_t(\beta)$. Let $t$ be the least such. Then $\Psi_\alpha(\beta) \neq \Psi_{\alpha+1}(\beta)$. By (7) for some $\beta^1 \leq \beta$ and $e^1 \leq e$, $\Theta(\beta^1) = \Theta(p(e^1))$ and (I) or (II) in the construction is applied at stage $t$. Note that for each $t$ such that $s' < t < u$, $e$ does not need attention at stage $t$ by the definition of $u$. So by (8-iii), $\Psi_t(\beta) = \Psi_t(\beta^1)$. If $\beta^1 = \beta$ then $\Theta(p(e)) = \Theta(\beta) = \Theta(\beta^1) = \Theta(p(e^1))$. So $e = e^1$. Then by the definition of $u$, if (I) is applied in the construction at stage $t$, then $\beta^1 = p(e)$ and $t = (e)_2$. If (II) in the construction is applied then $\beta^1 \neq p(e)$ and $t = \max\{(e)^2, |\beta^1|\}$. So $t = u$. This is a contradiction. Hence $\beta^1 < \beta$. So $\Theta(p(e^1)) = \Theta(\beta^1) < \Theta(\beta) = \Theta(p(e))$. If $\Theta(p(e^1)) = \Theta(p(f(e)))$, then $p(e^1) = p(f(e))$, and so $e^1 = f(e)$ by the definition of $p(e^1)$. So $\Theta(\beta^1) = \Theta(p(e^1)) = \Theta(\beta^1)$. By (5), $\Psi_{(\alpha+1)}(\beta^1) = \Psi_{\alpha+1}(p(f(e))) = \Psi_{\alpha+1}(\beta^1)$ and $\Psi_{\alpha+1}(\beta^1) = \Psi_{\alpha+1}(\beta^1) = \Psi_{\alpha+1}(\beta^1) = \Psi_{\alpha+1}(\beta^1) = \Psi_{\alpha+1}(\beta^1)$. This is a contradiction. So $\Theta(p(e)) > \Theta(\beta^1) > \Theta(p(f(e)))$. But this is a contradiction to the definition of $f(e)$. This proves (11-i). Then at stage $u$, if $\beta = p(e)$ then, for $e$ and $\beta$, (I) in the construction is applied at stage $u$. If $\beta \neq p(e)$ then, for $e$, (I) in the construction is applied and (II) in the construction is applied for $\beta$ at stage $u$. And $\Psi_\alpha(\beta) \in S(e)$ by the construction at stage $u$. By (8-i) $\Psi_\alpha(\beta) = \Psi_\alpha(\beta) \in S(e)$. This completes the proof of (11-ii). \[\square\]

Lemma 4. (1) $\Psi_{n+1}$ is consistent, i.e., if $\alpha \leq \beta$ then $\Psi_\alpha(\alpha) \leq \Psi_\alpha(\beta)$.

(2) Let $e = \langle \alpha, m, s \rangle$. If $\alpha$ is $m$-satisfied at stage $s$ for the first time, $\Theta(\beta) = \Theta(p(e))$ and $\Phi_m(\Psi(\beta)) = \Phi_m(\Psi(p(e)))$, then $\beta \equiv p(e)$.

Proof. (1) By induction on $n$. At stage $n+1$ assume for a contradiction that $\alpha \leq \beta$ and $\Psi_{n+1}(\alpha) \neq \Psi_{n+1}(\beta)$. Then let $\beta^1 \leq \beta$ and $n^1 \leq n+1$ be such that $\Psi_{n^1}(\beta^1) = \Psi_{n^1+1}(\alpha)$ and for some $e$ such that $\Theta(\beta^1) = \Theta(p(e))$, $\Psi_{n^1}(\beta^1)$ is explicitly defined at stage $n^1$. By Lemma 3(8-iii), $\beta^1 \neq \alpha$. So by (5) we can assume $\Theta(\alpha) < \Theta(\beta^1)$. Hence $\alpha < \beta^1$. By Lemma 3(8-iv), $\Psi_{n^1}(\beta^1) = \Psi_{n^1}(\beta^1)$. As $\Psi_{n^1}(\alpha) \neq \Psi_{n^1}(\beta^1)$ and $\Psi_{n^1}(\alpha) \neq \Psi_{n^1}(\beta^1)$, then $\Psi_{n^1}(\alpha) = \Psi_{n^1}(\beta^1)$ and $\Psi_{n^1}(\alpha) = \Psi_{n^1}(\beta^1)$ is explicitly defined at stage $n + 1$. As $\Theta(\alpha) < \Theta(\beta^1)$, $e' \leq e$ by Lemma 2(3). By Lemma 3(8-ii), $n + 1 < n^1$, a contradiction. Hence $\Psi_{n+1}$ is consistent.

(2) We first show that at stage $t < s$, for no $\beta$ such that $\Theta(\beta) \equiv \Theta(p(e))$, $\Psi_t(\beta)$ is explicitly defined. Assume for a contradiction that it is the case. Then (I) or (II) in the construction is applied at stage $t$ for $\beta$ and some $\beta'$ such that $\Theta(\beta') = \Theta(p(e'))$ and $\Theta(\beta) \equiv \Theta(p(e))$. So $\Theta(\alpha') = \Theta(p(e'))$ and $\Theta(\beta') = \Theta(p(e'))$ if (I) or (II) in the construction is applied. But this is a contradiction because $e = e'$. At stage $t = s$ if $\Psi_t(\beta)$ is explicitly defined for some $\beta$ such that $\beta \equiv p(e)$ and $\Theta(\beta) = \Theta(p(e))$ then (II) in the construction is applied at stage $t$. So $\Psi_t(\beta) = \lambda_{k_t,0}$ for some $k_t$, and by (1) in the construction, $\Psi_t(p(e)) = \lambda_{k_t,1}$ for some $k_t$, where...
Lemma 5. $\Psi(A)$ is total and $n$-generic.

Proof. Given $\alpha < A$, we show that $\alpha$ is $\Psi$-good. Given a string $\tau \supseteq \Psi(\alpha)$, let $m$ be such that

1. $\Phi_m$ is totally splittable above $\delta$ for any $\delta$ such that $\delta = \tau$ or $\delta \mid \Psi(\alpha)$.

2. For any $\sigma \supseteq \Psi(\alpha)$, $|\Phi_m(\sigma)| > 0$ implies $\sigma \supseteq \tau$ (i.e., there is no $\nu \supseteq \Psi(\alpha)$ such that $\Phi_m(\nu) \mid \Phi_m(\tau)$ and $\nu \mid \tau$).

Clearly such an $m$ exists. It suffices to show that for some $\alpha' \supseteq \alpha$, $\alpha' \in T_m$, since then, as $\Psi(\alpha') \supseteq \Psi(\alpha)$ and $|\Phi_m(\Psi(\alpha'))| > 0$, $\Psi(\alpha') \supseteq \tau$ by (2) in the definition of $m$. But then it suffices to show that $\alpha$ is $m$-satisfied at some stage $n$ since $p(\langle \alpha, m, n \rangle) \supseteq \alpha$. Assume for a contradiction that $\alpha$ is never $m$-satisfied. Let $n \supseteq m, m$ be such that for any $e' \equiv \langle \alpha, m, n \rangle$, $n \supseteq e'$, or $p(e') \supseteq p(\langle \alpha, m, n \rangle)$. Such an $n$ exists, for example, let $n = \max\{\langle e' \rangle \mid \langle \alpha, m, 0 \rangle > e'\} \cup \{\langle \alpha, m, 0 \rangle\}$. Then for any $e_0$ such that $(\langle \alpha, m, n \rangle) > (e_0)$, $e_0$ is of the form $(\langle \alpha, m, n' \rangle)$ for some $n' \leq n$ or $e_0 \equiv \langle \alpha, m, 0 \rangle$ by Lemma 2(4)(5). Let $e = \langle \alpha, m, n \rangle$. As $\alpha$ is never $m$-satisfied, $p(f(\langle \alpha, m, n \rangle) = p(\langle \alpha, m, 0 \rangle)$ by Lemma 2(5).

We first show that there is no $\sigma \in S(f(\langle \alpha, m, n \rangle))$ such that $\Psi(\alpha) \supseteq \sigma \mid \tau$. Let $(*)$ be this statement. To show this it suffices to show that $\Psi(\alpha) \in S(f(\langle \alpha, m, n \rangle))$ by Lemma 3(3). First let $\alpha' \equiv \alpha$ and $e'$ be such that $\Theta(\alpha') = p(e')$, $\Psi_k(\alpha') = \Psi(\alpha)$ and $\Psi_k(\alpha')$ is explicitly defined at some stage $k$. By Lemma 3(8-i), for all $k' \geq k$, $\Psi_k(\alpha') = \Psi_k(\alpha')$. And $(e')_0$ is $(e')_1$-satisfied at stage $(e')_2$ by Lemma 3(1). As $p(e') = \Theta(\alpha') \equiv \Theta(\alpha) \equiv \Theta(p(\langle \alpha, m, n \rangle))$ by Lemma 2(1), $e' \leq f(\langle \alpha, m, n \rangle)$ by Lemma 3(1) and the definition of $f$. As $\Psi_k(\alpha') = \Psi(\alpha)$, $\Psi_k(\alpha') = \Psi(\alpha)$, for no string $\alpha''$ such that $\alpha' < \alpha'' \equiv \alpha$, $\Psi_k(\alpha'')$ is explicitly defined at any stage $k' > k$ by Lemma 3(8-i). Also if for some string $\alpha''$ such that $\alpha' < \alpha'' \equiv \alpha$, $\Psi_k(\alpha'')$ is explicitly defined at some stage $k' = k$, then by Lemma 3(8-i) $\Theta(\alpha'') = \Theta(\alpha')$ and $\Psi_k(\alpha'') = \Psi_k(\alpha')$. Note $(f(e))_0$ is $(f(e))_1$-satisfied at stage $(f(e))_2$. As $\Theta(p(f(e))) \equiv \Theta(\alpha')$, there is $\alpha''$ such that $\alpha' \equiv \alpha'' \equiv \alpha$ and $\Theta(\alpha'') = \Theta(p(f(e)))$. So if $e' < f(e)$ then $\Theta(\alpha'') \equiv \Theta(\alpha')$ and $\alpha' < \alpha''$. So $\Psi(\alpha'')$ is not explicitly defined at any stage, a contradiction. So $e' = f(e)$. Hence $\Psi_k(\alpha') = \Psi(\alpha) \in S(f(e))$ by Lemma 3(5)(11-ii). We proved $(*)$. By $(*)$ let $n_1 \geq n$ be the least stage such that for each $\lambda_k \in S(f(\langle \alpha_1, m, n \rangle)) = \{\lambda_0, \ldots, \lambda_1\}$, there is $\lambda_{k, i} \equiv \lambda_k$ such that $|\lambda_{k, i}| \leq n_1$ and $\lambda_{k, i} \equiv \lambda_k$ and $\lambda_{k, i}$ are $\Phi_m$-split for each $\langle k_0, i \rangle \neq \langle k_1, j \rangle$, $0 \leq k_0, k_1 \leq 0$, and $0 \leq i, j \leq 1$. Such $\lambda_{k, i}$'s exist because of (1) in the definition of $m$. By Lemma 3(10-i), $f(\langle \alpha_1, m, n_1 \rangle) - f(\langle \alpha_1, m, n \rangle)$. So $\alpha_1$ is $m$-satisfied at stage $n_1$. This is a contradiction. \qed

Lemma 6. If $\Phi_m(\Psi(A))$ is nonrecursive then there is an $m'$ such that $T_{m'}$ is dense along $A$ and $\Phi_m(\Psi(A)) = \Phi_m(\Psi(A))$. 
**Proof.** If $\Phi_m(\Psi(A))$ is nonrecursive then by Lemma 1(2) let $m'$ be such that $\Phi_m(\Psi(A)) = \Phi_m(\Psi(A))$ and $\Phi_{m'}$ is totally $\psi$-splittable. Assume for a contradiction that there is an $\alpha_i < A$ such that no extension $\alpha$ of $\alpha_i$ is in $T_{m'}$. Let $n \equiv (\alpha_i, m')$ be such that for any $e' \equiv (\alpha_i, m', n)$, $n \equiv (e')$ or $p(e')$ or $p((\alpha_i, m', n))$. Let $n_1 \geq n$ be the least stage such that for each $\lambda_k \in S(f((\alpha_i, m', n))) = \{\lambda_0, \ldots, \lambda_i\}$, there is a $\lambda_k, i \equiv \lambda_k$ such that $|\lambda_k, i| \leq n_1$ and $\lambda_k, i$ and $\lambda_k, j$ are $\Phi_{m'}$-split for each $(k_0, i) \neq (k_1, j)$, $0 \leq k_0$, $k_1 \leq l$, and $0 \leq l$, $j \leq 1$. Such $\lambda_k, i$'s exist because $\Phi_m$ is totally $\psi$-splittable. By Lemma 3(10-i) $f((\alpha_i, m', n_1 - 1)) = f((\alpha_i, m', n))$. So $\alpha_1$ is $m'$-satisfied at stage $n_1$. Hence $\alpha_i \equiv p((\alpha_i, m', n)) \in T_{m'}$, which is a contradiction. So $T_{m'}$ is dense along $A$. □

**Lemma 7.** For any sets $B$ and $C$, if $\Theta(A) \equiv B < T A$ and $\emptyset < T C \equiv \Psi(A)$ then $B \oplus C \equiv T A$, and if $D \equiv T B$ and $D \equiv T C$ then $D$ is recursive.

**Proof.** Let $m$ be such that $\Phi_m(\Psi(A)) = C$ and $\Phi_m$ is totally $\psi$-splittable. Then $T_m$ is dense along $A$ by Lemma 6. As $A$ is 1-generic, $A$ extends infinitely many elements in $T_m$. So $\Theta(A) \oplus C \equiv T A$ by Lemma 4(2). So $B \oplus C \equiv T A$. Next assume for a contradiction that there is a nonrecursive set $D$ such that $D \equiv T B$ and $D \equiv T C$. As $\emptyset < T D \equiv T \Psi(A)$, $\Theta(A) \oplus D \equiv T A$. As $B \equiv T \Theta(A)$ and $D \equiv T B$, it follows that $B \equiv T A$, which is a contradiction. □

**References**