ELSEVIER

Discrete Mathematics 223 (2000) 1–12

DISCRETE MATHEMATICS

On Alspach's conjecture with two even cycle lengths

Peter Adams, Darryn E. Bryant, A. Khodkar *

Centre for Discrete Mathematics and Computing, Department of Mathematics, The University of Queensland, Queensland 4072, Australia

Received 13 May 1998; revised 23 December 1999; accepted 24 January 2000

Abstract

For m, n even and $n > m$, the obvious necessary conditions for the existence of a decomposition of the complete graph K_v when v is odd (or the complete graph with a 1-factor removed $K_v \setminus F$ when v is even) into r m-cycles and s n-cycles are shown to be sufficient if and only if they are sufficient for $v < 7n$. This result is used to settle all remaining cases with $m, n \leqslant 10$.

Keywords: Cycle systems; Cycle decompositions; m-cycle systems

1. Introduction

The obvious necessary conditions for the existence of a decomposition of the complete graph K_v into cycles $C_1, C_2, \ldots, C_t$, of lengths $m_1, m_2, \ldots, m_t$, whose edges partition the edge set of K_v are:

- $3 \leqslant m_i \leqslant v$ for $i = 1, 2, \ldots, t$;
- v is odd; and
- $m_1 + m_2 + \cdots + m_t = v(v-1)/2$.

When v is even, one may instead consider partitioning the edge set of the complete graph with a 1-factor removed $K_v \setminus F$ into cycles. In this case, the necessary conditions are:

- $3 \leqslant m_i \leqslant v$ for $i = 1, 2, \ldots, t$;
- v is even; and
- $m_1 + m_2 + \cdots + m_t = v(v-2)/2$.

☆ Research supported by the Australian Research Council.

* Corresponding author. Tel.: +07-33653277; fax: 07-33651477.
E-mail address: ak@maths.uq.edu.au (A. Khodkar).

0012-365X/00/$ - see front matter
PII: S0012-365X(00)00051-0

The proposition that the above necessary conditions are also sufficient is widely known as *Alspach's conjecture* (see [3]). Although the conditions have been shown to be sufficient in many cases, the question is still very much an open problem.

A great deal of work has been done on the special case where all of the cycles in the decomposition have the same length m. Such a decomposition is usually called an m-cycle system. For a survey of m-cycle systems see Lindner and Rodger [13]. It is proved in recent breakthrough results of Alspach and Gavlas [4] and Šajna [16] that Alspach's conjecture is true when the cycle lengths are all the same. It had already been shown that when $m_1, m_2, \dots, m_t \in \{m\}$, Alspach's conjecture is true for all v if and only if it is true for all $v \leqslant 3m$ (see [6,17] for the case m is even and [10] for the case m is odd). In this paper we obtain a similar result for the case where two even cycle lengths are permitted. We show (see Theorem 14) that for the case $m_1, m_2, \dots, m_t \in \{m, n\}$, m and n even and $m < n$, Alspach's conjecture is true for all v if and only if it is true for all $v < 7n$.

In the case of more than one permissible cycle length, Alspach's conjecture is known to be true for all $v \leqslant 10$ (see [15]) and for all v when

1. $m_i \in \{3, 4, 6\}$ for $i = 1, 2, \dots, t$ (see [9]);
2. $m_i \in \{3, 5\}$ for $i = 1, 2, \dots, t$ (see [2]); and when
3. $m_i \in \{4, 5\}$ for $i = 1, 2, \dots, t$ (see [7]).

By using our results and settling Alspach's conjecture for the necessary small values of v, we completely settle Alspach's conjecture for the five remaining cases ($\{m, n\} = \{4, 8\}, \{4, 10\}, \{6, 8\}, \{6, 10\}, \{8, 10\}$) where $m_1, m_2, \dots, m_t \in \{m, n\}$, m and n are even and $m, n \leqslant 10$ (see Section 4).

We need the following notation.

- If G_2 is a subgraph of G_1 we denote by $G_1 \backslash G_2$ the graph with vertex set $V(G_1 \backslash G_2) = V(G_1)$ and edge set $E(G_1 \backslash G_2) = E(G_1) \backslash E(G_2)$.
- When a graph G is the union of edge disjoint graphs $G_1, G_2, \dots, G_t$ we will write $G = G_1 + G_2 + \cdots + G_t$. The use of the $+$ symbol is restricted to the case in which the graphs $G_1, G_2, \dots, G_t$ are edge disjoint.
- An m-cycle on $\{a_1, a_2, a_3, \dots, a_m\}$ with edges $a_1a_2, a_2a_3, \dots, a_{m-1}a_m, a_ma_1$ will be denoted by $(a_1, a_2, a_3, \dots, a_m)$.
- An (m^r, n^s)-cycle system of a graph G is a set consisting of r m-cycles and s n-cycles whose edges partition $E(G)$.
- For any non-negative integer t, define $S_{m,n}(t) = \{(r, s) : mr + ns = t \text{ and } r, s \geqslant 0\}$ and for a given graph G, define $\mathrm{Type}_{m,n}(G) = \{(r, s) : \text{there exists an } (m^r, n^s)\text{-cycle system of } G\}$. Where it is clear what m and n are, we will omit the subscripts and just write $S(t)$ and $\mathrm{Type}(G)$.
- For $E \subseteq \mathbb{Z} \times \mathbb{Z}$ and $(r, s) \in \mathbb{Z} \times \mathbb{Z}$, define $(r, s) + E = \{(r + x, s + y) : (x, y) \in E\}$.
- For non-negative integers u and v with $v \geqslant u$, define $G_u^v = K_v \backslash K_u$ if u and v are odd, and $G_u^v = (K_v \backslash F_1) \backslash (K_u \backslash F_2)$ if u and v are even, where F_1 is a 1-factor of K_v and F_2 is a 1-factor of K_u with $F_2 \subseteq F_1$. If $u = 0$ or 1 then we use G^v.

2. Preliminary results

In this section we present results on the existence of various graphs which we need in later sections. We begin with the following result of Sotteau and then some results on m-cycle systems of K_v and $K_v \backslash F$.

Theorem 1 (Sotteau [17]). *Let m be even, and $x,\ y \geqslant 1$. There exists an m-cycle system of $K_{x,y}$ if and only if*

(1) $x,\ y \geqslant m/2$,
(2) $xy \equiv 0 \pmod m$, *and*
(3) $x \equiv y \equiv 0 \pmod 2$.

Theorem 2 (Alspach and Varma [5]). *Let $m \in \{4,6,8,10\}$ and let v be odd. Then there exists an m-cycle system of G^v for all $v \geqslant m$ such that $v(v-1) \equiv 0 \pmod{2m}$.*

Theorem 3 (Jackson [11], Kotzig [12], Rodger [14]). *Let $m \geqslant 4$ be even. There exists an m-cycle system of G^v for all $v \equiv 0$ or $1 \pmod{2m}$.*

We also need some results on m-cycle systems of G_u^v. The following lemma gives the necessary conditions for the existence of an m-cycle system of G_u^v when u and v are odd (see [8]).

Lemma 4. *If u and v are odd with $v > u \geqslant 1$ and there exists an m-cycle system of G_u^v then*

(1) $(v-m)(v-1) \geqslant u(u-1)$,
(2) $(v-u)(v+u-1) \equiv 0 \pmod{2m}$, *and*
(3) $v \geqslant (m+1)u/(m-1)+1$ *if m is odd.*

For the case u and v are even, it is straightforward to prove Lemma 5 in a similar manner.

Lemma 5. *If u and v are even with $v > u \geqslant 2$ and there exists an m-cycle system of G_u^v then*

(1) $(v-m)(v-2) \geqslant u(u-2)$,
(2) $(v-u)(v+u-2) \equiv 0 \pmod{2m}$, *and*
(3) $v \geqslant (m+1)u/(m-1)+2$ *if m is odd.*

Lemma 6 (Bryant et al. [8]). *Let m be even and let v and u be odd. Suppose $u=1$ or $u > m/2$ or $v > u$. If there exists an m-cycle system of G_u^v then there exists an m-cycle system of G_{u+ym}^{v+xm} with $x \geqslant y \geqslant 0$ and $x \equiv y \pmod 2$.*

Corollary 7 (Bryant [8]). *Let m be even and let v and u be odd. If $u=1$ or $u > m/2$, and $u \equiv v \pmod{2m}$ then there exists an m-cycle system of G_u^v.*

Using a similar proof to that of Lemma 6, we prove the following lemma for the case where u and v are even.

Lemma 8. *Let m, u and v be even. Suppose $u=0$ or $u \geqslant m/2$ or $v > u$. If there exists an m-cycle system of G_u^v then there exists an m-cycle system of G_{u+ym}^{v+xm} with $x \geqslant y \geqslant 0$ and $x \equiv y \pmod 2$.*

Proof. Let $(U \cup V, C)$ be an m-cycle system of G_u^v, where U is the set of u vertices in the hole and V is the remaining $v-u$ vertices. Let U' and V' be sets of ym and $(x-y)m$ further vertices, respectively. Form an m-cycle system $(U \cup U' \cup V \cup V', C')$ of G_{u+ym}^{v+xm} with hole $U \cup U'$ by defining $C' = C \cup C_1 \cup C_2 \cup C_3$ as follows.

Let $(U' \cup V, C_1)$ be an m-cycle system of $K_{ym,v-u}$ with bipartition U' and V. This exists by Theorem 1 since $v-u \geqslant m/2$ or $v-u=0$ by the necessary conditions, and since $v-u \equiv 0 \pmod 2$ by the hypothesis.

Let $(U \cup U' \cup V \cup V', C_2)$ be an m-cycle system of $K_{u+ym+v-u,(x-y)m}$ with bipartition $U \cup U' \cup V$ and V'. This exists by Theorem 1 since $u+ym+v-u=ym+v \equiv 0 \pmod 2$ and if $v \neq 0$ then $ym+v \geqslant m/2$ (either $v > u$ in which case $v-u \geqslant m/2$, or $v=u \geqslant m/2$ by the hypothesis of this theorem).

Let (V', C_3) be an m-cycle system of $G^{(x-y)m}$. This exists by Theorem 3, since $x \equiv y \pmod 2$. $\square$

Corollary 9. *Let m, u and v be even. If $u=0$ or $u \geqslant m/2$, and $u \equiv v \pmod{2m}$ then there exists an m-cycle system of G_u^v.*

Proof. There exists an m-cycle system of G_u^u since this graph has no edges. Therefore by Lemma 8 there exists an m-cycle system of G_u^{u+2xm} for all integers $x \geqslant 0$. $\square$

Theorem 10 (see Bryant et al. [7]). *Let u, v and w be non-negative integers. If*

(1) *there exists an m-cycle system of G_u^v;*
(2) *there exists an n-cycle system of G_w^v;*
(3) $\mathrm{Type}_{m,n}(G^u) = S_{m,n}(|E(G^u)|) \neq \emptyset$ *and* $\mathrm{Type}_{m,n}(G^w) = S_{m,n}(|E(G^w)|) \neq \emptyset$; *and*
(4) $(|E(G^w)| + |E(G^u)|) - |E(G^v)| \geqslant 0$,

then $\mathrm{Type}_{m,n}(G^v) = S_{m,n}(|E(G^v)|)$.

3. (m^r, n^s) -cycle systems of G^v when m and n are even

In this section we prove (see Theorem 14) that in the case $m_1, m_2, \ldots, m_t \in \{m, n\}$, where m and n are even and $4 \leqslant m < n$, Alspach's conjecture is true for all v if and only if it is true for all $v < 7n$. We need the following straightforward lemma.

Lemma 11. *If $G = G_1 + G_2 + \cdots + G_t$ and for $1 \leqslant i \leqslant t$, $(r_i, s_i) \in \mathrm{Type}(G_i)$, then $(\sum_{i=1}^t r_i, \sum_{i=1}^t s_i) \in \mathrm{Type}(G)$.*

Lemma 12. *Let m and n be even and $4 \leqslant m < n$. Suppose $v \geqslant 3n$, $u = v - 2m$ and $w = v - 2n$. If $S_{m,n}(|E(G^v)|) \neq \emptyset$ then $S_{m,n}(|E(G^u)|) \neq \emptyset$ and $S_{m,n}(|E(G^w)|) \neq \emptyset$*

Proof. Let $r_{\max} = \max\{r\colon (r,s) \in S_{m,n}(|E(G^v)|)\}$ and let $(r_{\max}, s_{\min}) \in S_{m,n}(|E(G^v)|)$. Moreover, let $\gcd(m,n)$ be the greatest common divisor for m and n. Now, $s_{\min} \leqslant m/d - 1$ where $d = \gcd(m,n)$ since otherwise $(r_{\max} + n/d, s_{\min} - m/d) \in S_{m,n}(|E(G^v)|)$ which is a contradiction. Now consider the case v is odd. Since $u = v - 2m$, it is straightforward to check that $|E(G_u^v)| = |E(G^v)| - |E(G^u)| = (2v - 2m - 1)m$ and so

$$|E(G^u)| = r_{\max}m + s_{\min}n - (2v - 2m - 1)m = (r_{\max} - 2v + 2m + 1)m + s_{\min}n.$$

Hence if we can show that $r_{\max} - 2v + 2m + 1 \geqslant 0$ then $((r_{\max} - 2v + 2m + 1), s_{\min}) \in S_{m,n}(|E(G^u)|)$ and $S_{m,n}(|E(G^u)|) \neq \emptyset$.

Since $v(v-1)/2 = r_{\max}m + s_{\min}n$ and $s_{\min} \leqslant m/d - 1$, it follows that $r_{\max} - 2v + 2m + 1 \geqslant (v(v-1) - 4mv - 2mn/d + 4m^2 + 2n + 2m)/2m$ and since $d \geqslant 2$ we have $r_{\max} - 2v + 2m + 1 \geqslant (v(v-1) - 4mv - mn + 4m^2 + 2n + 2m)/2m$. It is straightforward to check that this quadratic is non-negative for $v \geqslant 3n$. The case v is even can be proved in a similar manner. The fact that $S_{m,n}(|E(G^w)|) \neq \emptyset$ can also be proved in this manner. □

Theorem 13. *Let m and n be even and $4 \leqslant m < n$. Suppose*

$$\phi(v) = \begin{cases} v^2 - (4m + 4n + 1)v + 4(m^2 + n^2) + 2(m + n), & v \text{ odd; and} \\ v^2 - (4m + 4n + 2)v + 4(m^2 + n^2) + 4(m + n), & v \text{ even,} \end{cases}$$

and let v^* be an integer such that $\phi(v^*) < 0$ and $\phi(v) \geqslant 0$ for all $v > v^*$. If $\mathrm{Type}_{m,n}(G^v) = S_{m,n}(|E(G^v)|)$ for all $v \leqslant \max\{3n, v^*\}$ then $\mathrm{Type}_{m,n}(G^v) = S_{m,n}(|E(G^v)|)$ for all v.

Proof. The proof is by induction on v. Let $v > \max\{3n, v^*\}$ and assume that for all $v' < v$ $\mathrm{Type}_{m,n}(G^{v'}) = S_{m,n}(|E(G^{v'})|)$. If $S_{m,n}(|E(G^v)|) = \emptyset$ then there is nothing to prove. So let $S_{m,n}(|E(G^v)|) \neq \emptyset$. Define $u = v - 2m$ and $w = v - 2n$. We show that u, w and v satisfy the conditions of Theorem 10. Since $v \geqslant 3n$, by Corollaries 7 and 9 there exists an m-cycle system of G_u^v and an n-cycle system of G_w^v and so conditions (1) and (2) are satisfied. By Lemma 12 and the induction hypothesis we have condition (3). Since $v \geqslant v^*$ implies $\phi(v) \geqslant 0$, it is straightforward to check that condition 4 holds. □

Theorem 14. *Let n and m be even and $4 \leqslant m < n$. If $\mathrm{Type}_{m,n}(G^v) = S_{m,n}(|E(G^v)|)$ for all $v < 7n$ then $\mathrm{Type}_{m,n}(G^v) = S_{m,n}(|E(G^v)|)$ for all v.*

Proof. Let $\phi(v)$ and v^* be as defined in Theorem 13. It is straightforward to check that $\phi(v) \geqslant 0$ for all $v \geqslant 7n$. Hence $7n \geqslant \max\{3n, v^*\}$ and so the result follows by Theorem 13. □

4. The cases *m, n* even and $\leqslant 10$

We begin this section with the obvious necessary conditions for the existence of an (m^r, n^s)-cycle system of G^v.

Lemma 15. *If there exists an* (m^r, n^s)*-cycle system of* G^v *then* $v \geqslant \min\{m,n\}$ *and* $v(v-1) \equiv 0 \pmod{2d}$ *if* v *is odd, or* $v(v-2) \equiv 0 \pmod{2d}$ *if* v *is even, where* $d = \gcd(m,n)$.

We make use of the following lemma when v is small.

Lemma 16. *Let* u *and* w *be even and* $v = u + w + h$, *where* $h \in \{0,1,2\}$. *Suppose*

(1) $\mathrm{Type}_{m,n}(G^{u+h}) \neq \emptyset$ *and* $\mathrm{Type}_{m,n}(G^{w+h}) \neq \emptyset$;
(2) *there exists an m-cycle system of* $K_{u,w}$; *and*
(3) *there exists an n-cycle system of* $K_{u,w}$.

Then $\mathrm{Type}_{m,n}(G^{u+h}) + \mathrm{Type}_{m,n}(G^{w+h}) + \{(uw/m, 0), (0, uw/n)\} \subseteq \mathrm{Type}_{m,n}(G^v)$.

Proof. Since $G^v = G^{u+h} + G^{w+h} + K_{u,w}$ the result follows by Lemma 11. □

4.1. The case $(m,n) = (4,8)$

First note that by Lemma 15 the necessary conditions for the existence of a $(4^r, 8^s)$-cycle system of G^v are $v \equiv 0 \pmod 2$ or $v \equiv 1 \pmod 8$ and $v \geqslant 4$. Now by Theorem 13 we only need to prove $\mathrm{Type}(G^v) = S(|E(G^v)|)$ for all $4 \leqslant v \leqslant 40$.

Lemma 17. $\mathrm{Type}(G^v) = S(|E(G^v)|)$ *for all* $4 \leqslant v \leqslant 40$.

Proof. If v does not satisfy the necessary conditions then $\mathrm{Type}(G^v) = S(|E(G^v)|) = \emptyset$. Now let v satisfy the necessary conditions. For $v \leqslant 10$ the result follows by [15]. For $11 \leqslant v \leqslant 40$ we apply Lemma 16 with the values for u, w and h as shown in Table 1. For example, when $v=12$ we let $u=4$, $w=8$ and $h=0$. Since $\mathrm{Type}_{4,8}(G^4) = \{(1,0)\} \neq \emptyset$, $\mathrm{Type}_{4,8}(G^8) = \{(6,0),(4,1),(2,2),(0,3)\} \neq \emptyset$, and by Theorem 1 there exists a 4-cycle system of $K_{4,8}$ and an 8-cycle system of $K_{4,8}$, by Lemma 16 we have

$$\{(1,0)\} + \{(6,0),(4,1),(2,2),(0,3)\} + \{(8,0),(0,4)\} \subseteq \mathrm{Type}_{4,8}(G^{12}).$$

On the other hand, $\mathrm{Type}_{4,8}(G^{12}) = \{(15,0),(13,1),\ldots,(1,7)\}$. Therefore, $\mathrm{Type}_{4,8}(G^{12}) = S_{4,8}(|E(G^{12})|)$. Similarly, it is straightforward to check that this construction covers all types except the type $(0,17)$ when $v = 17$, which is settled by Theorem 2. □

Theorem 18. *Let* $r, s \geqslant 0$. *The necessary and sufficient conditions for the existence of a* $(4^r, 8^s)$*-cycle system of* G^v *are* $v \equiv 0 \pmod 2$ *or* $v \equiv 1 \pmod 8$, $v \geqslant 4$ *and* $4r + 8s = |E(G^v)|$.

Table 1

v	(u,w,h)	Remaining types	v	(u,w,h)	Remaining types
12	(4,8,0)	None	14	(4,10,0)	None
16	(4,12,0) (8,8,0)	None	17	(8,8,1)	(0,17)
18	(4,14,0) (8,10,0)	None	20	(4,16,0)	None
22	(4,18,0)	None	24	(8,16,0)	None
25	(8,16,1)	None	26	(8,18,0)	None
28	(4,24,0)	None	30	(4,26,0)	None
32	(8,24,0)	None	33	(16,16,1)	None
34	(8,26,0)	None	36	(4,32,0)	None
38	(4,34,0)	None	40	(8,32,0)	None

Proof. The necessary conditions follow by Lemma 15. To prove the conditions are sufficient we first apply Lemma 17 for $4 \leqslant v \leqslant 40$. Then Theorem 13 takes care of $v \geqslant 41$. $\square$

4.2. The case $(m,n) = (4,10)$

First note that by Lemma 15 the necessary conditions for the existence of a $(4^r, 10^s)$-cycle system of G^v are $v \equiv 0 \pmod 2$ or $v \equiv 1 \pmod 4$ and $v \geqslant 4$. Now by Theorem 13 we only need to prove $\mathrm{Type}(G^v) = S(|E(G^v)|)$ for all $4 \leqslant v \leqslant 47$.

Lemma 19. $\mathrm{Type}(G^v) = S(|E(G^v)|)$ *for all* $4 \leqslant v \leqslant 47$.

Proof. If v does not satisfy the necessary conditions then $\mathrm{Type}(G^v) = S(|E(G^v)|) = \emptyset$. Now let v satisfy the necessary conditions. For $v \leqslant 10$ the result follows by [15]. For $v=12$, we use the relation $G^{12} = G^{10} + K_{2,10}$ for types $(15,0)$, $(10,2)$ and $(5,4)$. The type $(0,6)$ is given in [1]. For $v \in \{13,17,21\}$ see [1]. For $v=25$, we use the relation $G^{25} = G^{17} + G_{17}^{25}$ for the types $(75,0),(70,2),(65,4),\ldots,(45,12)$, and we apply Theorem 2 for the type $(0,30)$. The remaining types are given in [1]. For the other values of v we apply Lemma 16 with the values for u, w and h as shown in Table 2. For example, when $v=14$ we let $u=4$, $w=10$ and $h=0$. Since $\mathrm{Type}_{4,10}(G^4) = \{(1,0)\} \neq \emptyset$, $\mathrm{Type}_{4,10}(G^{10}) = \{(10,0),(5,2),(0,4)\} \neq \emptyset$, and by Theorem 1 there exists a 4-cycle system of $K_{4,10}$ and a 10-cycle system of $K_{4,10}$, by Lemma 16 we have

$$\{(1,0)\} + \{(10,0),(5,2),(0,4)\} + \{(10,0),(0,4)\} \subseteq \mathrm{Type}_{4,10}(G^{14}).$$

On the other hand, $\mathrm{Type}_{4,10}(G^{14}) = \{(21,0),\ldots,(1,8)\}$. Therefore, $\mathrm{Type}_{4,10}(G^{14}) = S_{4,10}(|E(G^{14})|)$. Similarly, it is straightforward to check that this construction covers all types except the types $(1,14)$ for $v=18$, $(4,39)$ for $v=29$, $(132,0)$ for $v=33$, and $(205,0)$ for $v=41$. The last two follow by Theorem 2. Using the relation $G^{29} = K_{12,10} + G_7^{19} + G^{17}$ we see that $(4,39) \in \mathrm{Type}(G^{29})$. A 10-cycle system of G_7^{19} is given in [1]. Finally, [1] shows that $(1,14) \in \mathrm{Type}(G^{18})$. $\square$

Table 2

v	(u,w,h)	Remaining types	v	(u,w,h)	Remaining types
14	(4,10,0)	None	16	(6,10,0)	None
18	(8,10,0)	(1,14)	20	(10,10,0)	None
	(6,10,2)				
22	(10,12,0)	None	24	(10,14,0)	None
26	(6,20,0)	None	28	(10,18,0)	None
29	(8,20,1)	(4,39)	30	(10,20,0)	None
32	(10,22,0)	None	33	(12,20,1)	(132,0)
34	(10,24,0)	None	36	(6,30,0)	None
37	(16,20,1)	None	38	(10,28,0)	None
40	(10,30,0)	None	41	(20,20,1)	(205,0)
42	(10,32,0)	None	44	(10,34,0)	None
45	(20,24,1)	None	46	(6,40,0)	None

Theorem 20. *Let* $r,s\geqslant 0$. *The necessary and sufficient conditions for the existence of a* $(4^r,10^s)$-*cycle system of* G^v *is that* $v\equiv 0\,(\mathrm{mod}\,2)$ *or* $v\equiv 1\,(\mathrm{mod}\,4)$, $v\geqslant 4$ *and* $4r+10s=|E(G^v)|$.

Proof. The necessary conditions follow by Lemma 15. To prove the conditions are sufficient we first apply Lemma 19 for $4\leqslant v\leqslant 47$. Then Theorem 13 takes care of $v\geqslant 48$. □

4.3. The case $(m,n)=(6,8)$

First note that by Lemma 15 the necessary conditions for the existence of a $(6^r,8^s)$-cycle system of G^v are $v\equiv 0\,(\mathrm{mod}\,2)$ or $1\,(\mathrm{mod}\,4)$ and $v\geqslant 6$. Now by Theorem 13 we need only to prove $\mathrm{Type}(G^v)=S(|E(G^v)|)$ for all $6\leqslant v\leqslant 49$.

Lemma 21. $\mathrm{Type}(G^v)=S(|E(G^v)|)$ *for all* $6\leqslant v\leqslant 49$.

Proof. If v does not satisfy the necessary conditions then $\mathrm{Type}(G^v)=S(|E(G^v)|)=\emptyset$. Now let v satisfy the necessary conditions. For $v\leqslant 10$ the result follows by [15]. For $v\in\{13,17\}$ see [1]. For the other values of v we apply Lemma 16 with the values for u, w and h as shown in Table 3. For example, when $v=12$ we let $u=4$, $w=6$ and $h=2$. Since $\mathrm{Type}_{6,8}(G^6)=\{(2,0)\}\neq\emptyset$, $\mathrm{Type}_{6,8}(G^8)=\{(4,0),(0,3)\}\neq\emptyset$, and by Theorem 1 there exists a 6-cycle system of $K_{4,6}$ and an 8-cycle system of $K_{4,6}$, by Lemma 16 we have

$$\{(2,0)\}+\{(4,0),(0,3)\}+\{(4,0),(0,3)\}\subseteq\mathrm{Type}_{6,8}(G^{12}).$$

On the other hand, $\mathrm{Type}_{6,8}(G^{12})=\{(10,0),(6,3),(2,6)\}$. Therefore, $\mathrm{Type}_{6,8}(G^{12})=S_{6,8}(|E(G^{12})|)$. Similarly, it is straightforward to check that this construction covers all types except the types $(0,18)$ for $v=18$, $(0,66)$ for $v=33$ and $(0,147)$ for $v=49$. We use the relation $G^{18}=G^{10}+G^{10}+K_{8,8}$ for the type $(0,18)$ and apply Theorem 2 for the other two types. □

Table 3

v	(u,w,h)	Remaining types	v	(u,w,h)	Remaining types
12	(4,6,2)	None	14	(6,8,0)	None
16	(6,8,2)	None	18	(6,12,0)	(0,18)
20	(8,12,0)	None	21	(8,12,1)	None
22	(6,16,0)	None	24	(6,16,2)	None
25	(12,12,1)	None	26	(8,18,0)	None
28	(6,20,2)	None	29	(12,16,1)	None
30	(6,24,0)	None	32	(8,24,0)	None
33	(8,24,1)	(0,66)	34	(10,24,0)	None
36	(12,24,0)	None	37	(12,24,1)	None
38	(6,32,0)	None	40	(16,24,0)	None
41	(12,28,1)	None	42	(18,24,0)	None
44	(8,36,0)	None	45	(12,32,1)	None
46	(6,40,0)	None	48	(22,24,2)	None
49	(12,36,1)	(0,147)			

Theorem 22. *Let $r,s\geqslant 0$. The necessary and sufficient conditions for the existence of a $(6^r,8^s)$-cycle system of G^v are $v\equiv 0\,(\mathrm{mod}\,2)$ or $v\equiv 1\,(\mathrm{mod}\,4)$, $v\geqslant 6$ and $6r+8s=|E(G^v)|$.*

Proof. The necessary conditions follow by Lemma 15. To prove the conditions are sufficient we first apply Lemma 21 for $6\leqslant v\leqslant 49$. Then Theorem 13 takes care of $v\geqslant 50$. $\square$

4.4. The case $(m,n)=(6,10)$

First note that by Lemma 15 the necessary conditions for the existence of a $(6^r,10^s)$-cycle system of G^v are $v\equiv 0\,(\mathrm{mod}\,2)$ or $v\equiv 1\,(\mathrm{mod}\,4)$ and $v\geqslant 6$. Now by Theorem 13 we need only to prove $\mathrm{Type}(G^v)=S(|E(G^v)|)$ for all $6\leqslant v\leqslant 54$.

Lemma 23. $\mathrm{Type}(G^v)=S(|E(G^v)|)$ *for all* $6\leqslant v\leqslant 54$.

Proof. If v does not satisfy the necessary conditions then $\mathrm{Type}(G^v)=S(|E(G^v)|)=\emptyset$. Now let v satisfy the necessary conditions. For $v\leqslant 10$ the result follows by [15]. For $v\in\{12,13,14,17,21\}$ see [1]. For $v=20$, using relations $G^{20}=G^6+G^{14}+K_{6,14}$ and $G^{20}=G^{10}+G^{12}+K_{8,10}$ one can obtain all types. For $v=25$ we use the relation $G^{25}=G^{13}+G^{13}+K_{12,12}$ to obtain the types $(50,0)$, $(45,3)$, $(40,6)$, $(35,9)$ and $(30,12)$, and we apply Theorem 2 for the type $(0,30)$. The other types are given in [1]. For $v=29$, first we note that there exists a 6-cycle system of G^{29}_{17} (see Corollary 7) and a 10-cycle system of G^{29}_{17} since $G^{29}_{17}=K_{12,10}+G^{19}_{7}$. (See [1] for a 10-cycle system of G^{19}_{7}.) Now the relation $G^{29}=G^{29}_{17}+G^{17}$ covers all types except the types $(41,16)$, $(36,19)$, $(31,22)$ and $(26,25)$. For these types we use the relation $G^{29}=G^9+G^{21}+K_{8,20}$. For $v=37$, we use relations $G^{37}=G^{37}_{25}+G^{25}$ and $G^{37}=G^{21}+G^{17}+K_{20,16}$. For $v=41$,

Table 4

v	(u,w,h)	Remaining types	v	(u,w,h)	Remaining types
16	(6,10,0)	None	18	(6,10,2)	None
22	(10,12,0)	None	24	(10,12,2)	None
26	(6,20,0)	None	28	(10,18,0)	None
30	(10,18,2)	None	32	(12,20,0)	None
33	(12,20,1)	None	34	(10,24,0)	None
36	(6,30,0)	None	38	(8,30,0)	None
40	(10,30,0)	None	42	(12,30,0)	None
44	(14,30,0)	None	45	(20,24,1)	None
46	(6,40,0)	None	48	(18,30,0)	None
50	(20,30,0)	None	52	(10,42,0)	None
53	(12,40,1)	None	54	(22,30,2)	None

we use relations $G^{41}=G^{41}_{29}+G^{29}$ and $G^{41}=G^{21}+G^{21}+K_{20,20}$. Finally, for $v=49$, we use relations $G^{49}=G^{49}_{37}+G^{37}$ and $G^{49}=G^{29}+G^{21}+K_{20,28}$. It is easy to see that these constructions cover all types when $v\in\{37,41,49\}$. For the other values of v we apply Lemma 16 with the values for u, w and h as shown in Table 4. For example, when $v=16$ we let $u=6$, $w=10$ and $h=0$. Since $\mathrm{Type}_{6,10}(G^6)=\{(2,0)\}\neq\emptyset$, $\mathrm{Type}_{6,10}(G^{10})=\{(5,1),(0,4)\}\neq\emptyset$, and by Theorem 1 there exists a 6-cycle system of $K_{6,10}$ and a 10-cycle system of $K_{6,10}$, by Lemma 16 we have

$$\{(2,0)\}+\{(5,1),(0,4)\}+\{(10,0),(0,6)\}\subseteq\mathrm{Type}_{6,10}(G^{16}).$$

On the other hand, $\mathrm{Type}_{6,10}(G^{16})=\{(17,1),\ldots,(2,10)\}$. Therefore, $\mathrm{Type}_{6,10}(G^{16})=S_{6,10}(|E(G^{16})|)$. Similarly, it is straightforward to check that this construction covers all types. □

Theorem 24. *Let $r,s\geqslant 0$. The necessary and sufficient conditions for the existence of a $(6^r,10^s)$-cycle system of G^v are $v\equiv 0\,(\mathrm{mod}\,2)$ or $v\equiv 1\,(\mathrm{mod}\,4)$, $v\geqslant 6$ and $6r+10s=|E(G^v)|$.*

Proof. The necessary conditions follow by Lemma 15. To prove the conditions are sufficient we first apply Lemma 23 for $6\leqslant v\leqslant 54$. Then Theorem 13 takes care of $v\geqslant 55$. □

4.5. The case $(m,n)=(8,10)$

First note that by Lemma 15 the necessary conditions for the existence of a $(8^r,10^s)$-cycle system of G^v are $v\equiv 0\,(\mathrm{mod}\,2)$ or $1\,(\mathrm{mod}\,4)$ and $v\geqslant 8$. Now by Theorem 13 we need only to prove $\mathrm{Type}(G^v)=S(|E(G^v)|)$ for all $8\leqslant v\leqslant 63$.

Lemma 25. $\mathrm{Type}(G^v)=S(|E(G^v)|)$ *for all* $8\leqslant v\leqslant 63$.

Table 5

v	(u,w,h)	Remaining types	v	(u,w,h)	Remaining types
18	(8,10,0)	None	20	(8,10,2)	None
22	(10,12,0)	None	26	(10,16,0)	None
28	(8,20,0)	None	30	(10,20,0)	None
32	(12,20,0)	(60,0)	33	(12,20,1)	(66,0)
34	(10,24,0)	None	36	(14,20,2)	None
37	(16,20,1)	None	38	(8,30,0)	None
40	(8,30,2)	None	41	(20,20,1)	None
42	(10,32,0)	None	44	(12,30,2)	None
45	(20,24,1)	None	46	(10,36,0)	None
48	(8,40,0)	None	49	(20,28,1)	(147,0)
50	(10,40,0)	None	52	(12,40,0)	None
53	(12,40,1)	(171,1)	54	(10,44,0)	None
56	(16,40,0)	None	57	(16,40,1)	None
58	(8,50,0)	None	60	(20,40,0)	None
61	(20,40,1)	None	62	(10,52,0)	None

Proof. If v does not satisfy the necessary conditions then $\mathrm{Type}(G^v)=S(|E(G^v)|)=\emptyset$. Now let v satisfy the necessary conditions. For $v\leqslant 10$ the result follows by [15]. For $v\in\{12,13,14,16,17,21,25\}$ see [1]. For $v=24$ we use the relations $G^{24}=G^{12}+G^{14}+K_{10,12}$ and $G^{24}=G^8+G^{16}+K_{8,16}$. For $v=29$ we use the relations $G^{29}=G^{29}_{17}+G^{17}$ and $G^{29}=G^{13}+G^{17}+K_{12,16}$. Note that there exists a 10-cycle system of G^{29}_{17} (see Lemma 23). The remaining type, $(22,23)$, is given in [1]. For the other values of v we apply Lemma 16 with the values for u, w and h as shown in Table 5. For example, when $v=18$ we let $u=8$, $w=10$ and $h=0$. Since $\mathrm{Type}_{8,10}(G^8)=\{(3,0)\}\neq\emptyset$, $\mathrm{Type}_{8,10}(G^{10})=\{(5,0),(0,4)\}\neq\emptyset$, and by Theorem 1 there exists an 8-cycle system of $K_{8,10}$ and a 10-cycle system of $K_{8,10}$, by Lemma 16 we have

$$\{(3,0)\}+\{(5,0),(0,4)\}+\{(10,0),(0,8)\}\subseteq \mathrm{Type}_{8,10}(G^{18}).$$

On the other hand, $\mathrm{Type}_{8,10}(G^{18})=\{(18,0),\ldots,(3,12)\}$. Therefore, $\mathrm{Type}_{8,10}(G^{18})=S_{8,10}(|E(G^{18})|)$. Similarly, it is straightforward to check that this construction covers all types except the ones shown in Table 5. For $v=32$ the remaining type, $(60,0)$, can be found using the relation $G^{32}=G^{16}+G^{16}+K_{16,16}$. The remaining types $(66,0)$ for $v=33$ and $(147,0)$ for $v=49$ are covered by Theorem 2. Finally, we use the relation $G^{53}=G^{53}_{37}+G^{37}$ to find the remaining type $(171,1)$ for $v=53$. □

Theorem 26. *Let $r,s\geqslant 0$. The necessary and sufficient conditions for the existence of a $(8^r,10^s)$-cycle system of G^v are $v\equiv 0\ (\mathrm{mod}\ 2)$ or $v\equiv 1\ (\mathrm{mod}\ 4)$, $v\geqslant 8$ and $8r+10s=|E(G^v)|$.*

Proof. The necessary conditions follow by Lemma 15. To prove the conditions are sufficient we first apply Lemma 25 for $8\leqslant v\leqslant 63$. Then Theorem 13 takes care of $v\geqslant 64$. □

References

[1] P. Adams, D.E. Bryant, A. Khodkar, On Alspach's conjecture with two even cycle lengths (m,n), where $(m,n) \in \{(4,10),(6,8),(6,10),(8,10)\}$, Research Report, Department of Mathematics, University of Queensland, 1998. (See http://www.maths.uq.edu. au/~ak/Report/report2.ps.)
[2] P. Adams, D.E. Bryant, A. Khodkar, 3, 5-cycle decompositions, J. Combin. Des. 6 (1998) 91–110.
[3] B. Alspach, Research problems, Problem 3, Discrete Math. 36 (1981) 333.
[4] B. Alspach, H. Gavlas, Cycle decompositions of K_n and $K_n - I$, submitted for publication.
[5] B. Alspach, B.N. Varma, Decomposing complete graphs into cycles of length $2p^e$, Ann. Discrete Math. 9 (1980) 155–169.
[6] J.C. Bermond, C. Huang, D. Sotteau, Balanced cycle and circuit designs: even cases, Ars Combin. 5 (1978) 293–318.
[7] D.E. Bryant, H.L. Fu, A. Khodkar, (m,n)-cycle systems, J. Statist. Plann. Inference, to appear.
[8] D.E. Bryant, C.A. Rodger, E.R. Spicer, Embeddings of m-cycle systems and incomplete m-cycle systems: $m \leqslant 14$, Discrete Math. 171 (1997) 55–75.
[9] K. Heinrich, P. Horak, A. Rosa, On Alspach's conjecture, Discrete Math. 77 (1989) 97–121.
[10] D.G. Hoffman, C.C. Lindner, C.A. Rodger, On the construction of odd cycle systems, J. Graph Theory 13 (1989) 417–426.
[11] B.W. Jackson, Some cycle decompositions of complete graphs, J. Combin. Inform. System Sci. 13 (1988) 20–32.
[12] A. Kotzig, On decompositions of the complete graph into 4-gons, Mat. Fyz. Casopis 15 (1965) 227–233.
[13] C.C. Lindner, C.A. Rodger, Decomposition into cycles II, Cycle systems, in: J.H. Dinitz, D.R. Stinson (Eds.), Contemporary Design Theory: a Collection of Surveys, Wiley, New York, pp. 325–369.
[14] C.A. Rodger, Problems on cycle systems of odd length, Congr. Numer. 61 (1988) 5–22.
[15] A. Rosa, Alspach's conjecture is true for $n \leqslant 10$, Mathemtical Reports, McMaster University, to appear.
[16] M. Šajna, Cycle decompositions of K_n and $K_n - I$, Ph.D. Thesis, Simon Fraser University, July 1999.
[17] D. Sotteau, Decompositions of $K_{m,n}$ $(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory, Ser. B 30 (1981) 75–81.