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PARABOLIC PROBLEMS WITH MIXED VARIABLE LATERAL CONDITIONS: AN ABSTRACT APPROACH

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ABSTRACT. – We study the initial value problem for parabolic second order equations with mixed and time-dependent boundary conditions obtaining optimal regularity results under weak assumptions on the data and on the geometrical behavior of the boundary. An approximation approach to abstract evolution equations on variable domains is the basic tool we develop; an application to parabolic problems in non-cylindrical domains is also given.

0. Introduction

Let Ω be a uniformly $C^{1,1}$ open set of $\mathbb{R}^N$ with boundary $\Gamma = \partial\Omega$; for a fixed positive number $T > 0$ we set

$$Q = \Omega\times]0,T[, \qquad \Sigma = \Gamma\times]0,T[,$$

and we choose a uniform family of $C^{1,1}$ submanifolds (with boundary) $\Gamma_0^t \subset \partial\Omega$, t varying in $[0,T]$; Σ_0 will be the subset of Σ covered by this family, that is:

$$\Sigma_0 = \bigcup_{t\in]0,T[} \Gamma_0^t \times \{t\}, \qquad \Sigma_1 = \Sigma \setminus \overline{\Sigma}_0.$$

We want to study the mixed boundary value Cauchy problem

$$(PP) \qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} + Au(x,t) = f(x,t), & \text{in } Q, \\ u(x,t) = g_0(x,t), & \text{on } \Sigma_0, \\ \dfrac{\partial u(x,t)}{\partial \nu_A} = g_1(x,t), & \text{on } \Sigma_1, \\ u(x,0) = u_0(x), & \text{on } \Omega. \end{cases}$$

Here A is a uniform elliptic second order operator with variable coefficients of the type

$$(0.1) \qquad Au = -\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a^{ij}(x,t)\frac{\partial u}{\partial x_j}\Big) + \sum_i b^i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

with $a^{ij} \in W^{1,\infty}(Q), b^i, c \in L^\infty(Q)$, $a^{ij} = a^{ji}$, and

$$(0.2) \qquad \exists\, \alpha > 0: \quad \sum_{i,j} a^{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \quad \forall (x,t) \in Q;$$

$\nu_A = \nu_A(x,t)$ is the related conormal vector to $\Gamma\times]0,T[$, f, g_0, g_1 and u_0 are the data given in suitable Sobolev spaces of functions defined on Q and its boundary.

Problems of this kind have been studied for long time from many points of view. Among the first contributions (whose references can be found in [32]), we quote a uniqueness [31] and an existence [32] result by Magenes, the latter one holding when Σ_0 is of cylindrical type, *i.e.* Γ_0^t is independent of time. This particular case can also be studied either in the natural variational framework *via* the standard theory of abstract evolution equations (*see* [26], [23], [29]) or by a more direct analysis in suitable weighted function spaces, which take into account the lack of regularity near the interface between Σ_0 and Σ_1 (*see* [38] and the references quoted by [12]).

These techniques (analogous to the Vishik-Eskin's ones [37] for the elliptic case) are further developed by [11], [12] and consequently adapted to the case of time dependent mixed conditions; here Σ_0, Σ_1 have to be C^∞ submanifolds of Σ and their interface must never be tangent to the hyperplanes $t = \text{const}$ (except for $t = 0$, in [12]), so that a careful change of variable transforms the problem in the previous cylindrical form and the solution will belong to function spaces closely connected to the geometrical structure of the boundaries.

On the other hand, it is interesting to know existence and regularity properties of the solution in spaces independent of the geometry involved and under weaker assumptions on the data and on the boundary. Thanks to a general result about evolution equations in variable Hilbert domains, Baiocchi obtained in [7] a theorem of existence and uniqueness of the solution of (PP) with homogeneous lateral boundary conditions (g_0, $g_1 = 0$) under very weak geometric assumptions; more precisely, if f, u_0 are in $L^2(Q)$ and $L^2(\Omega)$ respectively, then a suitable weak formulation of (PP) admits a unique solution u belonging to the class

$$H^{1,1/2}(Q) = L^2(0,T; H^1(\Omega)) \cap H^{1/2}(0,T; L^2(\Omega)) \quad (^1).$$

A remarkable fact is that at this level of regularity no smoothness of Σ_0, Σ_1 (and also of the a^{ij}) is needed.

Other weak results of this kind could be obtained in more general Banach frameworks by applying the widely developed abstract theory (*see* [24], [23], [18], [1] and the references quoted therein (2): for a comparison of the various hypotheses *see* [2]): differently from the Baiocchi's work, however, all these more technical results require careful elliptic-type

(1) *See* [29] for the complete definitions and the properties of the $H^s, H^{r,s}$ hilbertian families of function spaces; we shall recall some of them in the next sections.

(2) The nonlinear case is deeply studied in [13] and [27]; for evolution equations of hyperbolic type with variable domain we refer to [4], [15].

estimates linked to the behavior of the boundary, which must be regular in some sense. In any case the singular nature of the mixed conditions does not allow to recover either a *strong* solution (*i.e.* time differentiable for a.e. $t \in]0,T[$ in some L^p space of the x-variable) or the expected maximal regularity supplied by the data.

We will be concerned with these last two related questions in the simpler variational Hilbert context of [7]; more precisely, we are interested in sufficiently wide conditions on $\{\Gamma_0^t\}_{t\in[0,T]}$ in order to obtain stronger regularity of the type ([3])

$$(0.3)\quad f \in L^2(Q),\quad u_0, g_0, g_1 \text{ in suitable trace spaces } \Rightarrow \begin{cases} \dfrac{\partial u}{\partial t},\ Au \in L^2(Q), \\ \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \in L^\infty(0,T). \end{cases}$$

We have already noticed that for cylindrical Σ_i the abstract variational theory works well, so that (0.3) holds; a partial extension of this result is given by [10] (*see* also [21]), where Γ_0^t must not increase with respect to t.

Our aim is to show that (0.3) also holds if we assume that the excess

$$e(\Gamma_0^t, \Gamma_0^s) = \sup_{x\in\Gamma_0^t} d(x, \Gamma_0^s), \qquad s,t \in [0,T],$$

for $t > s$ can be controlled by the uniform linear bound ([4])

$$(0.4)\qquad e(\Gamma_0^t, \Gamma_0^s) \le K(t-s), \qquad \forall s < t,$$

for a constant $K > 0$ independent of s and t. Let us remark that this condition includes the monotone previous one, since the points of Γ_t which also belong to Γ_s do not affect the excess in (0.4); so we are only imposing a one-side condition on the growth of Γ_0^t and we could say that the points of Γ_0^t "go away with a bounded speed" as the time t increases. Of course, smooth manifolds (in space and time) are allowed and the same is true in the case of a Lipschitz time dependence of Γ_0^t with respect to the Hausdorff distance between the subsets of Γ; in these conditions better regularity properties can be derived.

Our proof is characterized by two different features:

I. A new regularity and perturbation result for the solution of an elliptic problem with mixed boundary conditions proved in [34];

II. A simple approximation procedure of (PP) by the backward Euler scheme, which is also interesting from a numerical point of view; we shall apply this technique in the abstract framework proposed by [7] since the structural hypotheses suggested by the previous point I are common to very different situations as parabolic equations in non cylindrical domains.

([3]) The choice of the good spaces for the boundary data is suggested by the theory for the pure Cauchy-Dirichlet and Cauchy-Neumann problems (*see* [29]): we will detail it in Sect. 4.

([4]) But weaker conditions could be given; *see* Sect. 4.

Let us describe this framework in the case of (PP) with $g_0, g_1 \equiv 0$ and A independent of time. On the Hilbert space $V = H^1(\Omega)$ we introduce the bilinear form associated to A:

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j} a^{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} + \sum_i b^i(x) \frac{\partial u(x)}{\partial x_i} v(x) + c(x)u(x)v(x) \right\} dx$$

which we can always assume to be coercive.

We consider u, f as functions of the time with values in V and $H = L^2(\Omega)$ respectively. The homogeneous Dirichlet condition will be imposed by requiring that for a.e. $t \in]0, T[$

$$u(t) \in V_t = H^1_{\Gamma_0^t}(\Omega) = \left\{ v \in H^1(\Omega) : v_{|\Gamma_0^t} \equiv 0 \text{ in the sense of traces} \right\}, \tag{0.5}$$

and the equation together with the "natural" Neumann condition will be recovered by the variational formulation

$$(u'(t), v)_H + a(u(t), v) = (f(t), v)_H, \qquad \forall v \in V_t, \quad \text{for a.e. } t \in]0, T[. \tag{0.6}$$

If $f \in L^2(0,T;H)$ and $u_0 \in V_0$ we ask for $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$ satisfying (0.5), (0.6) and $u(0) = u_0$.

Substantially, the known abstract theory assumes the monotonicity of V_t (*see* [10]) or the continuity (or even the Hölderianity) of the time derivative of the resolvent operators associated to a and the family V_t in the space of the linear and bounded operators of H (*see* [26], [18]), a condition which does not hold in our concrete case ([5]) and is not compatible with the previous monotonicity one.

We overcome the use of the derivative of these operators by directly comparing two different solutions of the family of elliptic time dependent problems

$$\text{given } L \in H \text{ find } u \in V_t : \quad a(u,v) = (L,v)_H, \qquad \forall v \in V_t. \tag{0.7}$$

Obviously the difficult lies on the varying test-functions sets; the idea is to measure their difference by considering the "residual" functional on V

$$v \in V \mapsto a(u,v) - (L,v)_H = R_{L,t}(v).$$

Of course R is identically zero on V_t; we shall show that it is enough to control it on those elements w of V_s which solve the analogous of (0.7) at the time $s < t$. In other words, if

$$w \in V_s : \quad a(w,v) = (L',v)_H, \qquad \forall v \in V_s, \quad \text{with } L' \in H,$$

then we ask that

$$R_{L,t}(w) = a(u,w) - (L,w)_H \leq K|t-s| \, \|L\|_H \, \|L'\|_H^{1-\theta} \|w\|_V^\theta, \tag{0.8}$$

([5]) [26] derives it by a time-differentiability property of the projections on the V_t: *see* the related comparison remarks of [10].

for some $\theta \in]0,1]$ and $K > 0$ independent of t, s and the data. In our concrete case this estimate is exactly proved by [34] for $\theta = 1/2$.

An interesting fact is that (0.8) holds also for a suitable abstract formulation of parabolic equations with Cauchy-Dirichlet conditions in non cylindrical domains under simple geometric assumptions quite similar to (0.4) ([6]); moreover, (0.8) is a good assumption in order to prove the stability and the convergence of the simplest discrete scheme we can use to approximate (0.6). We conclude this introduction with a brief sketch of this approach, coming back to the concrete version of (PP).

We divide the time interval $]0,T]$ in $\kappa > 0$ subintervals of equal length $\tau = T/\kappa$ and we choose suitable approximations $f_\tau^n(x), g_\tau^n(x)$ of $f(t,x), g_1(t,x)$ at the nodes $t = n\tau$, $n = 0, 1, \ldots, \kappa$; then we solve recursively the elliptic problems in the unknowns $u_\tau^n(x)$:

$$(EP_n) \qquad \begin{cases} \dfrac{1}{\tau}(u_\tau^n(x) - u_\tau^{n-1}(x)) + Au_\tau^n(x) = f_\tau^n(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \Gamma_0^{n\tau}, \\ \dfrac{\partial u(x)}{\partial \nu_A} = g_\tau^n(x), & \text{on } \Gamma_1^{n\tau}, \\ u_\tau^{-1}(x) = u_0(x), & \text{in } \Omega. \end{cases}$$

The values $\{u_\tau^n\}_{n=0,1,\ldots,\kappa}$ give raise to a continuous and piecewise linear (with respect to time) function $\hat{u}_\tau(x,t)$ which takes $u_\tau^n(x)$ at $t = n\tau$ and we shall show that $\hat{u}_\tau$ converges to the solution u of (PP) as $\tau = T/\kappa$ goes to 0.

The plain of this paper is the following: first we develop the abstract theory stating in that context the approximation and regularity results we need; proofs are given in the next two sections and the last one is devoted to the applications to (PP) and to parabolic equations in non cylindrical domains.

1. The abstract theory

Let us give two separable Hilbert spaces $V \subset H$ with continuous and dense inclusion, let $\|\cdot\|$ and $|\cdot|$ be their norms and $(\cdot,\cdot)$ the scalar product of H. As usual we identify H with its dual H', so that H can be densely embedded in V' and its scalar product can be uniquely extended to the duality pairing between V' and V. Furthermore, we are given

$$\text{a family } \{V_t\}_{t \in [0,T]} \text{ of closed subspaces of } V$$

and

$$\text{a family of continuous bilinear forms } a(t;\cdot,\cdot) : V \times V \mapsto \mathbb{R}, \quad t \in [0,T].$$

([6]) For this kind of equations we could repeat almost the same previous remarks; *see* e.g. [19], [16]; we shall detail our results in Sect. 4.

We consider the following:

PROBLEM 1. – *Given* $u_0 \in H$ *and* $L :]0,T[\mapsto V'$ *find* $u : [0,T] \mapsto V$ *such that for a.e.* $t \in]0,T[$

$$(PP') \qquad \begin{cases} u(t) \in V_t, \\ \big(u'(t), v\big) + a(t; u(t), v) = \big(L(t), v\big), \quad \forall\, v \in V_t, \\ u(0) = u_0. \end{cases}$$

We have already said in the introduction that the existence of a weak solution of (PP') is proved in [7] ([7]) whereas a stronger solution can be found in [10] assuming that the V_t are non decreasing; in this case [21] gives some other results of regularity and proves the convergence of a penalization scheme for (PP').

We follow a different procedure, requiring some compatibility and regularity conditions on the V_t-family and the bilinear forms $a(t;\cdot,\cdot)$. First of all we assume that:

$$(H1) \qquad \begin{cases} \text{for every } t \in [0,T] \ a(t;\cdot,\cdot) \text{ is } \textit{symmetric and coercive} \text{ on } V: \\ \exists\, \alpha > 0 : \ a(t;u,u) \geq \alpha \|u\|^2, \quad \forall\, u \in V \end{cases}$$

and we impose a one-side control on the time dependence of a:

$$(H2) \qquad \begin{cases} \text{there exists a bounded measure } \mu \text{ on } [0,T] \text{ such that} \\ a(t;u,u) - a(s;u,u) \leq \mu\big(]s,t]\big)\|u\|^2, \qquad \forall\, u \in V, \quad 0 \leq s \leq t \leq T. \end{cases}$$

We shall see later how these two conditions could be relaxed; we just note that $(H2)$ allows non increasing quadratic forms.

1.1. REMARK. – It is easy to see that $(H1\text{-}2)$ imply the uniform boundedness of the family $a(t;\cdot,\cdot)$:

$$(1.1) \qquad \exists\, \beta > 0 : \ a(t;u,v) \leq \beta \|u\| \, \|v\|, \quad \forall\, u, v \in V.$$

Moreover, we shall show that there exists a countable set S_a such that, for every choice of $u, v \in V$, the mapping

$$(1.2) \qquad t \mapsto a(t;u,v) \text{ is continuous for every } t \in [0,T] \setminus S_a.$$

In particular $a(t;\cdot,\cdot)$ is weakly measurable (*see* [26]). □

([7]) The basic assumption of this work, besides the coercivity of a, is the existence of a closed vector space $\hat{V}$ contained in each V_t, such that $(\hat{V}, V')_{1/2,2} = H$.

We consider now the behavior of V_t, via the following construction. To every functional L in V' and to every time t we associate the unique solution (thanks to the coercivity assumption $(H1)$) $u = u_L(t)$ of

$$u \in V_t; \quad a(t; u, v) = (L, v), \quad \forall v \in V_t \tag{1.3}$$

and the corresponding residual $R = R_L(t)$ in V'

$$\big(R, v\big) = a(t; u, v) - \big(L, v\big), \qquad \forall v \in V. \tag{1.4}$$

We will assume that the restriction of R on a suitable subspace of V_s with $s < t$ is of the same order of $t - s$ as $s \to t_-$, if L belongs to a space W "better" than V'.

Therefore we fix a Hilbert space W between H and V':

$$H \subset W \subset V'$$

and we denote by D_t the domain in V_t of the bilinear form $a(t; \cdot, \cdot)$ with respect to W:

$$D_t = \{u \in V_t : \ a(t; u, v) = (L, v), \forall v \in V_t \text{ with } L \in W\} \tag{1.5}$$

which is an Hilbert space if it is endowed with its natural norm

$$\|u\|_{D_t} = \inf\big\{\|L\|_W, \ L \text{ satisfying } (1.5)\big\}. \tag{1.6}$$

We assume

$$(H3) \qquad \begin{cases} \text{There exist a positive number } K \text{ and a } \theta \in]0, 1] \text{ such that} \\ \text{for every } L \in W, \ \ t \in]0, T], \ \ v \in D_s \text{ with } s \leq t, \\ \big(R_L(t), v\big) \leq K(t - s)\|L\|_W \, \|v\|^\theta \, \|v\|_{D_s}^{1-\theta}. \end{cases}$$

1.2. Remark. – If V_t are not decreasing, then $(H3)$ is trivially satisfied. On the other hand, it is interesting to study what kind of better properties follow by assuming that $(H2\text{-}3)$ hold also for $s > t$ (with the obvious changes, of course). We shall refer to this case as $(H2')$ and $(H3')$ respectively. □

1.3. Remark. – In the previous formula we can restrict s in the range $[t - h_0, t]$ for a fixed $h_0 > 0$; this will be useful in order to apply the estimates of [34]. □

1.4. Remark. – Thanks to a standard interpolation result ([8]) $(H3)$ is equivalent to

$$\big(R_L(t), v\big) \leq K(t - s)\|L\|_W \, \|v\|_{(D_s, V)_{\theta,1}}, \quad \forall v \in (D_s, V)_{\theta,1} \quad \square$$

([8]) We use the real interpolation functor of J.-L. Lions and J. Peetre [30] $(\cdot,\cdot)_{\theta,p}$: *see* [9], [14].

1.5. REMARK. – We can give another version of $(H3)$ assuming for simplicity a independent of time and $W \equiv H$. Define D as in (1.5) with the whole V instead of V_t and substitute (1.3) with:

$$(1.7)\qquad \begin{cases} \tilde u \in V: & a(\tilde u, v) = (L, v), \quad \forall v \in V; \\ u = u_L(t) \in V_t: & a(u(t) - \tilde u, v) = 0, \quad \forall v \in V_t. \end{cases}$$

We are taking the projection of $\tilde u$ on V_t with respect to the scalar product $a(\cdot,\cdot)$: let us denote by P_t this linear operator, which maps D onto D_t too. Thanks to the properties of P_t $(H3)$ becomes

$$(1.8)\qquad \begin{aligned} a(P_t\tilde u - \tilde u, v) &= a(P_t\tilde u - \tilde u, v - P_t v) = a(\tilde u, v - P_t v) \\ &\le K(t-s)\|\tilde u\|_D \|v\|_{(D_s,V)_{\theta,1}}. \end{aligned}$$

Since

$$|v| = \sup_{u\in D, u\neq 0} \frac{a(v,u)}{\|u\|_D}, \qquad \forall v \in V,$$

the previous formula can be rewritten in the more readable form

$$(1.9)\qquad v \in (D_s, V)_{\theta,1} \;\Rightarrow\; |v - P_t v| \le K(t-s)\|v\|_{(D_s,V)_{\theta,1}}, \quad s < t.$$

When $(H3')$ holds too, as in [34] we deduce that

$$\begin{aligned} \alpha\,\|P_t\tilde u - P_s\tilde u\|^2 &\le a(P_t\tilde u - P_s\tilde u, P_t\tilde u - P_s\tilde u) \\ &= a(\tilde u - P_s\tilde u, P_t\tilde u) + a(\tilde u - P_t\tilde u, P_s\tilde u) \le 2\,K\,|t-s|\,\|\tilde u\|_D^{2-\theta}\,\|\tilde u\|^\theta \end{aligned}$$

for every s and t in $[0,T]$, $\tilde u \in D$. □

$(H3)$ has interesting (and, in a certain sense, unexpected) consequences on the "regularity" of the family V_t, which better clarify some properties of the solution of (PP') we shall see in a moment. Following [25], we define

$$s\liminf_{t\to t_0} V_t \ \text{ as the set of the limits of the families } v_t \in V_t \text{ as } t \to t_0$$

and

$$s\limsup_{t\to t_0} V_t \ \text{ as the set of the cluster points of the families } v_t \in V_t \text{ as } t \to t_0$$

in the strong topology of V. Replacing "strong" by "weak" we obtain the corresponding notions of $w\liminf_{t\to t_0}$, $w\limsup_{t\to t_0}$; we use the symbol of limit when the two sets are equal. The definition of the left and right limits are straightforward as well as the following inclusions

$$\liminf_{s\to t} V_s \subset \limsup_{s\to t} V_s, \qquad \text{(both with } s \text{ or } w\text{)},$$
$$s\liminf_{s\to t} V_s \subset w\liminf_{s\to t} V_s, \qquad s\limsup_{s\to t} V_s \subset w\limsup_{s\to t} V_s.$$

We have

THEOREM 1. – *If* $(H1\text{-}3)$ *hold, then for all* t *we have:*

$$w\limsup_{s\to t^-} V_s \subset V_t \subset s\liminf_{s\to t^+} V_s. \tag{1.10}$$

Moreover the family V_t *is strongly* V*-measurable in the sense that* (see [17]).

$$\forall u \in V \text{ the mapping } [0,T] \ni t \mapsto d(u,V_t) = \inf_{v\in V_t} \|u-v\| \text{ is measurable.} \tag{1.11}$$

1.6. REMARK. – Thanks to the general results of [17], it would not be difficult to show that in (1.10) we can replace the inclusions with identities for a.e. $t \in]0,T[$. □

1.7. REMARK. – In the case of a non decreasing family of spaces

$$s \le t \quad \Rightarrow \quad V_s \subset V_t$$

(1.10) becomes obvious since

$$\lim_{s\to t^-} V_s = \overline{\bigcup_{s<t} V_s}; \qquad \lim_{s\to t^+} V_s = \bigcap_{s>t} V_s \quad \square$$

1.8. REMARK. – If $(H2'\text{-}3')$ hold then we easily deduce

$$\lim_{s\to t} V_s = V_t, \qquad \forall t \in [0,T] \tag{1.12}$$

both in the strong and in the weak topology of V. □

We can prove:

THEOREM 2. – *(Existence.) With the previous hypotheses* $(H1\text{-}3)$ *let us assume that*

$$(D1)\qquad L = f + g, \qquad f \in L^2(0,T;H), \quad g \in L^2(0,T;W)\cap W^{1,1}(0,T;V') \quad (^9)$$

and

$$(D2)\qquad u_0 \in V_0.$$

(9) For a generic Hilbert space $\mathcal{H}$, $L^p(0,T;\mathcal{H})$, $1 \le p \le \infty$, is the Banach space of the (strongly) measurable $\mathcal{H}$-valued functions whose $\mathcal{H}$-norm is in $L^p(0,T)$; the corresponding (first order) Sobolev spaces are:

$$W^{1,p}(0,T;\mathcal{H}) = \{f \text{ absolutely continuous in } [0,T] \text{ with values in } \mathcal{H} : f' \in L^p(0,T;\mathcal{H})\}.$$

As usual $H^1(0,T;\mathcal{H}) = W^{1,2}(0,T;\mathcal{H})$.

Then Problem 1 admits a unique solution $u \in H^1(0,T;H) \cap L^\infty(0,T;\mathcal{V})$ ([10]). *Moreover* u *satisfies*

$$(1.13) \qquad u(t) \in V_t \ \textit{for every}\ t \in]0,T].$$

1.9. REMARK. – We shall show that $(D2)$ can be replaced by the weaker

$$(D2') \qquad u_0 \in V_0^+ = w \limsup_{s \to 0^+} V_s \quad \square$$

1.10. REMARK. – Let us recall that the functions of $H^1(0,T;H) \cap L^\infty(0,T;V)$ are continuous with respect to the *weak* topology of V so that the range of the trace operator

$$H^1(0,T;H) \cap L^\infty(0,T;\mathcal{V}) \ni v \mapsto v(0) \in V$$

is contained in V_0^+. From Theorem 2 and the previous remark it follows that this operator is a surjection on V_0^+ and by (1.13) we have:

$$V_0^+ = w \liminf_{s \to 0^+} V_s = w \lim_{s \to 0^+} V_s \quad ([11]) \quad \square$$

1.11. REMARK. – Let us point out that from the equation we easily read that u belongs to

$$(1.14) \qquad L^2(0,T;\mathcal{D}) = \begin{cases} u \in L^2(0,T;\mathcal{V}) \text{ such that } \exists\, L \in L^2(0,T;W) \text{ with} \\ a(t;u(t),v) = \big(L(t),v\big),\ \forall v \in V_t, \text{ for a.e. } t \in]0,T[. \end{cases} \quad \square$$

1.12. REMARK. – Theorem 2 shows a natural "semigroup" property for the solution u of (PP'): if we split the interval $[0,T]$ into $[0,s]$ and $[s,T]$, the restriction of u to the second interval can be recovered solving (PP') in $[s,T]$ with respect to the initial datum given by the right trace of $u|_{[0,s]}$ at s. $\square$

As we said in the introduction, we approximate the solution of Problem 1 by the backward Euler method: we divide the interval $]0,T]$ in κ subintervals

$$I_\tau^n =](n-1)\tau, n\tau], \qquad n = 1,\dots,\kappa,$$

of equal size $\tau = T/\kappa$ and we look for a sequence $\{u_\tau^n\}_{n=0,1,\dots,\kappa}$ of points of V which is a suitable approximation of the values of u at the nodes $n\tau$.

With this aim we consider the sequence of variational problems $(n = 0,\dots,\kappa)$

$$(AP_n) \qquad \begin{cases} \text{Find } u_\tau^n \in V_{n\tau} \text{ such that.} \\ u_\tau^{-1} = u_0,\ L_\tau^0 = 0. \\ \left(\dfrac{u_\tau^n - u_\tau^{n-1}}{\tau}, v\right) + a_\tau^n(u_\tau^n, v) = \big(L_\tau^n, v\big), \qquad \forall v \in V_{n\tau}. \end{cases}$$

([10]) We write $L^p(0,T;\mathcal{V})$, $p \in [1,\infty]$, for the space $\{u \in L^p(0,T;V) : u(t) \in V_t \text{ a.e.}\}$.

([11]) This property holds for every right limit of the family $\{V_t\}_{t \in [0,T]}$, of course.

where

$$(1.15)\qquad a_\tau^n(\cdot,\cdot)=a(n\tau;\cdot,\cdot)\qquad\text{and}\qquad L_\tau^n=\frac{1}{\tau}\int_{I_\tau^n}L(t)\,dt\in W.$$

The coercivity assumption ensures that (AP_n) can be uniquely solved so that it defines recursively the sequence u_τ^n; we introduce the piecewise constant and linear interpolant of the values $\{u_\tau^n\}$

$$(1.16)\qquad u_\tau(t)=u_\tau^n,\quad \hat u_\tau(t)=(t/\tau-n+1)u_\tau^n+(n-t/\tau)u_\tau^{n-1},\qquad\text{if}\quad t\in I_\tau^n$$

and we have:

THEOREM 3. – *(Approximation) With the same hypotheses of the previous theorem, as τ goes to 0 $\hat u_\tau$ converges to the solution u of Problem 1 in the "energy norm" of $C^0(0,T;H)\cap L^2(0,T;V)$ and in the weak* topology of $H^1(0,T;H)\cap L^\infty(0,T;V)$. Moreover $u_\tau(t)$ and $\hat u_\tau(t)$ converge to $u(t)$ in V for every V-continuity point t of u in* $[0,T]\setminus S_a$ (see (1.2)).

1.13. REMARK. – For other approximation results see the next section; following the approach of [21] it is possible to give a more precise estimate of the convergence in the energy norm. We also observe that the scheme (AP_n) requires neither a preliminary regularization procedure of the family V_t nor a penalization technique. Of course the estimates are strongly dependent on $(H3)$. □

We give now other information about the regularity:

THEOREM 4. – *(Regularity) The solution u given by the previous theorems belongs also to $B^{1/2}_{2\,\infty}(0,T;V)$* ([12]) *and it is* right *continuous with respect to the* strong *topology of V at every point of $[0,T[$, the discontinuity set being at most countable. Moreover, if $(H2'$-$3')$ hold, then u is strongly V-continuous in the whole interval $[0,T]$.*

1.14. REMARK. – A simple consequence of this result is:

$$\exists\lim_{s\to t^+}V_s,\qquad\forall t\in[0,T[,$$

both in the strong and in the weak topology of V. □

([12]) For $0<s<1$, $B^s_{2\infty}(0,T;\mathcal{H})$ can be defined as the Banach space of the $L^2(0,T;\mathcal{H})$-functions v such that the seminorm

$$[v]^2_{B^s_{2\infty}}=\sup_{0<h<T}\int_h^T\left\|\frac{v(t)-v(t-h)}{h^s}\right\|^2_{\mathcal{H}}dt$$

is finite; as usual, the norm of this space is obtained by adding the $L^2(0,T;\mathcal{H})$-one. We recall that an equivalent definition follows by real interpolation

$$B^s_{2\,\infty}(0,T;\mathcal{H})=\big(L^2(0,T;\mathcal{H}),H^1(0,T;\mathcal{H})\big)_{s,\infty}$$

with the continuous inclusions

$$H^s(0,T;\mathcal{H})\subset B^s_{2\,\infty}(0,T;\mathcal{H})\subset H^{s-\varepsilon}(0,T;\mathcal{H}),\qquad\forall\varepsilon>0.$$

We refer to [36], [33] for analogous examples of this kind of intermediate regularity in the framework of abstract evolution equations and inequalities.

We make a few comments about some easy extensions of these Theorems:

1.15. EXTENSION. – The assumptions ($H1$-2) on the bilinear form a can be weakened assuming that

$$\begin{cases} a = a_0 + a_1, & a_0 \text{ satisfying } (H1\text{-}2) \text{ and} \\ a_1 \text{ being } \textit{uniformly bounded} \text{ on } V \times H; \end{cases} \tag{1.17}$$

in particular we can consider the case of a weakly coercive bilinear form. Observe that we can limit ourselves to check ($H3$) only on the principal part a_0. The proof of this case follows by the usual method of continuity in a parameter (*see* [6], Sect. 5 for a similar application). □

1.16. EXTENSION. – In ($H3$) the term $K(t-s)$ can be substituted by the integral

$$\int_s^t \rho(\xi)\, d\xi \quad \text{for a fixed non negative function} \quad \rho \in L^{2/\theta}(0,T). \tag{1.18}$$

Our simpler initial choice corresponds obviously to $\rho \in L^\infty(0,T)$. □

1.17. EXTENSION. – Following [21] we could also replace $\|L\|_W$ in ($H3$) by different intermediate norms between W and V', obtaining better summability exponents in (1.18), but requiring stronger "elliptic" estimates. In order to fix our ideas, let us assume $W \equiv H$ and substitute the last line of ($H3$) by:

$$\big(R_L(t), v\big) \le (\|L\|_{V'}^{\sigma}\|L\|_H^{1-\sigma})(\|v\|^{\theta}\|v\|_{D_s}^{1-\theta}) \int_s^t \rho(\xi)\, d\xi \tag{1.19}$$

with $\theta, \sigma \in [0,1]$, $\theta + \sigma > 0$; in this case we can allow

$$\rho \in L^{2/(\sigma+\theta)}(0,T). \tag{1.20}$$

In the framework of Remark 1.5 (1.19) can be rewritten as:

$$v \in \big(D_s, V\big)_{\theta,1} \Rightarrow \|v - P_t v\|_{(H,V)_{\sigma,\infty}} \le \|v\|_{(D_s,V)_{\theta,1}} \int_s^t \rho(\xi)\, d\xi, \tag{1.21}$$

obtaining a finer scale of conditions in order to evaluate the time dependence of the projectors P_t. Of course, combinations of the various assumptions are possible. □

1.18. EXTENSION. – In ($D1$) we could replace the absolutely continuous functions of $W^{1,1}(0,T;V')$ by the bounded variation ones of $BV(0,T;V')$ (as in [8] and [33]); since we are also interested in the V-continuity properties of the solution, we do not insist with this setting. □

2. Preliminary results

The aim of this section is to prove some properties of the bilinear forms $a(t;\cdot,\cdot)$ and of the family of spaces $\{V_t\}_{t\in[0,T]}$.

In order to have a shorter notation we denote by $a(s;\cdot)$ the quadratic form associated to $a(s;\cdot,\cdot)$

$$a(s;u)=a(s;u,u);$$

we also assume that the three imbeddings

$$V\hookrightarrow H\hookrightarrow W\hookrightarrow V'$$

have norms ≤ 1 and in our arguments we take account of extension 1.16; (1.19) only requires minor changes, as detailed in [21].

2.1. Proposition. – *Assume* $(H1\text{-}2)$*; then there exists a countable set* $S_a\subset[0,T]$ *such that*

$$[0,T]\ni t\mapsto a(t;u,v)\ \textit{are continuous in}\ [0,T]\setminus S_a,\quad \forall\, u,v\in V. \tag{2.1}$$

Proof. – Let M be the countable set of $]0,T]$ where "μ jumps":

$$M=\{t\in]0,T]:\mu\{t\}>0\}.$$

We note that for every choice of $v\in V$ the mapping

$$[0,T]\ni t\mapsto a(t;v)+\mu(]0,t])\|v\|^2$$

is not increasing so that it is continuous except at a countable set; denoting by G_v the union of this set with M, the map $t\mapsto a(t;v)$ is surely continuous outside G_v.

Let us fix now a countable dense subset $\tilde V$ of the unit closed ball of V and define

$$S_a=\bigcup_{v\in\tilde V}G_v.$$

The family $\{t\mapsto a(t;v)\}_{v\in V,\|v\|\leq 1}$ is the closure of $\{t\mapsto a(t;v)\}_{v\in\tilde V}$ in the topology of the uniform convergence, so that its elements are continuous outside S_a. By homogeneity we deduce the same property for every $a(t;v)$, $v\in V$, and by the standard polarization identity ([13]) we prove it also for the associated symmetric bilinear form. ■

([13]) That is

$$a(t;u,v)=\frac{1}{4}\big[a(t;u+v)-a(t;u-v)\big],\qquad \forall\, u,v\in V.$$

2.2. REMARK. – As a consequence of the previous proof we find that for every $u, v \in V$ there exists the limit

$$\lim_{s \to t^+} a(s; u, v) = \bar{a}(t; u, v)$$

and it defines a bounded symmetric bilinear form which coincides with $a(t; u, v)$ outside S_a, is right continuous and satisfies:

$$\bar{a}(t; u) \leq a(t; u); \quad a(t; u) - \bar{a}(s; u) \leq \mu(]s, t])\|u\|^2, \qquad \forall u \in V,\ s \leq t. \quad \square$$

We can obtain a sort of uniformity of the limit outside S_a:

2.3. PROPOSITION. – *Let $t \notin S_a$ be a "regular" point for a and let $t_n \in [0, T]$, $u_n \in V$ be two sequences such that:*

$$\lim_{n \to \infty} t_n = t, \qquad \|u_n\| \textit{ is bounded by a constant } U < +\infty.$$

Then for any $v \in V$

$$\lim_{n \to \infty} |a(t_n; u_n, v) - a(t; u_n; v)| = 0. \tag{2.2}$$

Proof. – We observe that the bilinear form

$$q(u, v) = a(s; u, v) - a(t; u, v) + \mu(]s, t])\,(u, v)_V, \qquad s \leq t,$$

is positive by $(H2)$ so that by Schwarz inequality $|q(u, v)|^2 \leq q(u, u) q(v, v)$ we get:

$$\begin{aligned} &|a(s; u, v) - a(t; u, v)| \\ &\quad \leq \mu(]s, t])\|u\|\|v\| + \left[a(s; u) - a(t; u) + \mu(]s, t])\|u\|^2\right]^{1/2} \\ &\qquad \times \left[a(s; v) - a(t; v) + \mu(]s, t])\|v\|^2\right]^{1/2}. \end{aligned}$$

In our situation, denoting by I_n the interval limited by t and t_n, we obtain

$$\begin{aligned} &|a(t_n; u_n, v) - a(t; u_n, v)| \\ &\quad \leq U\|v\|\mu(I_n) + U\sqrt{2\beta + \mu(I_n)}[|a(t; v) - a(t_n; v)| + \mu(I_n)\|v\|^2]^{1/2}. \end{aligned}$$

As $n \to \infty$ we have

$$\mu(I_n) \to 0, \quad |a(t; v) - a(t_n; v)| \to 0$$

since $t \notin S_a \supset M$; we conclude. ■

2.4. COROLLARY. – *With the same notation of the previous Proposition, let us suppose that $u_n \rightharpoonup u$ in V. Then*

$$\liminf_{n \to \infty} a(t_n; u_n) \geq a(t; u) \tag{2.3}$$

and

$$\limsup_{n\to\infty} a(t_n;u_n) \le a(t;u) \quad \Rightarrow \quad \lim_{n\to\infty} \|u_n - u\| = 0. \tag{2.4}$$

Proof. – Since $t \notin S_a$, the difference between $a(t_n;u_n)$ and $a(t;u)$ has the same behavior of

$$a(t_n;u_n) - a(t_n;u) \tag{2.5}$$

as n goes to ∞. Now we write

$$\begin{aligned} a(t_n;u_n) - a(t_n;u) &= a(t_n;u_n - u) + 2\,a(t_n,u,u_n - u) \\ &\ge \alpha\|u_n - u\|^2 + 2[a(t_n;u,u_n-u) - a(t;u,u_n-u)] \\ &\quad + 2\,a(t;u,u_n-u) \end{aligned}$$

and apply (2.2) together with the weak convergence of u_n. ■

2.5. REMARK. – In the previous two results, if we replace a by $\bar{a}$ (*see* Remark 2.2) and we impose t_n greater than t, then (2.2), (2.3) and (2.4) hold for *every* $t \in [0,T[$. □

Now we study the measurability properties of the family $\{V_t\}$, by using an approximation procedure based on the family of linear operators $\{J^\varepsilon(t);\ \varepsilon > 0,\ t \in [0,T]\}$, which send an element v of H into the solution $v^\varepsilon(t)$ of

$$v^\varepsilon(t) \in V_t; \quad \big(v^\varepsilon(t) - v, w\big) + \varepsilon a(t;v^\varepsilon(t),w) = 0, \qquad \forall\, w \in V_t. \tag{2.6}$$

These estimates are well known (*see* [28]).

2.6. LEMMA. – *For any choice of ε, t we have $v^\varepsilon(t) \in D_t$ with:*

$$\begin{cases} |v^\varepsilon(t)|^2 + 2\alpha\varepsilon\|v^\varepsilon(t)\|^2 \le |v|^2, \qquad \|v^\varepsilon\|_{D_t} \le \left|\dfrac{v^\varepsilon - v}{\varepsilon}\right|; \\ v \in \overline{V_t}^H \ \Rightarrow\ \lim\limits_{\varepsilon\to 0} |v^\varepsilon(t) - v| = 0; \\ v \in V_t \ \Rightarrow\ \dfrac{2}{\varepsilon}|v^\varepsilon(t) - v|^2 + a(t;v^\varepsilon(t)) \le a(t;v), \qquad \lim\limits_{\varepsilon\to 0} \|v^\varepsilon(t) - v\| = 0. \end{cases} \tag{2.7}$$

In particular, D_t is dense in V_t, for all t. Finally, if $v \in D_s$ with $t - h_0 \le s \le t$ we have:

$$\varepsilon^{-1}|v^\varepsilon(t) - v|^2 + a(t;v^\varepsilon(t)) + \alpha\|v^\varepsilon(t) - v\|^2 \le a(t;v) + E^2(t-s)\|v\|^{2\theta}\,\|v\|_{D_s}^{2-2\theta} \tag{2.8}$$

with $E^2 = \int_0^T \rho^2(\xi)\,d\xi$ (see 1.16).

Proof. – We have to prove only this last formula; by $(H3)$ with $L = -(v^\varepsilon - v)/\varepsilon$ we obtain

$$\begin{aligned}2\varepsilon\left|\frac{v^\varepsilon(t)-v}{\varepsilon}\right|^2 &+ a(t;v^\varepsilon(t)) + a(t;v^\varepsilon(t)-v)\\ &\le a(t;v) + 2\left|\frac{v^\varepsilon(t)-v}{\varepsilon}\right| \|v\|_{D_s}^{1-\theta}\|v\|^\theta \int_s^t \rho(\xi)\,d\xi\\ &\le a(t;v) + E^2(t-s)\|v\|^{2\theta}\,\|v\|_{D_s}^{2-2\theta} + \varepsilon\left|\frac{v^\varepsilon(t)-v}{\varepsilon}\right|^2. \qquad \blacksquare\end{aligned}$$

2.7. COROLLARY. – *For every $t \in [0,T[$ we have:*

$$V_t \subset s\liminf_{s\to t^+} V_s.$$

Proof. – It is sufficient to show that $D_t \subset s\liminf_{s\to t^+} V_s$; for a given $v \in D_t$ we choose $v_h = J^h(t+h)v$ and we deduce that $v_h \to v$ as $h\to 0^+$ by applying the last formula and taking into account Remark 2.5. ■

A consequence of these estimates is the following theorem:

2.8. THEOREM. – *Let us given a sequence $v_n \in V_{t_n}$ such that*

$$t_n \le t, \quad t_n \to t \in]0,T]; \quad v_n \rightharpoonup v \in V. \tag{2.9}$$

Then v belongs to V_t. In other words,

$$w\limsup_{s\to t^-} V_s \subset V_t.$$

Proof. – Let us set $v_n^\varepsilon = J^\varepsilon(t_n)v_n$, $v^\varepsilon = J^\varepsilon(t)v$; by $(H3)$ in the modified form of 1.16 we have the estimate

$$(v^\varepsilon - v, v^\varepsilon - v_n^\varepsilon) + \varepsilon a(t;v^\varepsilon, v^\varepsilon - v_n^\varepsilon) \le \varepsilon\alpha^{-\theta}\left|\frac{v^\varepsilon - v}{\varepsilon}\right|\left|\frac{v_n^\varepsilon - v_n}{\varepsilon}\right|\int_{t_n}^t \rho(\xi)\,d\xi.$$

Now we write $-v_n^\varepsilon$ in the first term as $-v + (v - v_n) + (v_n - v_n^\varepsilon)$ obtaining

$$\begin{aligned}\frac{1}{2}|v^\varepsilon - v|^2 &+ \frac{\varepsilon}{2}a(t;v^\varepsilon)\\ &\le \frac{\varepsilon}{2}a(t;v_n^\varepsilon) + \frac{1}{2}|v_n - v_n^\varepsilon|^2 + (v^\varepsilon - v, v_n - v)\\ &\quad + \varepsilon\alpha^{-\theta}\left|\frac{v^\varepsilon - v}{\varepsilon}\right|\left|\frac{v_n^\varepsilon - v_n}{\varepsilon}\right|\int_{t_n}^t \rho(\xi)\,d\xi.\end{aligned}$$

We pass to the limit as $n \to \infty$ in the right hand member, observing that the last two terms go to 0 whereas

$$|v^n - v_n^\varepsilon|^2 + \varepsilon a(t; v_n^\varepsilon) \leq C\varepsilon \|v_n\|^2,$$

with C independent of ε and n. We deduce that

$$|v^\varepsilon - v|^2 + \varepsilon a(t; v^\varepsilon) \leq C\varepsilon \liminf_{n\to\infty} \|v_n\|^2$$

and v^ε converges to v in H and weakly in V, as $\varepsilon \to 0$, since $\|v_n\|$ is bounded; being V_t (weakly) closed, we conclude. ■

2.9. COROLLARY. – *The mapping*

$$[0, T] \ni t \mapsto d(u, V_t) = \inf_{v \in V_t} \|u - v\|$$

is measurable for all $u \in V$.

Proof. – We shall show that the functions $t \mapsto d(u, V_t)$ are *left* lower semicontinuous and therefore measurable ([14]).

Let us fix u in V, $t \in]0, T[$ and choose $u(s) \in V_s$, $s < t$ so that

$$\|u - u(s)\| \leq d(u, V_s) + t - s;$$

$u(s)$ is bounded in V; hence from every sequence s_n converging to t from the left as $n \to \infty$ we can extract a subsequence (still denoted by s_n) such that $u(s_n)$ weakly converges to $\bar{u}$. By the previous Theorem, $\bar{u}$ belongs to V_t and we obtain

$$d(u, V_t) \leq \|u - \bar{u}\| \leq \liminf_{n\to\infty} \|u - u(s_n)\| = \liminf_{n\to\infty} d(u, V_{s_n})$$

by the lower semicontinuity of the V-norm with respect to the weak convergence. ■

- *The proof of* Theorem 1 *is then complete.*

The importance of this last property is highlighted by the following result:

2.10. PROPOSITION. – *Assume that v is an H-valued measurable function; then the map*

$$t \mapsto [J^\varepsilon v](t) = J^\varepsilon(t) v(t)$$

is also measurable ([15]).

([14]) A left lower semicontinuous function $f :]0, T[\mapsto \mathbb{R}$ is measurable since the inverse images

$$F_c = \{t \in]0, T[: f(t) > c\}$$

are left-open subsets of $]0, T[$, in the sense that

$$x \in F_c \Rightarrow \exists \varepsilon > 0 :]x - \varepsilon, x] \subset F_c.$$

Now, a left-open set is a countable union of a family of left-open intervals (the connected components) and consequently it is measurable.

([15]) In this case strong and weak measurability coincide by Pettis' theorem.

Proof. – For a constant bilinear form $a(\cdot,\cdot)$ this property follows from the general results of [17], being $J^\varepsilon v(t)$ the pointwise projection on V_t with respect to the scalar product:

$$(u,v)+\varepsilon a(u,v) \tag{2.10}$$

of the V-measurable function $\tilde{v}$ defined as in (1.7) with (2.10) instead of $a(\cdot,\cdot)$. In the case of a time dependent form, we consider the step function

$$t_\tau \equiv n\tau \quad \text{on } I_\tau^n$$

and we define z_τ as

$$z_\tau(t)\in V_t;\quad (z_\tau(t)-v(t),w)+\varepsilon a(t_\tau;z_\tau(t),w)=0,\qquad \forall w\in V_t,$$

which is measurable by the previous remark, being t_τ constant on I_τ^n. Applying 2.3 we easily find that for $t\notin S_a$ $z_\tau(t)$ weakly converges to $[J^\varepsilon v](t)$ as $\tau\to 0$. ■

2.11. Corollary. – *For all $p\in[1,\infty]$ J^ε maps $L^p(0,T;H)$ into $L^p(0,T;\mathcal{D})$; moreover, if v belongs to $L^p(0,T;\mathcal{V})$, $p<\infty$, then $J^\varepsilon v$ converges to v in $L^p(0,T;\mathcal{V})$ and a.e. as $\varepsilon\to 0$. In particular $L^p(0,T;\mathcal{D})$ is dense in $L^p(0,T;\mathcal{V})$.*

3. Proof of Theorems 2-4

Let u_τ^n be defined by (AP_n) and let us consider the functions u_τ and $\hat{u}_\tau$ as in (1.16); we want to show that as τ goes to 0 there exists the limit of $\hat{u}_\tau$ in the weak* topology of $H^1(0,T;H)\cap L^\infty(0,T;V)$ and it defines the solution of (PP'). The following Proposition gives the basic estimate we need:

3.1. Proposition. – *There exists a constant $C>0$ such that, for $\tau<h_0$ (16) we have:*

$$\left.\begin{array}{c}\displaystyle\int_0^T |\hat{u}_\tau'(t)|^2\,dt\\ \displaystyle\sup_{t\in[0,T]}\|\hat{u}_\tau(t)\|^2\end{array}\right\}\leq C[\|u_0\|^2+\|f\|^2_{L^2(0,T;H)}+\|g\|^2_{L^2(0,T;W)\cap W^{1,1}(0,T;V')}]. \tag{3.1}$$

(16) h_0 is given by Remark 1.3; from now on we shall denote by C the constants independent of the data and of the parameter τ.

Proof. – We observe that the solution u_τ^n of (AP_n) belongs to $D_{n\tau}$ so that we have, for $n \geq 1$ ([17]):

$$
\begin{aligned}
(3.2)\qquad & a_\tau^n(u_\tau^n, u_\tau^n - u_\tau^{n-1}) + \left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau} - L_\tau^n, u^n - u_\tau^{n-1}\right) \\
&= a_\tau^n(u_\tau^n, -u_\tau^{n-1}) + \left(\frac{u_\tau^n - u_\tau^{n-1}}{\tau} - L_\tau^n, -u_\tau^{n-1}\right) \\
&\leq \tau\rho_\tau^n \left\|\frac{u_\tau^n - u_\tau^{n-1}}{\tau} - L_\tau^n\right\|_W \|u_\tau^{n-1}\|^\theta \left\|\frac{u_\tau^{n-1} - u_\tau^{n-2}}{\tau} - L_\tau^{n-1}\right\|_W^{1-\theta},
\end{aligned}
$$

where we set (*see* 1.16)

$$
(3.3)\qquad \rho_\tau^n = \frac{1}{\tau}\int_{I_\tau^n} \rho(\xi)\, d\xi.
$$

The last term of (3.2) can be easily bounded by

$$
\begin{aligned}
&C\tau|\rho_\tau^n|^{2/\theta}\|u_\tau^{n-1}\|^2 + \frac{\tau}{8}\left(\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 + \left|\frac{u_\tau^{n-1} - u_\tau^{n-2}}{\tau}\right|^2\right) \\
&\quad + 4\tau(\|L_\tau^n\|_W^2 + \|L_\tau^{n-1}\|_W^2)
\end{aligned}
$$

whereas the first one is greater than ([18])

$$
\begin{aligned}
&\frac{3\tau}{4}\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 + \frac{1}{2}a_\tau^n(u_\tau^n) - \frac{1}{2}a_\tau^n(u_\tau^{n-1}) + \frac{1}{2}a_\tau^n(u_\tau^n - u_\tau^{n-1}) \\
&\quad - \tau|f_\tau^n|^2 - \tau\left(g_\tau^n, \frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right)
\end{aligned}
$$

Now setting $\mu_\tau^n = \frac{\mu(I_\tau^n)}{\tau}$ we can substitute the term $-\frac{1}{2}a_\tau^n(u_\tau^{n-1})$ in the last formula by

$$
-\frac{1}{2}a_\tau^{n-1}(u_\tau^{n-1}) - \tau\mu_\tau^n\|u_\tau^{n-1}\|^2.
$$

([17]) For $n = 0$ we simply have

$$
\frac{1}{2}a(0; u_\tau^0) + \frac{1}{2}a(0; u_\tau^0 - u_0) + \tau\left|\frac{u_\tau^0 - u_0}{\tau}\right|^2 = \frac{1}{2}a(0; u_0).
$$

([18]) With obvious notation, we split $L_\tau^n = f_\tau^n + g_\tau^n$.

Summing up from $n = 0$ (*see* note [17]) to $m \le \kappa$ we obtain:

$$\frac{\tau}{4}\sum_{n=0}^{m}\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 + \sum_{n=0}^{m}\frac{\alpha}{2}\|u_\tau^n - u_\tau^{n-1}\|^2 + \frac{\alpha}{2}\|u_\tau^m\|^2$$
$$\le \frac{1}{2}a(0;u_0) + \tau\sum_{n=1}^{m}[|f_\tau^n|^2 + \|g_\tau^n\|_W^2 + \big(\mu_\tau^n + |\rho_\tau^n|^{2/\theta}\big)\|u_\tau^{n-1}\|^2 + \big(g_\tau^n, u_\tau^n - u_\tau^{n-1}\big)].$$

Since

$$\tau\sum_{n=1}^{\kappa}(\mu_\tau^n + |\rho_\tau^n|^{2/\theta}) \le \mu(]0,T]) + \int_0^T |\rho(\xi)|^{2/\theta}\,d\xi,$$

by the application of a discrete version of the Gronwall lemma we find:

$$\frac{\tau}{4}\sum_{n=0}^{m}\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 + \frac{\alpha}{2}\sum_{n=0}^{m}\|u_\tau^n - u_\tau^{n-1}\|^2 + \frac{\alpha}{2}\|u_\tau^m\|^2$$
$$\le e^{CT}\left[\frac{1}{2}a(0;u_0) + \tau\sum_{n=1}^{m}(|f_\tau^n|^2 + \|g_\tau^n\|_W^2) + \sup_{1\le s\le m}\left|\sum_{n=1}^{s}\big(g_\tau^n, u_\tau^n - u_\tau^{n-1}\big)\right|\right]. \tag{3.4}$$

Now recalling that

$$\sum_{n=1}^{s}\big(g_\tau^n, u_\tau^n - u_\tau^{n-1}\big) = \big(g_\tau^s, u_\tau^s\big) - \big(g_\tau^1, u_\tau^0\big) - \sum_{n=1}^{s-1}\big(g_\tau^{n+1} - g_\tau^n, u_\tau^n\big)$$
$$\le \|g\|_{W^{1,1}(0,T;V')}\sup_{0\le n\le\kappa}\|u_\tau^n\|$$

and

$$\tau\sum_{n=1}^{\kappa}\Big(|f_\tau^n|^2 + \|g_\tau^n\|_W^2\Big) \le \int_0^T \big(|f(t)|^2 + \|g(t)\|_W^2\big)\,dt,$$

we obtain the final

$$\tau\sum_{n=0}^{\kappa}\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 + \alpha\sup_{0\le n\le\kappa}\|u_\tau^n\|^2 + \alpha\sum_{n=0}^{\kappa}\|u_\tau^n - u_\tau^{n-1}\|^2$$
$$\le C[\|u_0\|^2 + \|f\|_{L^2(0,T;H)}^2 + \|g\|_{L^2(0,T;W)}^2 + \|g\|_{W^{1,1}(0,T;V')}^2]. \tag{3.5}$$

Since $\hat u_\tau'(t) \equiv \frac{u_\tau^n - u_\tau^{n-1}}{\tau}$ if $t \in I_\tau^n$ we get

$$\tau\sum_{n=1}^{\kappa}\left|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right|^2 = \int_0^T |\hat u'(t)|^2\,dt$$

and analogously

$$\sup_{0\le n\le \kappa} \|u_\tau^n\| = \|\hat{u}_\tau\|_{L^\infty(0,T;V)} \ge \|u_\tau\|_{L^\infty(0,T;V)}.$$

In this way (3.1) is equivalent to (3.5). ■

3.2. COROLLARY. – *The families $\hat{u}_\tau$ and u_τ have at least one common weak* accumulation point $u \in L^\infty(0,T;V)$ which also belongs to $H^1(0,T;H)$; moreover we have:*

$$\left.\begin{array}{l} \lim\limits_{\tau\to 0} \|\hat{u}_\tau - u_\tau\|_{L^\infty(0,T;H)} \\ \lim\limits_{\tau\to 0} \|\hat{u}_\tau - u_\tau\|_{L^2(0,T;V)} \\ \lim\limits_{\tau\to 0} \|u_\tau(t) - u_\tau(t-\tau)\|_{L^2(0,T;V)} \end{array}\right\} = 0. \tag{3.6}$$

Proof. – It is sufficient to note that

$$u_\tau(t) - \hat{u}_\tau(t) = \tau \ell_\tau(t)\hat{u}_\tau'(t) = \ell_\tau(t)[u_\tau(t) - u_\tau(t-\tau)] \tag{3.7}$$

with $0 \le \ell_\tau(t) \le 1$; then we use (3.5). ■

Now we want to show that the function u given by this Corollary solves Problem 1.

To this end we observe that u_τ and $\hat{u}_\tau$ satisfy a suitable approximate problem; in order to describe it, we introduce the spaces (*see* ([9]) and 1.10) for $p \in [1,\infty]$

$$L^p(0,T;\mathcal{V}_\tau) = \{v \in L^p(0,T;V) \text{ such that } v(t) \in V_{n\tau}, \text{ for a.e. } t \in I_\tau^n\}$$

and

$$L^p(0,T;\mathcal{D}_\tau) = \begin{cases} u \in L^p(0,T;\mathcal{V}_\tau) \text{ such that } \exists\, L \in L^p(0,T;W) \text{ with} \\ a_\tau(t;u(t),v) = \big(L(t),v\big),\ \forall v \in V_{n\tau}, \text{ for a.e. } t \in I_\tau^n, \end{cases}$$

with the corresponding natural norms. With this notation u_τ and $\hat{u}_\tau$ satisfy

$$(AP_\tau) \qquad \begin{cases} u_\tau \in L^\infty(0,T;\mathcal{V}_\tau), \qquad \hat{u}_\tau \in H^1(0,T;H) \cap C^0([0,T];V), \\ \hat{u}_\tau(0) = u_0^\tau, \\ \displaystyle\int_0^T \big[\big(\hat{u}_\tau', {}^\tau v\big) + a_\tau(t;u_\tau,{}^\tau v) - \big(L_\tau,{}^\tau v\big)\big]\,dt = 0, \qquad \forall\, {}^\tau v \in L^1(0,T;\mathcal{V}_\tau), \end{cases}$$

where of course $L_\tau(t) \equiv L_\tau^n$ on I_τ^n. If we want to pass to the limit with respect to τ in (AP_τ) we have to answer the following questions:

[$Q1$] does u belong to $L^\infty(0,T;\mathcal{V})$?

[$Q2$] are all the elements of $L^1(0,T;\mathcal{V})$ approximable in the norm of $L^1(0,T;V)$ by a family of ${}^\tau v \in L^1(0,T;\mathcal{V}_\tau)$ so that they are admissible test functions in the limit formulation of (AP_τ)?

[$Q3$] can we pass to the limit in the bilinear term $a_\tau(t;u_\tau,{}^\tau v)$?

An affirmative reply to them gives immediately the proof of:

3.3. COROLLARY. – *Any weak* cluster point u solves the following weak form of Problem 1:*

$$(wPP')\qquad \begin{cases} u \in L^\infty(0,T;\mathcal{V}) \cap H^1(0,T;H), \\ u(0) = u_0, \\ \displaystyle\int_0^T \left[\left(u', v\right) + a(t;u,v) - \left(L, v\right) \right] dt = 0, \qquad \forall v \in L^1(0,T;\mathcal{V}). \end{cases}$$

Of course, it is not restrictive to assume $v \in L^\infty(0,T;\mathcal{V})$ in $[Q2]$ and in the last formula.

$[Q1]$ *$u(t)$ belongs to V_t for every $t \in [0,T]$.*

Proof. – We know that $u_\tau(t)$ weakly converges to $u(t)$ in V for all t. The case $t = 0$ being trivial, we can assume $t > 0$ and we observe that also $u_\tau(t-\tau)$ weakly converges to $u(t)$. We already denoted by

(3.8) $t_\tau = \tau \min\{n : t \le n\tau\}$, $s_\tau = t_\tau - \tau$, with the property $t \in]s_\tau, t_\tau] \in \{I_\tau^n\}_{n=1,\dots,\kappa}$

and we have $u_\tau(t-\tau) \in V_{s_\tau}$; now we can apply Theorem 2.8. ■

$[Q2]$ *For each function $v \in L^\infty(0,T;\mathcal{V})$, there exists a uniformly bounded family ${}^\tau v \in L^\infty(0,T;\mathcal{D}_\tau)$ converging a.e. to v; in particular ${}^\tau v \to v$ in $L^p(0,T;V)$ for all $p \in [1,\infty[$.*

Proof. – By 2.11 we can assume $v \in L^\infty(0,T;\mathcal{D})$. For the Lebesgue points $t \in I_\tau^n$ of v we define:

$${}^\tau v(t) \in V_{n\tau} \qquad \text{as} \quad J^\tau(n\tau)v(t) = J^\tau(t_\tau)v(t)$$

and we apply (2.8) obtaining a uniformly bounded family in $L^\infty(0,T;V)$ with

$${}^\tau v \to v \quad \text{in } L^\infty(0,T;H) \qquad \text{and} \qquad {}^\tau v(t) \rightharpoonup v(t) \quad \text{in } V.$$

We conclude if we show that ${}^\tau v(t)$ strongly converges to $v(t)$ for a.e. $t \in]0,T[$.

We apply the final estimate of Lemma 2.6 obtaining

$$\alpha \|{}^\tau v(t) - v(t)\|^2 \le a(t_\tau; v(t)) - a(t_\tau; {}^\tau v(t)) + C\tau$$

and we recall (2.3). ■

Finally we have:

$[Q3]$ *Let $v, {}^\tau v$ be given as in the previous $[Q2]$ and u as in Corollary 3.3; then we have*

$$\lim_{\tau \to 0} \int_0^T |a_\tau(t; u_\tau, {}^\tau v) - a(t; u, v)|\, dt = 0. \tag{3.9}$$

Proof. – Being the integrand in (3.9) uniformly bounded we have only to prove its a.e. convergence.

We know that there exists a negligible set $S \subset [0,T]$ such that (2.1) holds and ${}^\tau v(t) \to v(t)$ if $t \notin S$. For a given $t \notin S$ we bound the modulus in (3.9) by the sum:

$$|a_\tau(t; u_\tau, {}^\tau v - v)| + |a_\tau(t; u_\tau, v) - a(t; u_\tau, v)| + |a(t; u_\tau, v) - a(t; u, v)|.$$

The last term goes to 0 since $u_\tau(t)$ weakly converges to $u(t)$ in V; the same holds for the first one, by the strong convergence of ${}^\tau v(t)$. The estimate of

$$|a_\tau(t; u_\tau, v) - a(t; u_\tau, v)| = |a(t_\tau; u_\tau, v) - a(t; u_\tau, v)|$$

is given in 2.3. ■

It is a straightforward consequence that u solves also the pointwise formulation of Problem 1 (*see* i.e. [21]). Since at this level of regularity the uniqueness of the solution is immediate, we deduce that the whole family $\hat{u}_\tau$ converges to u in the weak* topology of $H^1(0,T;H) \cap L^\infty(0,T;V)$.

In order to see that u_0 can be chosen in V_0^+, let $u_n \in V_{t_n}$, $t_n \to 0$, be a sequence such that $u_n \rightharpoonup u$ in V; let us consider the corresponding solutions $u_n(t)$ of Problem 1 starting from the initial condition $u_n(t_n) = u_n$ and extended to the whole interval $[0,T]$ by setting

$$u_n(t) \equiv u_n, \quad \text{if } t \in [0, t_n[.$$

Of course $u_n(t)$ is uniformly bounded in $H^1(0,T;H) \cap L^\infty(0,T;V)$ and satisfies:

$$\begin{cases} u_n \in L^\infty(t_n, T; \mathcal{V}) \cap H^1(0,T;H), \qquad u_n(0) = u_n; \\ \left| \displaystyle\int_0^T \left[\left(u_n', v\right) + a(t; u_n, v) - \left(L, v\right) \right] dt \right| \leq C\sqrt{t_n} \|v(t)\|_{L^\infty(0,t_n;V)}, \\ \qquad \forall\, v \in L^\infty(0,T;\mathcal{V}) \end{cases}$$

It is easy to see that a weak* accumulation point u of u_n satisfies (wPP') and then (PP').

• *This concludes the proof of Theorem* 2 *and the related Remark* 1.9.

Theorem 3 is almost completely proved, too; the strong convergence in $L^2(0,T;V)$ and in $L^\infty(0,T;H)$ of $\hat{u}_\tau$ is a standard fact: choose $v = u_\tau$ in (AP_τ) and recall that by (3.7)

$$\left(\hat{u}_\tau'(t), u_\tau(t)\right) \geq \left(\hat{u}_\tau'(t), \hat{u}_\tau(t)\right), \qquad \text{a.e. in }]0,T[.$$

We obtain

$$\begin{aligned} (3.10) \quad & \frac{1}{2}|\hat{u}_\tau(T)|^2 + \int_0^T a(t; u_\tau(t))\, dt \\ & \leq \frac{1}{2}|u_0^\tau|^2 + \int_0^T \left(L_\tau(t), u_\tau(t)\right) dt + \int_0^T \left[a(t; u_\tau(t)) - a_\tau(t; u_\tau(t))\right] dt \end{aligned}$$

whereas u satisfies:

$$\frac{1}{2}|u(T)|^2 + \int_0^T a(t; u(t))\, dt \leq \frac{1}{2}|u_0|^2 + \int_0^T \left(L(t), u(t)\right) dt.$$

Since

$$\liminf_{\tau\to 0} |\hat u_\tau(T)|^2 \ge |u(T)|^2; \quad \lim_{\tau\to 0} \|L_\tau - L\|_{L^2(0,T;V')} = 0; \quad \lim_{\tau\to 0} \|u_0^\tau - u_0\| = 0;$$

the strong convergence in $L^2(0,T;V)$ follows if we show that

$$\limsup_{\tau\to 0} \int_0^T \left[a(t;u_\tau(t)) - a_\tau(t;u_\tau(t))\right] dt \le 0. \tag{3.11}$$

We extend trivially $u_\tau(t)$ and $a(t;\cdot,\cdot)$ outside $[0,T]$ and we split the integrand as:

$$\begin{aligned}
&[a(t;u_\tau(t)) - a(s_\tau;u_\tau(t))] + [a(s_\tau;u_\tau(t)) - a(s_\tau;u_\tau(t-\tau))] \\
&\qquad + [a(s_\tau;u_\tau(t-\tau)) - a(t_\tau;u_\tau(t))] \\
&\quad \le U^2\mu(]s_\tau,t]) + 2\beta U\|u_\tau(t) - u_\tau(t-\tau)\| + [a(s_\tau;u_\tau(t-\tau)) - a(t_\tau;u_\tau(t))],
\end{aligned}$$

where U is an upper bound of $\sup_{[0,T]}\|u_\tau\|$. We integrate from 0 to T and we observe that

$$\lim_{\tau\to 0}\int_0^T [\mu(]s_\tau,t]) + \|u_\tau(t) - u_\tau(t-\tau)\|]\, dt = 0$$

by Lebesgue convergence theorem and (3.6); finally

$$\int_0^T [a(s_\tau;u_\tau(t-\tau)) - a(t_\tau;u_\tau(t))]\, dt = -\int_{T-\tau}^T a(T;u_\tau(t))\, dt \le 0.$$

Combining these results we obtain (3.11).

At this point the uniform boundedness of $\hat u_\tau$ in $H^1(0,T;H)$ implies the uniform convergence in $L^\infty(0,T;H)$. In order to prove the strong V convergence in every V-continuity point of the solution, we can just repeat the argument of [21], Thm. 3.8. ■

We consider now the *proof* of Theorem 4 adapting an idea developed in [21]; for the sake of simplicity, we initially assume

$$u_0 \in D_0, \qquad L \in L^2(0,T;H)$$

and we call $U = \|u\|_{L^\infty(0,T;V)}$.

We choose a number $r \in]0,T[$ and set $v = u(t) - u(t-h)$, $0 < h < h_0$ ([19]) obtaining by $(H3)$:

$$\begin{aligned}
&2(u'(t), u(t) - u(t-h)) + a(t;u(t)) + a(t;u(t) - u(t-h)) \\
&\quad \le a(t;u(t-h)) + 2(L(t), u(t) - u(t-h)) \\
&\qquad + C_U\, h(|u'(t)|^2 + |u'(t-h)|^2 + \|L(t)\|_W^2 + \|L(t-h)\|_W^2)\int_{t-h}^t \rho^{2/\theta}(s)\, ds \\
&\quad \le \bar a(t-h;u(t-h)) + \mu(]t-h,t])U^2 + 2(L(t), u(t) - u(t-h)) \\
&\qquad + C_U\, h(|u'(t)|^2 + |u'(t-h)|^2 + \|L(t)\|_W^2 + \|L(t-h)\|_W^2)\int_{t-h}^t \rho^{2/\theta}(s)\, ds.
\end{aligned}$$

([19]) We set $u(t) \equiv u_0$ for $t < 0$; being $u_0 \in D_0$ there exists an $L_0 \in W$ such that $a(u_0, v) = (L_0, v)$ for $v \in V_0$ and consequently we define $L(t) \equiv L_0$ for $t < 0$.

We can replace $a(t; u(t))$ in the left hand member by $\bar{a}(t; u(t))$ and we integrate from 0 to $r + h \leq T$ obtaining:

$$
\begin{aligned}
(3.12) \int_0^{r+h} \left[(u'(t), u(t) - u(t-h)) + \alpha \|u(t) - u(t-h)\|^2\right] dt + \int_r^{r+h} \bar{a}(t; u(t))\, dt \\
\leq h\, \bar{a}(0; u(0)) + hU^2 \mu(]0, r+h]) + 2 \int_0^{r+h} \big(L(t), u(t) - u(t-h)\big)\, dt \\
+ 2\, C_U\, h \left\{ \int_0^{r+h} \left(|u'(t)|^2 + \|L(t)\|_W^2\right) dt + h\|L_0\|_W^2 \right\} \cdot \int_0^{r+h} \rho^{2/\theta}(s)\, ds.
\end{aligned}
$$

Finally we divide by h and pass to the limit as h goes to 0 applying Fatou's lemma:

$$
\begin{aligned}
(3.13) \qquad & \int_0^r |u'(t)|^2\, dt + \bar{a}(r; u(r)) \\
& \leq \bar{a}(0; u_0) + C_{U,\rho} \int_0^r \left[\|L(t)\|_W^2 + |u'(t)|^2\right] dt \\
& \quad + 2 \int_0^r \big(L(t), u'(t)\big)\, dt + U^2 \mu(]0, r]),
\end{aligned}
$$

obtaining a relation which does not depend on the additional hypothesis $u_0 \in D_0$. This relation gives the right continuity of $u(t)$ in V at $t = 0$ thanks to (2.4) and Remark 2.5, since we obtain

$$
\limsup_{r \to 0+} \bar{a}(r; u(r)) \leq \bar{a}(0; u_0).
$$

Moreover, in this argument the choice of the initial time $t = 0$ plays no role by the semigroup property, so that we deduce the right continuity at all points of $[0, T[$.

Rewriting (3.13) starting from an initial time $s < r$ we get

$$
\bar{a}(r; u(r)) \leq \bar{a}(s; u(s)) + \nu(]s, r]),
$$

where ν is a finite measure on $]0, T]$ depending on μ, L and u; this relation implies that the mapping $t \mapsto \bar{a}(t; u(t))$ is of bounded variation and u is continuous except at a countable subset thanks to 2.1 and 2.4.

In order to obtain the $B^{1/2}_{2\,\infty}(0, T; V)$ estimate we recall that it is sufficient to prove that the seminorm

$$
[u]^2_{B^{1/2}_{2\,\infty}(0,T;V)} = \sup_{0 \leq h \leq h_0} \frac{1}{h} \int_h^T \|u(t) - u(t-h)\|^2\, dt
$$

is finite: this is given by (3.12) choosing $r = T - h$.

Finally, when $(H2'\text{-}3')$ hold too, we can apply our arguments to the function

$$\tilde{u}(t) = u(T-t) \in \tilde{V}_t = V_{T-t}$$

which solves

$$(\tilde{u}'(t), v) + a(T-t; \tilde{u}, v) = (L(T-t) - 2u'(T-t), v), \qquad \forall v \in \tilde{V}_t \quad \blacksquare$$

3.4. REMARK. – When L admits the decomposition $(D1)$, in (3.12) we have to control the additional term

$$\int_0^{r+h} (g(t), u(t) - u(t-h))\,dt$$
$$= \int_0^r \big(g(t) - g(t+h), u(t)\big)\,dt + \int_r^{r+h} \big(g(t), u(t)\big)\,dt - \int_0^h \big(g(t), u_0\big)\,dt$$

which is obviously bounded by $C_U\, h\|g\|_{W^{1,1}(0,T;V')}$ and becomes, after the previous limit process leading to (3.13),

$$-\int_0^r (g'(t), u(t))\,dt + (g(r), u(r)) - (g(0), u_0)).$$

Being g absolutely continuous, this quantity tends to 0 as r goes to 0. $\blacksquare$

4. Applications to parabolic problems

Application 1

Let us deal with the equation (PP) stated in the introduction, under the regularity hypothesis (0.4), or better:

$$\text{(4.1)} \qquad \exists\, \rho \in L^4(0,T): \quad e(\Gamma_0^t, \Gamma_0^s) \leq \int_s^t \rho(\xi)\,d\xi, \qquad 0 \leq s < t \leq T.$$

4.1. THEOREM. – *If we are given:*

$$f \in L^2(Q), \quad u_0 \in H^1(\Omega), \quad g_0 \in H^{3/2,3/4}(\Sigma_0), \quad g_1 \in H^{1/2,1/4}(\Sigma_1) \quad [20]$$

[20] *i.e.* g_0, g_1 admit an extension to functions in $H^{3/2,3/4}(\Sigma)$, $H^{1/2,1/4}(\Sigma)$ respectively, where

$$H^{r,s}(\Sigma) = L^2(0,T; H^r(\Gamma)) \cap H^s(0,T; L^2(\Gamma)).$$

We recall that these trace assumptions (together with the possible required compatibility conditions) give the exact regularity of (0.3) in the case of pure Dirichlet or Neumann boundary value problems; in those cases we obviously deduce also $u \in H^{2,1}(Q)$, which is in general false when mixed conditions occurs, even in the cylindrical framework.

satisfying the initial compatibility condition

$$u_0(x) = g_0(x,0) \quad on \quad \Gamma_0^0,$$

then (PP) *has a unique solution* u *satisfying:*

$$\frac{\partial u}{\partial t},\ Au \in L^2(Q), \quad u \in L^\infty(0,T;H^1(\Omega)) \cap B^{1/2}_{2\infty}(0,T;H^1(\Omega)).$$

Moreover, if (4.1) *holds also for the Hausdorff distance (instead of* $e(\cdot,\cdot)$*) we have* $u \in C^0(0,T;H^1(\Omega))$.

Proof. – By the trace result of [29] chap. 4, sect. 2.5, it is not restrictive to assume $g_0, g_1 \equiv 0$, so that the Dirichlet condition becomes

$$u(x,t) = 0 \qquad \text{on} \quad \Gamma_0^t.$$

We choose

$$V = H^1(\Omega), \qquad H = L^2(\Omega), \qquad V_t = H^1_{\Gamma_0^t}(\Omega)$$

and the bilinear form

$$a(t;u,v) = \int_\Omega \left\{ \sum_{i,j} a^{ij}(x,t) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_i b^i(x,t)\frac{\partial u}{\partial x_i} v + c(x,t) u\, v \right\} dx,$$

which is admissible thanks to extension 1.15 (we assumed a global Lipschitz condition on the coefficient a^{ij}). $(H3)$ is satisfied with $\theta = 1/2$ thanks to the estimates of [33], thm. 5, and (4.1) corresponds to 1.16. ■

4.2. Remark. – We can apply our abstract theory in a more direct way by choosing:

$$W = \begin{cases} L \in \left(H^1(\Omega)\right)' : (L,v) = \displaystyle\int_\Omega f(x)\, u(x)\, dx + \int_\Gamma g(x) u(x)\, d\mathcal{H}^{n-1}x, \\ \text{with } f \in L^2(\Omega),\ g \in H^{1/2}(\Gamma), \end{cases}$$

with the norm induced by $L^2(\Omega) \times H^{1/2}(\Gamma)$; in this case g_1 is the restriction to Σ_1 of a function

$$\tilde{g}_1 \in L^2(0,T;H^{1/2}(\Gamma)) \cap W^{1,1}(0,T;H^{-1/2}(\Gamma)) \quad \square$$

4.3. Remark. – The time regularity assumptions on the differential operator A could be weakened: for instance, if $A = -a(t)\Delta$ then every function $a(\cdot) \geq \alpha > 0$ of bounded variation in $[0,T]$ is allowed. □

Application 2

Let us given a uniform family of $C^{1,1}$ open sets $\Omega_t \subset \mathbb{R}^N$ for $t \in [0,T]$ and consider the following subsets of $\mathbb{R}^N \times]0,T[$:

$$Q = \bigcup_{t \in]0,T[} \Omega_t \times \{t\}, \qquad \Sigma = \bigcup_{t \in]0,T[} \partial\Omega_t \times \{t\}.$$

We suppose that Q is open and we consider the following boundary value problem

$$(PP_2) \qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} + Au(x,t) = f(x,t), & \text{in } Q, \\ u(x,t) = 0, & \text{on } \Sigma, \\ u(x,0) = u_0(x), & \text{on } \Omega_0. \end{cases}$$

We can apply the abstract results quoted in the introduction also in this case. We assume that A is defined by (0.1) in all $\mathbb{R}^N \times]0,T[$ ([21]) and

$$f \in L^2(Q), \qquad u_0 \in H^1_0(\Omega_0).$$

The family Ω_t have to satisfy a condition analogous to (4.1) ([22]): there exists a function $\rho \in L^2(0,T)$ with

$$(4.2) \qquad e(\Omega_s, \Omega_t) = \sup_{x \in \Omega_s \setminus \Omega_t} d(x, \Omega_t) \le \int_s^t \rho(\xi)\, d\xi, \qquad \text{if} \quad 0 \le s < t \le T.$$

We have:

4.4. THEOREM. – *With the previous hypotheses, there exists a unique solution u of (PP_2) satisfying*

$$(4.3) \qquad \frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(Q)$$

$$u(\cdot,t) \in H^1_0(\Omega_t); \quad \exists\, C > 0 : \ \|\nabla u(\cdot,t)\|_{L^2(\Omega_t)} \le C, \qquad \forall t \in [0,T].$$

Moreover, if (4.2) *holds for the Hausdorff distance between Ω_s and Ω_t, then the trivial extension of u outside Q belongs to $C^0(0,T; H^1(\mathbb{R}^N)) \cap B^{1/2}_{2\,\infty}(0,T; H^1(\mathbb{R}^N))$.*

Proof. – We extend u and f to 0 outside Q and we set $V = H^1(\mathbb{R}^N), H = L^2(\mathbb{R}^N)$ and

$$V_t = \{u \in H^1(\mathbb{R}^N) : \operatorname{supp}(u) \subset \overline{\Omega}_t\} = \{u \in H^1(\mathbb{R}^N) : u_{|\mathbb{R}^N \setminus \Omega_t} \equiv 0\}.$$

For the sake of simplicity, we will sometimes identify V_t with $H^1_0(\Omega_t)$. Let us check $(H3)$ for the bilinear form (*see* 1.15):

$$(4.4) \qquad a_0(t; u, v) = \int_{R^N} \left\{ \sum_{i,j} a^{ij}(x,t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + uv \right\} dx$$

([21]) For simplicity; in fact a cylinder containing Q is sufficient.

([22]) As in the other case, a monotone family (non decreasing) is allowed.

observing that the standard regularity theory ensures

$$D_t = H^2(\Omega_t) \cap H_0^1(\Omega_t)$$

with a uniform bound of the respective norms. So we fix $F \in L^2(\mathbb{R}^N)$ and we consider the solution $u \in V_t$ of

$$a(t; u, w) = \int_{\mathbb{R}^N} F(x)\, w(x)\, dx, \quad \forall\, w \in V_t.$$

By the usual Green's formula, we have for a given $v \in D_s$:

$$\begin{aligned} \big(R_F(t), v\big) &= a(t; u, v) - \int_{\Omega_t} F\, v\, dx - \int_{\Omega_s \setminus \Omega_t} F\, v\, dx \\ &= \int_{\partial\Omega_t} \frac{\partial u}{\partial \nu_a} v\, d\mathcal{H}^{N-1} - \int_{\Omega_s \setminus \Omega_t} F\, v\, dx \\ &= \int_{\partial\Omega_t \cap \Omega_s} \frac{\partial u}{\partial \nu_a} v\, d\mathcal{H}^{N-1} - \int_{\Omega_s \setminus \Omega_t} F\, v\, dx. \end{aligned}$$

By [34], Lemma 3.10, being $\{\Omega_t\}_{t \in [0,T]}$ a uniformly $C^{1,1}$ regular family, if $e(\Omega_s, \Omega_t)$ is small enough we have:

$$\sup_{x \in \Omega_s \setminus \Omega_t} d(x, \partial\Omega_t) + \sup_{x \in \Omega_s \setminus \Omega_t} d(x, \partial\Omega_s) \le C\, e(\Omega_s, \Omega_t)$$

with C only depending on the $C^{1,1}$ character of $\{\Omega_t\}_{t \in [0,T]}$. We have (*see* [36], [21])

$$\begin{aligned} &\left| \int_{\partial\Omega_t \cap \Omega_s} \frac{\partial u}{\partial \nu_a} v\, d\mathcal{H}^{N-1} \right|^2 \le C \|u\|_{H^2(\Omega_t)} \|u\|_{H^1(\Omega_t)} \|v\|_{L^2(\Omega_s \setminus \Omega_t)} \|\nabla v\|_{L^2(\Omega_s \setminus \Omega_t)} \\ &\le C(\|u\|_{D_t} \|u\|_V) \|\nabla v\|^2_{L^2(\Omega_s \setminus \Omega_t)}\, e(\Omega_s, \Omega_t) \\ &\le C(\|u\|_{D_t} \|u\|_V) \|\nabla v\|_{L^2(\Omega_s)} \|v\|_{H^2(\Omega_s)}\, e(\Omega_s, \Omega_t)^2 \\ &\le C(\|u\|_{D_t}\, \|u\|_V)(\|v\|_V\, \|v\|_{D_s}) e(\Omega_s, \Omega_t)^2 \end{aligned}$$

and

$$\int_{\Omega_s \setminus \Omega_t} \big|F\, v\big|\, dx \le e(\Omega_s, \Omega_t) \|F\|_{L^2(\mathbb{R}^N)} \|v\|_V,$$

so that

$$\big(R_F(t), v\big) \le C(\|F\|_H^{1/2} \|u\|_V^{1/2}\ \|v\|_{D_s}^{1/2} \|v\|_V^{1/2} + \|F\|_H \|v\|_V) \int_s^t \rho(\xi)\, d\xi.$$

Applying Remark 1.17 with $\theta = \sigma = 1/2$ and with $\theta = 1, \sigma = 0$ we conclude. ■

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