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FINITE ELEMENT CONVERGENCE ON A FIXED GRID

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1. INTRODUCTION

Let D denote a polygonal domain on the plane and $P(D)$ the family of all functions $f(x, y)$ which are

(i) continuous on D

(ii) there exists a triangular mesh $\mu_f = \{T_1, T_2, \ldots, T_n\}$, $D = \cup T_i$ such that for each triangle T_k the restriction $f|T_k$ is a polynomial $(1 \leqq k \leqq n)$. $P(D)$ is usually referred as the space of C_0 continuous piecewise polynomial functions on D.

The standard finite element approximation method (to solve second order problems) selects a sequence $f_1 \ldots, f_n, \ldots \in P(D)$ with the additional requirement $f_n|\partial D = 0$ (in case of homogeneous boundary conditions) and achieves accuracy by a refining sequence of meshes while keeping the degrees of the polynomials $f_n|T_{nj}$ fixed. Another way recently developed at Washington University has proved to be advantageous in many instances. Keeping the mesh fixed and increasing the degree of the approximating polynomials provides the user with

(i) a comparatively small number of elements to deal with

(ii) a hierarchal system of basis functions which save computational time by using the results of the previous steps

(iii) fast convergence.

Putting

$$P_0(D)^n = \{f \in P(D), f|_{\partial D} = 0, \text{degree } f \leqq n\}$$

and

$$P_0(D)_\mu = \{f \in P(D), f|_{\partial D} = 0, f \text{ is a piecewise polynomial over the mesh } \mu\},$$

the conventional approximation is based upon $P_0(D)^n$ with a fixed integer n, while the new approach makes use of $P_0(D)_\mu$ fixing the mesh μ. As opposed to the new method, the results of which rely mostly on empirical studies and numerical evidences, the standard finite element analysis has a penetrating and elaborate mathematical background. It is well known, for instance, that $P_0(D)^n$ is everywhere dense in the Sobolev space $H_0^1(D)$ for any fixed $n = 1, 2, \ldots$; this establishes the convergence for the way of obtaining finite element approximations by refining the mesh. To the best knowledge of this author, however, the corresponding theorem has not been established which proves convergence for the procedure in which the triangulation is fixed and the degrees of approximating piecewise polynomials are increased. It is the aim of this paper to prove that this is indeed the case, i.e. $P_0(D)_\mu$ is indeed everywhere dense in $H_0^1(D)$. To this end it is enough to show that each fixed $f \in P_0(D)^1$ can be approximated by elements of $P_0(D)_\mu$, and it is an obvious consequence of the following.

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THEOREM. Let $f \in P_0(D)^1$ and mesh μ on D be given. For any $\epsilon > 0$ there exists a function $g \in P_0(D)_\mu$ with

$$\iint_D \{|f-g|^2 + |f_x - g_x|^2 + |f_y - g_y|^2\}\, dx\, dy < \epsilon^2. \tag{1.1}$$

2. BERNSTEIN POLYNOMIALS ON A TRIANGLE AND PIECEWISE BERNSTEIN POLYNOMIALS

To carry out the proof we define Bernstein polynomials on triangles. The one-dimensional Bernstein polynomials are well known to be applicable to the simultaneous approximation of a function and its derivatives (see [1]) and also a higher dimensional generalization has been considered in [2]. Usually the n-dimensional Bernstein polynomials are defined on the unit cube by

$$\sum_{k_1=0}^{N_1} \cdots \sum_{k_n=0}^{N_n} f\left(\frac{k_1}{N_1}, \ldots, \frac{k_n}{N_n}\right)\binom{N_1}{k_1} \cdots \binom{N_n}{k_n} x_1^{k_1}(1-x_1)^{N_1-k_1} \ldots x_n^{k_n}(1-x_n)^{N_n-k_n}$$

It is equally natural (and for our purposes it is only natural) to define the Bernstein polynomials on simplexes. Returning to the 2-dimensional case, let T be a triangle with vertices A_1, A_2, A_3, let the equations $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ define the straight lines A_2A_3, A_3A_1, A_1A_2, respectively, and we can suppose that, the linear function L_1, L_2, L_3 are normalized by $L_1(A_1) = L_2(A_2) = L_3(A_3) = 1$. Then the triplet (L_1, L_2, L_3) represents the points of the plane in triangular co-ordinates, and the identity

$$L_1 + L_2 + L_3 = 1$$

holds.

In the points of T we also have $L_1 \geqq 0$, $L_2 \geqq 0$, $L_3 \geqq 0$. Any function $f(x, y)$ defined on T, can be represented as a function of the triangular co-ordinates:

$$f(x, y) = \hat{f}(L_1, L_2, L_3).$$

Now we define the Bernstein polynomials of f by

$$B(L_1, L_2, L_3; f, n) = \sum_{k+j+\nu=n} \hat{f}\left(\frac{k}{n}, \frac{j}{n}, \frac{\nu}{n}\right) \frac{n!}{k!j!\nu!} L_1^k L_2^j L_3^\nu \quad (n = 1, 2, \ldots).$$

In analogy to the one-dimensional use, this sum is a weighted average of the values $\hat{f}(k/n, j/n, \nu/n)$ because of

$$1 = (L_1 + L_2 + L_3)^n = \sum_{k+j+\nu=n} \frac{n!}{k!j!\nu!} L_1^k L_2^j L_3^\nu.$$

If we confine the polynomial B to one of the sides of T, say $L_3 = 0$, then all the terms with $\nu > 0$ vanish and we get

$$B(L_1, L_2, 0) = \sum_{k=0}^{n} \hat{f}\left(\frac{k}{n}, \frac{n-k}{n}, 0\right)\binom{n}{k} L_1^k (1-L_1)^{n-k} \tag{2.1}$$

that is, the one-dimensional Bernstein polynomial of f on the 'unit interval' $0 \leqq L_1 \leqq 1$. It is immediate now that, if another triangle T^* is attached to T with the common side A_1A_2 and f is defined on both triangles then the corresponding n^{th} order Bernstein polynomials agree on A_1A_2. More generally, if a function f and a triangular mesh μ is given on a polygonal domain D, then the Bernstein polynomials of degree n taken on each triangular elements of μ define together a continuous piecewise polynomial function $\beta_n(x, y)$. Under the additional hypothesis $f|_{\partial D} = 0$ we have $\beta_n|_{\partial D} = 0$ as well as an obvious consequence of (2.1), that is $\beta_n \in P_0(D)_\mu$. It remains to show that $\beta_n \to f$ in the norm of the Sobolev space $H^1(D)$, if only f is a piecewise linear function on D.

3. PROOF OF THE THEOREM

Let f be a piecewise linear function over a mesh μ^* and let μ be another fixed triangular mesh on D. We can suppose that μ^* is a refinement of μ since f is piecewise linear on the common refinement of μ and μ^* as well. It is enough to establish the convergence on the triangular elements of μ, and the case of a general triangle can be reduced to a standard triangle by an affine transformation. That is, we can, for the sake of convenience, confine ourselves to consider the triangle $T: A_1 = (1, 0), A_2 = (0, 1), A_3 = (0,0)$. In this case the triangular co-ordinates (L_1, L_2, L_3) agree with the standard co-ordinates $(x, y, 1-x-y)$, and the Bernstein polynomials of a function $f(x, y)$ defined on this triangle take the form

$$B_n(x, y) = \sum f\left(\frac{k}{n}, \frac{j}{n}\right) \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j}.$$

The remaining part of the proof runs on the same line as in the one-dimensional case, thus we omit the detailed calculations. The reader can easily check our formulae following the well known one-dimensional operations.

For fixed x and y putting briefly

$$p_{k,j} = \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j}$$

we have

$$\sum (k, j) p_{k,j} = (nx, ny)$$

and

$$\sigma^2 = \sum |(k, j) - (nx, ny)|^2 p_{k,j} = n[x(1-x) + y(1-y)]$$

(these are the expectation and variance of the random vector (k, j) with probability $p_{k,j}$). Thus Chebyshev's inequality runs as follows:

$$\sum p_{kj} \leq \frac{1}{t^2} \qquad (t > 0) \tag{3.1}$$

$$\text{if } \left(\frac{k}{n} - x\right)^2 + \left(\frac{j}{n} - y\right)^2 \geq \frac{t}{n}[x(1-x) + y(1-y)].$$

There is nothing to be added to the one-dimensional proof to show that $B_m \to f$ uniformly on T, if f is a continuous function. Therefore only the contributions of the partial derivatives in (1.1) have to be checked. By straightforward differentiation of the $n+1^{\text{th}}$ order Bernstein polynomial we get easily

$$\frac{\partial B_{n+1}(x, y)}{\partial x} = \sum_{k+j \leq n} \frac{f\left(\frac{k+1}{n+1}, \frac{j}{n+1}\right) - f\left(\frac{k}{n+1}, \frac{j}{n+1}\right)}{\frac{1}{n+1}} \cdot \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j},$$

that is we obtain a Bernstein polynomial corresponding to the discrete function

$$\delta(k, j) = \frac{f\left(\frac{k+1}{n+1}, \frac{j}{n+1}\right) - f\left(\frac{k}{n+1}, \frac{j}{n+1}\right)}{\frac{1}{n+1}} \qquad (k + j \leq n)$$

Since f is piecewise linear on T, there exists a triangulation $T = \cup T_i$, such that $f|_{T_i}$ is linear. If both points $[(k+1)/(n+1), j/(n+1)]$ and $[k/(n+1), j/(n+1)]$ belong to the same element T_i, then

$$\delta(k, j) = f_x|_{T_i} \quad \text{(constant)}.$$

The derivative $f_x(x, y)$ is a step function on T (piecewise constant) and denoting by

$$m = \max|f_x(x, y)|$$

we have

$$\frac{1}{h}\{f(x+h, y) - f(x, y)\} \leqq m$$

for any $(x, y) \in T$, $(x+h, y) \in T$. Hence

$$\left|\frac{\partial B_n}{\partial x}\right| \leqq m \text{ and } |\delta(k, j)| \leqq m$$

for any choice of $n = 1, 2, \ldots$ and $(x, y) \in T$, and (k, j).

For each elementary triangle T_i we draw another smaller triangle T_i' with sides running parallel to those of T_i such that the distance of the corresponding sides is $\rho > 0$. The value of ρ is to be chosen later. We remark that if $n \geqq 3/\rho$ then

$$(x, y) \in T_i' \text{ and } \left|(x, y) - \left(\frac{k}{n}, \frac{j}{n}\right)\right| \leqq \rho/3$$

imply

$$\left(\frac{k}{n+1}, \frac{j}{n+1}\right) \in T_i \text{ and } \left(\frac{k+1}{n+1}, \frac{j}{n+1}\right) \in T_i,$$

(and hence $\delta(k, j) = f_x(x, y)$).

In fact, by recalling $0 \leqq k \leqq n, 0 \leqq j \leqq n$ we have for $\theta = 0, 1$

$$\left|(x, y) - \left(\frac{k+\theta}{n+1}, \frac{j}{n+1}\right)\right| \leqq \left|(x, y) - \left(\frac{k}{n}, \frac{j}{n}\right)\right| + \left|\left(\frac{k}{n}, \frac{j}{n}\right) - \left(\frac{k+\theta}{n+1}, \frac{j}{n+1}\right)\right|$$

$$\leqq \frac{\rho}{3} + \left|\left(\frac{k-n\theta}{n(n+1)}, \frac{j}{n(n+1)}\right)\right| \leqq \frac{\rho}{3} + \frac{|k-n\theta| + j}{n(n+1)} \leqq \frac{\rho}{3} + \frac{2n}{n(n+1)}$$

$$< \frac{\rho}{3} + \frac{2\rho}{3} = \rho,$$

thus $[(k+\theta)/(n+1), j/(n+1)]$ falls inside of T_i.

To estimate $|f_x - \partial B_n/\partial x|$ we consider first the points of the smaller triangles. Suppose $(x, y) \in T_i'$ then

$$\left|f_x(x, y) - \frac{\partial B_{n+1}}{\partial x}(x, y)\right| = \left|\sum \{f_x(x, y) - \delta(k, j)\}p_{k,j}\right| \leqq \sum |f_x(x, y) - \delta(k, j)|p_{k,j}$$

$$= \sum_{|(k/n,\, j/n)-(x,\, y)| \leqq \rho/3} |f_x - \delta(k, j)|p_{k,j} + \sum_{|(k/n,\, j/n)-(x,\, y)| > \rho/3} |f_x - \delta(k, j)|p_{k,j}.$$

The first sum has zero terms only, since by the remark above $|(x, y) - (k/n, j/n)| \leqq \rho/3$ implies $f_x(x, y) = \delta(k, j)$.

To estimate to second sum we apply $|f_x| \leqq m$, $|\delta(k, j)| \leqq m$ and inequality (3.1) with

$$t = \frac{n\rho}{3\{x(1-x)+y(1-y)\}}$$

and obtain

$$\sum_{|(k/n, j/n)-(x,y)|>\rho/3} |f_x - \delta(k, j)| p_{k,j} \leqq 2m \sum_{|(k/n, j/n)-(x,y)|>\rho/3} p_{k,j} \leqq \frac{2m9[x(1-x)+y(1-y)]^2}{n^2\rho^2} \leqq \frac{9m}{2n^2\rho^2},$$

since $x(1-x)+y(1-y) \leqq \frac{1}{2}$ for any $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

For any $(x, y) \in T$ we have

$$\left| f_x - \frac{\partial B_{n+1}}{\partial x} \right| \leqq 2m$$

thus

$$\iint_T \left(f_x - \frac{\partial B_{n+1}}{\partial x} \right)^2 = \iint_{T_i'} + \iint_{T \setminus \cup T_i'}$$

$$\leqq \left(\frac{9}{2} m \right)^2 \frac{1}{n^4\rho^4} \operatorname{area}(T) + (2m)^2 \operatorname{area}(T \setminus \cup T_i')$$

$$\leqq \left(\frac{9}{2} m \right)^2 \frac{1}{n^4\rho^4} \operatorname{area}(T) + (2m)^2\, 2l\rho$$

where l denotes the total length of all dividing segments occurring in the mesh $\{T_1, T_2, \ldots\}$.

If we choose $\rho = n^{-4/5}$ then $n \geqq 3/\rho$ holds for $n \geqq 243 = 3^5$ and we obtain

$$\iint \left(f_x - \frac{\partial B_{n+1}}{\partial x} \right)^2 \mathrm{d}x\, \mathrm{d}y \leqq \frac{C}{n^{4/5}}$$

where constant C depends on f and on the mesh only.

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