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A summation formula and Ramanujan type series ☆

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ABSTRACT

Using some properties of the general rising shifted factorial and the gamma function we derive a variant form of Dougall's ${}_5F_4$ summation for the classical hypergeometric functions. This variant form allows us to derive easily many Ramanujan type series for $1/\pi$ and Ramanujan type series for some other constants.

1. Introduction

The gamma function $\Gamma(z)$ can be defined by the formula [27, p. 76]

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},$$

where γ is the Euler constant defined as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).$$

$\Gamma(z)$ is meromorphic in the entire complex plane and has simple poles at $z = 0, -1, -2, \ldots.$ It is easy to verify that $\Gamma(1) = 1$ and $\Gamma(z)$ satisfies the recurrence relation $\Gamma(z+1) = z\Gamma(z)$. It follows that for every positive integer n, $\Gamma(n) = (n-1)!$. It is also well-known that $\Gamma(1/2) = \sqrt{\pi}$, and for every positive integer n, we have [27, p. 79]

$$\Gamma(n+1/2) = \frac{(2n)!}{4^n n!}\sqrt{\pi}, \qquad \Gamma(-n+1/2) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}. \tag{1.1}$$

For any complex α, we define the general rising shifted factorial by

$$(z)_\alpha = \Gamma(z+\alpha)/\Gamma(z). \tag{1.2}$$

It follows that $(z)_0 = 1$ and for every positive integer n, we have

$$(z)_n = z(z+1)\cdots(z+n-1), \qquad (z)_{-n} = \frac{1}{(z-1)\cdot(z-2)\cdots(z-n)}. \tag{1.3}$$

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Euler's reflection formula for the gamma function is stated in the following proposition [1, p. 9], [27, p. 78].

Proposition 1.1 *(Euler's reflection formula).*

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}.$$

In his famous paper [25], Ramanujan recorded a total of 17 series for $1/\pi$ without proofs. These series were not extensively studied until around 1987. The Borwein brothers [8,9] provided rigorous proofs of Ramanujan's series for the first time and also obtained many new series of Ramanujan type for $1/\pi$. Some remarkable extensions of them were given by the Chudnovsky brothers [18].

Many new Ramanujan type series for $1/\pi$ have been published recently, see for example, [2–4,6,10–15,17,19,21,23,26,28]. For more details, please refer to the survey paper [5].

Some mathematicians before Ramanujan had also derived some series expansions for $1/\pi$, notably [7] and [20].

In [24], the author used the general rising shifted factorial and the Gauss summation formula to prove the following four-parameter series expansion formula which implies infinitely many Ramanujan type series for $1/\pi$.

Theorem 1.1. *For any complex α and $Re(c-a-b)>0$, we have*

$$\sum_{n=0}^{\infty}\frac{(\alpha)_{a+n}(1-\alpha)_{b+n}}{n!\Gamma(c+n+1)}=\frac{(\alpha)_a(1-\alpha)_b\Gamma(c-a-b)}{(\alpha)_{c-b}(1-\alpha)_{c-a}}\times\frac{\sin\pi\alpha}{\pi}.$$

Motivated by [24], in this paper we prove the following variant from of Dougall's ${}_5F_4$ summation for the classical hypergeometric functions, which allows us to derive many Ramanujan type series for $1/\pi$ and Ramanujan type series for some other constants.

Theorem 1.2. *If $Re(a+b+c+d+\alpha-\beta-\gamma-\delta+1)>0$, then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\alpha+a+2n)(\alpha)_{a+n}(\beta)_{n-b}(\gamma)_{n-c}(\delta)_{n-d}}{n!(1+\alpha-\beta)_{a+b+n}(1+\alpha-\gamma)_{a+c+n}(1+\alpha-\delta)_{a+d+n}}\\&\quad=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\delta)\Gamma(2+\alpha-\beta-\gamma-\delta)}{\Gamma(\alpha)\Gamma(1+\alpha-\beta-\gamma)\Gamma(1+\alpha-\beta-\delta)\Gamma(1+\alpha-\gamma-\delta)}\\&\qquad\times\frac{(\beta)_{-b}(\gamma)_{-c}(\delta)_{-d}(2+\alpha-\beta-\gamma-\delta)_{a+b+c+d-1}}{(1+\alpha-\beta-\gamma)_{a+b+c}(1+\alpha-\beta-\delta)_{a+b+d}(1+\alpha-\gamma-\delta)_{a+c+d}}.\end{aligned}$$

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. In Section 3 we discuss some applications of Theorem 1.2 to Ramanujan type series for $1/\pi$. Theorem 1.2 allows us to derive some nontrivial series expansions for π^2 in Section 4. A general series expansion formula for $\Gamma^{-3}(\frac{2}{3})$ is given in Section 5. In Section 6, a general series expansion formula for $1/\sqrt{\pi}\Gamma^2(\frac{3}{4})$ is derived. In Section 7, Theorem 1.2 is used to derive Ramanujan type series for $1/\pi^2$.

2. Proof of Theorem 1.2

To prove Theorem 1.2 we need Dougall's ${}_5F_4$ summation (see [1, p. 71]) which is stated in the following theorem.

Theorem 2.1 *(Dougall's ${}_5F_4$ summation). If $Re(a+b+c+d+1)>0$, then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(a+2n)\Gamma(a+n)\Gamma(n-b)\Gamma(n-c)\Gamma(n-d)}{n!\Gamma(a+b+n+1)\Gamma(a+c+n+1)\Gamma(a+d+n+1)}\\&\quad=\frac{\Gamma(-b)\Gamma(-c)\Gamma(-d)\Gamma(a+b+c+d+1)}{\Gamma(a+b+c+1)\Gamma(a+b+d+1)\Gamma(a+c+d+1)}.\end{aligned}$$

Now we begin to prove Theorem 1.2 using Theorem 2.1 and some properties of the gamma function.

Proof. Using the general rising shifted factorial in (1.2), it is easily seen that

$$\Gamma(a+\alpha+n)=(\alpha)_{a+n}\Gamma(\alpha),\qquad \Gamma(n-b+\beta)=(\beta)_{n-b}\Gamma(\beta),$$
$$\Gamma(n-c+\gamma)=(\beta)_{n-c}\Gamma(\gamma),\qquad \Gamma(n-d+\delta)=(\delta)_{n-d}\Gamma(\delta),$$
$$\Gamma(\beta-b)=(\beta)_{-b}\Gamma(\beta),\qquad \Gamma(\gamma-c)=(\gamma)_{-c}\Gamma(\gamma),\qquad \Gamma(\delta-d)=(\delta)_{-d}\Gamma(\delta),$$
$$\Gamma(a+b+\alpha-\beta+n+1)=(\alpha-\beta+1)_{a+b+n}\Gamma(\alpha-\beta+1),$$
$$\Gamma(a+c+\alpha-\gamma+n+1)=(\alpha-\gamma+1)_{a+c+n}\Gamma(\alpha-\gamma+1),$$
$$\Gamma(a+d+\alpha-\delta+n+1)=(\alpha-\delta+1)_{a+d+n}\Gamma(\alpha-\delta+1),$$
$$\Gamma(a+b+c+\alpha-\beta-\gamma+1)=(\alpha-\beta-\gamma+1)_{a+b+c}\Gamma(\alpha-\beta-\gamma+1),$$
$$\Gamma(a+b+d+\alpha-\beta-\delta+1)=(\alpha-\beta-\delta+1)_{a+b+d}\Gamma(\alpha-\beta-\delta+1),$$
$$\Gamma(a+c+d+\alpha-\gamma-\delta+1)=(\alpha-\gamma-\delta+1)_{a+c+d}\Gamma(\alpha-\gamma-\delta+1),$$
$$\Gamma(a+b+c+d+\alpha-\beta-\gamma-\delta+1)=(\alpha-\beta-\gamma-\delta+2)_{a+b+c+d-1}\Gamma(\alpha-\beta-\gamma-\delta+2).$$

Replacing (a,b,c,d) by $(a+\alpha, b-\beta, c-\gamma, d-\delta)$ in Theorem 2.1 and then substituting the above fourteen identities into the resulting equation and simplifying, we complete the proof of Theorem 1.2. □

3. Ramanujan type series for $1/\pi$

In this section we will use Theorem 1.2 to prove the following general series expansion formula for $1/\pi$.

Theorem 3.1. *If $Re(a+b+c+d)>0$, then we have the series expansion*

$$\sum_{n=0}^{\infty}\frac{(4n+2a+1)(\frac{1}{2})_{n+a}(\frac{1}{2})_{n-b}(\frac{1}{3})_{n-c}(\frac{2}{3})_{n-d}}{n!(1)_{a+b+n}(\frac{7}{6})_{a+c+n}(\frac{5}{6})_{a+d+n}}=\frac{(\frac{1}{2})_{-b}(\frac{1}{3})_{-c}(\frac{2}{3})_{-d}(1)_{a+b+c+d-1}}{\sqrt{3}\pi(\frac{2}{3})_{a+b+c}(\frac{1}{3})_{a+b+d}(\frac{1}{2})_{a+c+d}}.$$

Proof. We let $(\alpha,\beta,\gamma,\delta)=(1/2,1/2,1/3,2/3)$ in Theorem 1.2 and multiplying both sides of the resulting equation by 2. We then use $\Gamma(1/2)=\sqrt{\pi}$ and $6\Gamma(7/6)=\Gamma(1/6)$ to find that for $Re(a+b+c+d)>0$,

$$\sum_{n=0}^{\infty}\frac{(4n+2a+1)(\frac{1}{2})_{n+a}(\frac{1}{2})_{n-b}(\frac{1}{3})_{n-c}(\frac{2}{3})_{n-d}}{n!(1)_{a+b+n}(\frac{7}{6})_{a+c+n}(\frac{5}{6})_{a+d+n}}=\frac{\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})(\frac{1}{2})_{-b}(\frac{1}{3})_{-c}(\frac{2}{3})_{-d}(1)_{a+b+c+d-1}}{3\pi\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})(\frac{2}{3})_{a+b+c}(\frac{1}{3})_{a+b+d}(\frac{1}{2})_{a+c+d}}.$$

Setting $z=1/6$ and $z=1/3$ respectively in the Euler reflection formula in Proposition 1.1, we find that

$$\Gamma(1/6)\Gamma(5/6)=2\pi,\qquad \Gamma(1/3)\Gamma(2/3)=2\pi/\sqrt{3}.$$

Combining the above two equations, we complete the proof of Theorem 3.1. □

When a, b, c and d are integers such that $Re(a+b+c+d)>0$, it is obvious that every term of the series on the left hand side of the equation in Theorem 3.1 is a rational function of n. Thus Theorem 3.1 allows us to derive infinitely many series for $1/\pi$.

If we take $(a,b,c,d)=(1,0,0,0)$ in Theorem 3.1 and simplifying, we obtain

$$\frac{3\sqrt{3}}{\pi}=1+\sum_{n=0}^{\infty}\frac{(3+13n+12n^2)}{(n+1)^2(6n+1)(6n+5)}\frac{(\frac{1}{2})_n^2(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2(\frac{1}{6})_n(\frac{5}{6})_n}. \tag{3.1}$$

Putting $(a,b,c,d)=(0,0,0,1)$ in Theorem 3.1 and using $(2/3)_{-1}=-3$ in the resulting equation, we deduce that

$$\frac{5}{\sqrt{3}\pi}=1-\frac{5}{18}\sum_{n=1}^{\infty}\frac{(4n+1)(\frac{1}{2})_n^2(\frac{1}{3})_n(\frac{2}{3})_{n-1}}{n!^2(\frac{7}{6})_n(\frac{5}{6})_{n+1}}. \tag{3.2}$$

4. A general series expansion for π^2

In this section, we will prove the following series expansion for π^2 by using Theorem 1.2.

Theorem 4.1. *If $Re(a+b+c+d-1/2)>0$, then we have the identity*

$$\frac{\pi^2(\frac{1}{2})_{-b}(\frac{1}{2})_{-c}(\frac{1}{2})_{-d}(\frac{1}{2})_{a+b+c+d-1}}{(1)_{a+b+c-1}(1)_{a+b+d-1}(1)_{a+c+d-1}}=\sum_{n=0}^{\infty}\frac{(a+2n)(1)_{a+n-1}(\frac{1}{2})_{n-b}(\frac{1}{2})_{n-c}(\frac{1}{2})_{n-d}}{n!(\frac{1}{2})_{a+b+n}(\frac{1}{2})_{a+c+n}(\frac{1}{2})_{a+d+n}}.$$

Proof. Using the recurrence relation $\Gamma(z+1)=z\Gamma(z)$, we can rewrite the equation in Theorem 1.2 as

$$\sum_{n=0}^{\infty}\frac{(\alpha+a+2n)(\alpha+1)_{a+n-1}(\beta)_{n-b}(\gamma)_{n-c}(\delta)_{n-d}}{n!(1+\alpha-\beta)_{a+b+n}(1+\alpha-\gamma)_{a+c+n}(1+\alpha-\delta)_{a+d+n}}$$
$$=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\delta)\Gamma(2+\alpha-\beta-\gamma-\delta)}{\Gamma(\alpha+1)\Gamma(2+\alpha-\beta-\gamma)\Gamma(2+\alpha-\beta-\delta)\Gamma(2+\alpha-\gamma-\delta)}$$
$$\times\frac{(\beta)_{-b}(\gamma)_{-c}(\delta)_{-d}(2+\alpha-\beta-\gamma-\delta)_{a+b+c+d-1}}{(2+\alpha-\beta-\gamma)_{a+b+c-1}(2+\alpha-\beta-\delta)_{a+b+d-1}(2+\alpha-\gamma-\delta)_{a+c+d-1}}.$$

Taking $(\alpha,\beta,\gamma,\delta)=(0,1/2,1/2,1/2)$ in the above equation and using $\Gamma(1/2)=\sqrt{\pi}$ in the resulting equation, we complete the proof of Theorem 4.1. □

Setting $a=1$ and $b=c=d=0$ in Theorem 4.1, we immediately obtain the following well-known identity:

$$\frac{\pi^2}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}. \tag{4.1}$$

Letting $a=b=c=1$ and $d=0$ in Theorem 4.1 and simplifying, we find that

$$\frac{3\pi^2}{256}=\sum_{n=0}^{\infty}\frac{1}{(2n-1)^2(2n+1)^2(2n+3)^2}. \tag{4.2}$$

Choosing $a=b=c=d=1$ in Theorem 4.1 and simplifying, we conclude that

$$\pi^2=\frac{4096}{405}-\frac{4096}{15}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3(2n+1)^2(2n+3)^3}. \tag{4.3}$$

5. A general series expansion formula for $\Gamma^{-3}(\frac{2}{3})$

We begin this section by proving the following theorem with Theorem 1.2 which contains infinitely many Ramanujan type series for $\Gamma^{-3}(\frac{2}{3})$.

Theorem 5.1. *If* $Re(a+b+c+d+1/3)>0$, *then we have the formula*

$$\frac{3(\frac{1}{3})_{-b}(\frac{1}{3})_{-c}(\frac{1}{3})_{-d}(\frac{1}{3})_{a+b+c+d}}{\Gamma^3(\frac{2}{3})(\frac{2}{3})_{a+b+c}(\frac{2}{3})_{a+b+d}(\frac{2}{3})_{a+c+d}}=\sum_{n=0}^{\infty}\frac{(6n+3a+1)(\frac{1}{3})_{a+n}(\frac{1}{3})_{n-b}(\frac{1}{3})_{n-c}(\frac{1}{3})_{n-d}}{n!(1)_{a+b+n}(1)_{a+c+n}(1)_{a+d+n}}.$$

Proof. Letting $(\alpha,\beta,\gamma,\delta)=(1/3,1/3,1/3,1/3)$ in Theorem 1.2 and simplifying, we complete the proof of Theorem 5.1. □

When $a=b=c=d=0$ Theorem 5.1 immediately reduces to the following identity:

$$\frac{3}{\Gamma^3(\frac{2}{3})}=\sum_{n=0}^{\infty}\frac{(6n+1)(\frac{1}{3})_n^4}{n!^4}. \tag{5.1}$$

Setting $a=1$ and $b=c=d=0$ in Theorem 5.1, we deduce that

$$\frac{27}{16\Gamma^3(\frac{2}{3})}=\sum_{n=0}^{\infty}\frac{(3n+2)(\frac{1}{3})_{n+1}(\frac{1}{3})_n^3}{n!(n+1)!^3}. \tag{5.2}$$

6. A general series expansion formula for $1/\sqrt{\pi}\Gamma^2(\frac{3}{4})$

In this section we will prove the following theorem with Theorem 1.2 which contains infinitely series expansion formulas for $1/\sqrt{\pi}\Gamma^2(\frac{3}{4})$.

Theorem 6.1. *If* $Re(a+b+c+d+1/2)>0$, *then we have the formula*

$$\frac{2\sqrt{2}(\frac{1}{4})_{-b}(\frac{1}{4})_{-c}(\frac{1}{4})_{-d}(\frac{1}{2})_{a+b+c+d}}{\sqrt{\pi}\Gamma^2(\frac{3}{4})(\frac{3}{4})_{a+b+c}(\frac{3}{4})_{a+b+d}(\frac{3}{4})_{a+c+d}}=\sum_{n=0}^{\infty}\frac{(8n+4a+1)(\frac{1}{4})_{a+n}(\frac{1}{4})_{n-b}(\frac{1}{4})_{n-c}(\frac{1}{4})_{n-d}}{n!(1)_{a+b+n}(1)_{a+c+n}(1)_{a+d+n}}.$$

Proof. Taking $(\alpha, \beta, \gamma, \delta) = (1/4, 1/4, 1/4, 1/4)$ in Theorem 1.2 and using $\Gamma(1/2) = \sqrt{\pi}$, we deduce that

$$\frac{4\sqrt{\pi}(\frac{1}{4})_{-b}(\frac{1}{4})_{-c}(\frac{1}{4})_{-d}(\frac{1}{2})_{a+b+c+d}}{\Gamma(\frac{1}{4})\Gamma^3(\frac{3}{4})(\frac{3}{4})_{a+b+c}(\frac{3}{4})_{a+b+d}(\frac{3}{4})_{a+c+d}} = \sum_{n=0}^{\infty} \frac{(8n+4a+1)(\frac{1}{4})_{a+n}(\frac{1}{4})_{n-b}(\frac{1}{4})_{n-c}(\frac{1}{4})_{n-d}}{n!(1)_{a+b+n}(1)_{a+c+n}(1)_{a+d+n}}.$$

Setting $z = 1/4$ in the well-known identity $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, we find that

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi.$$

Combining the above two equations, we complete the proof of Theorem 6.1. □

Setting $a = b = c = d = 0$ in Theorem 6.1, we find that [1, Exercise 26(b), p. 182]

$$\frac{2\sqrt{2}}{\sqrt{\pi}\Gamma^2(\frac{3}{4})} = \sum_{n=0}^{\infty} \frac{(8n+1)(\frac{1}{4})_n^4}{n!^4}. \tag{6.1}$$

Setting $a = 1$ and $b = c = d = 0$ in Theorem 6.1, we conclude that

$$\frac{27\sqrt{2}}{64\sqrt{\pi}\Gamma^2(\frac{3}{4})} = \sum_{n=0}^{\infty} \frac{(8n+5)(\frac{1}{4})_{n+1}(\frac{1}{4})_n^3}{(n+1)^3 n!^4}. \tag{6.2}$$

7. Ramanujan type series for $1/\pi^2$

The main result of this section is the following general expansion formula for $1/\pi^2$.

Theorem 7.1. *If* $Re(a+b+c+d) > 0$, *then we have the formula*

$$\frac{2(\frac{1}{2})_{-b}(\frac{1}{2})_{-c}(\frac{1}{2})_{-d}(1)_{a+b+c+d-1}}{\pi^2(\frac{1}{2})_{a+b+c}(\frac{1}{2})_{a+b+d}(\frac{1}{2})_{a+c+d}} = \sum_{n=0}^{\infty} \frac{(4n+2a+1)(\frac{1}{2})_{a+n}(\frac{1}{2})_{n-b}(\frac{1}{2})_{n-c}(\frac{1}{2})_{n-d}}{n!(1)_{a+b+n}(1)_{a+c+n}(1)_{a+d+n}}.$$

Proof. Taking $(\alpha, \beta, \gamma, \delta) = (1/2, 1/2, 1/2, 1/2)$ in Theorem 1.2 and using $\Gamma(1/2) = \sqrt{\pi}$ in the resulting equation, we obtain Theorem 7.1. □

Remark 7.1. A similar result has been obtained by Chu [16, Theorem 2]. It is obvious that Theorem 7.1 is more elegant than Chu's theorem. A few examples of Ramanujan type series for $1/\pi^2$ have been found in [22].

If we let $b = c = d = -a$ in Theorem 7.1, we obtain the following corollary.

Corollary 7.1. *If* $Re(a) < 0$, *then we have the identity*

$$\frac{2(\frac{1}{2})_a^3(1)_{-2a-1}}{\pi^2(\frac{1}{2})_{-a}^3} = \sum_{n=0}^{\infty} \frac{(4n+2a+1)(\frac{1}{2})_{n+a}^4}{n!^4}.$$

Setting $a = -1$ in the above equation and using $(1/2)_{-1} = -2$, we find the following identity of Glaisher [20] (see also [16, Eq. (7)]):

$$\frac{128}{\pi^2} = 16 - \sum_{n=1}^{\infty} \frac{(4n-1)(\frac{1}{2})_{n-1}^4}{n!^4}. \tag{7.1}$$

Putting $a = -2$ in Proposition 7.1 and simplifying, we deduce that

$$\frac{1024}{99\pi^2} = 1 + \frac{27}{176} \sum_{n=0}^{\infty} \frac{(4n+5)(\frac{1}{2})_n^4}{(n+2)!^4}. \tag{7.2}$$

Next we will continue to discuss some special cases of Theorem 7.1. If we choose $(a, b, c, d) = (k, 0, 0, 0)$ in Theorem 7.1 and simplifying, we can easily find the following proposition.

Proposition 7.1. *If k is a positive integer, then we have*

$$\frac{2(k-1)!}{(\frac{1}{2})_k^3\pi^2}=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_{n+k}(\frac{1}{2})_n^3(4n+2k+1)}{n!(n+k)!^3}.$$

When $k=1$, the above equation reduces to the following beautiful identity of Glaisher [20, Eq. (ix), p. 194], which was first derived by Glaisher using Fourier–Legendre expansions:

$$\frac{32}{\pi^2}=\sum_{n=0}^{\infty}\frac{(2n+1)(4n+3)(\frac{1}{2})_n^4}{(n+1)^3 n!^4}. \tag{7.3}$$

Letting $k=2$ in Proposition 7.1, we deduce that

$$\frac{512}{27\pi^2}=\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)(4n+5)(\frac{1}{2})_n^4}{n!(n+2)!^3}. \tag{7.4}$$

Proposition 7.2. *If k is a nonnegative integer, then we have the formula*

$$\frac{4k!}{\pi^2(\frac{1}{2})_k(\frac{1}{2})_{k+1}^2}=\frac{(4k+2)(\frac{1}{2})_k}{(k+1)!k!^2}-\sum_{n=1}^{\infty}\frac{(4n+2k+1)(\frac{1}{2})_{k+n}(\frac{1}{2})_{n-1}(\frac{1}{2})_n^2}{n!(k+n+1)!(k+n)!^2}.$$

Proof. Choosing $(a,b,c,d)=(k,1,0,0)$ in Theorem 7.1, we find that

$$\frac{2k!(\frac{1}{2})_{-1}}{\pi^2(\frac{1}{2})_k(\frac{1}{2})_{k+1}^2}=\frac{(2k+1)(\frac{1}{2})_k(\frac{1}{2})_{-1}}{(k+1)!k!^2}+\sum_{n=1}^{\infty}\frac{(2k+4n+1)(\frac{1}{2})_{k+n}(\frac{1}{2})_{n-1}(\frac{1}{2})_n^2}{n!(k+n+1)!(k+n)!^2}.$$

Substituting $(1/2)_{-1}=-2$ in the above equation and multiplying both sides of the resulting equation by -1, we arrive at Proposition 7.2. □

Putting $k=0$ and $k=1$ in Proposition 7.2 respectively, we conclude that

$$\frac{16}{\pi^2}=2-\sum_{n=1}^{\infty}\frac{(4n+1)(\frac{1}{2})_{n-1}(\frac{1}{2})_n^3}{(n+1)!n!^3}, \tag{7.5}$$

$$\frac{128}{9\pi^2}=\frac{3}{2}-\sum_{n=1}^{\infty}\frac{(4n+3)(\frac{1}{2})_{n-1}(\frac{1}{2})_{n+1}(\frac{1}{2})_n^2}{n!(n+2)!(n+1)!^2}. \tag{7.6}$$

Choosing $(a,b,c,d)=(k,1,1,0)$ in Theorem 7.1 and using the same argument that we used to prove Proposition 7.2, we obtain the following proposition.

Proposition 7.3. *If $k\geqslant 0$ is an integer, then we have*

$$\frac{8(k+1)!}{\pi^2(\frac{1}{2})_{k+2}(\frac{1}{2})_{k+1}^2}=\frac{(8k+4)(\frac{1}{2})_k}{k!(k+1)!^2}+\sum_{n=1}^{\infty}\frac{(4n+2k+1)(\frac{1}{2})_{k+n}(\frac{1}{2})_{n-1}^2(\frac{1}{2})_n}{n!(n+k)!(n+k+1)!^2}.$$

Setting $k=0$ in the above equation, we find the following identity:

$$\frac{128}{3\pi^2}=4+\sum_{n=1}^{\infty}\frac{(4n+1)(\frac{1}{2})_{n-1}^2(\frac{1}{2})_n^2}{n!^2(n+1)!^2}. \tag{7.7}$$

Taking $(a,b,c,d)=(k,1,1,1)$ in Theorem 7.1, we obtain the following proposition.

Proposition 7.4. *If k is a nonnegative integer, then we have*

$$\frac{16(k+2)!}{\pi^2(\frac{1}{2})_{k+2}^3}=\frac{(16k+8)(\frac{1}{2})_k}{k!^3}-\sum_{n=1}^{\infty}\frac{(4n+2k+1)(\frac{1}{2})_{k+n}(\frac{1}{2})_{n-1}^3}{n!(k+n+1)!^3}.$$

Letting $k=0$ in Proposition 7.4, we immediately deduce that

$$\frac{2048}{27\pi^2}=8-\sum_{n=1}^{\infty}\frac{(4n+1)(\frac{1}{2})_n(\frac{1}{2})_{n-1}^3}{n!(n+1)!^3}. \tag{7.8}$$

Taking $(a,b,c,d)=(k,-1,-1,-1)$ in Theorem 7.1, we obtain the following proposition.

Proposition 7.5. *If $k\geqslant 4$ is an integer, then we have*

$$\frac{(k-4)!}{4(\frac{1}{2})_{k-2}^3\pi^2}=\sum_{n=0}^{\infty}\frac{(4n+2k+1)(\frac{1}{2})_{k+n}(\frac{1}{2})_{n+1}^3}{n!(k+n-1)!^3}.$$

Letting $k=4$ in the above proposition, we find that

$$\frac{16}{27\pi^2}=\sum_{n=0}^{\infty}\frac{(4n+9)(\frac{1}{2})_{n+4}(\frac{1}{2})_{n+1}^3}{n!(n+3)!^3}. \tag{7.9}$$

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia Math. Appl., vol. 71, Cambridge Univ. Press, Cambridge, 1999.
[2] N.D. Baruah, B.C. Berndt, Ramanujan's series for $1/\pi$ arising from his cubic and quartic theories of elliptic functions, J. Math. Anal. Appl. 341 (2008) 357–371.
[3] N.D. Baruah, B.C. Berndt, Ramanujan's Eisenstein series and new hypergeometric-like series for $1/\pi^2$, J. Approx. Theory 160 (2009) 135–153.
[4] N.D. Baruah, B.C. Berndt, Eisenstein series and Ramanujan-type series for $1/\pi$, Ramanujan J. 23 (2010) 17–44.
[5] N.D. Baruah, B.C. Berndt, H.H. Chan, Ramanujan's series for $1/\pi$: a survey, Amer. Math. Monthly 116 (2009) 567–587.
[6] N.D. Baruah, N. Nayak, New hypergeometric-like series for $1/\pi^2$ arising from Ramanujan's theory of elliptic functions to alternative base 3, Trans. Amer. Math. Soc. 363 (2011) 887–900.
[7] G. Bauer, Von den Coefficienten der Reihen von Kugelfunctionen einer Variabeln, J. Reine Angew. Math. 56 (1859) 101–121.
[8] J.M. Borwein, P.B. Borwein, Class number three Ramanujan type series for $1/\pi$, J. Comput. Appl. Math. 46 (1993) 281–290.
[9] J.M. Borwein, P.B. Borwein, Pi and the AGM, Wiley, New York, 1987.
[10] H.H. Chan, S.H. Chan, Z. Liu, Domb's numbers and Ramanujan–Sato type series for $1/\pi$, Adv. Math. 186 (2004) 396–410.
[11] H.H. Chan, S. Cooper, W.-C. Liaw, The Rogers–Ramanujan continued fraction and a quintic iteration for $1/\pi$, Proc. Amer. Math. Soc. 135 (2007) 3417–3424.
[12] H.H. Chan, W.C. Liaw, V. Tan, Ramanujan's class invariant λ_n and a new class of series for $1/\pi$, J. London Math. Soc. 64 (2) (2001) 93–106.
[13] H.H. Chan, K.P. Loo, Ramanujan's cubic continued fraction revisited, Acta Arith. 126 (2007) 305–313.
[14] H.H. Chan, H. Verrill, The Apéry numbers, the Almkvist–Zudilin numbers and new series for $1/\pi$, Math. Res. Lett. 16 (2009) 405–420.
[15] H.H. Chan, W. Zudilin, New representations for Apéry-like sequences, Mathematika 56 (2010) 107–117.
[16] W. Chu, π-formulas implied by Dougall's summation theorem for ${}_5F_4$-series, Ramanujan J. 26 (2011) 251–255.
[17] W. Chu, Dougall's bilateral ${}_2H_2$ series and Ramanujan-like π formulas, Math. Comp. 80 (2011) 2223–2251.
[18] D.V. Chudnovsky, G.V. Chudnovsky, Approximations and complex multiplication according to Ramanujan, in: G.E. Andrews, B.C. Berndt, R.A. Rankin (Eds.), Ramanujan Revisited, Proceedings of the Centenary Conference, Urbana–Champaign, 1987, Academic Press, Boston, 1988, pp. 375–472.
[19] S. Cooper, Series and iterations for $1/\pi$, Acta Arith. 141 (2010) 33–58.
[20] J.W.L. Glaisher, On series for $1/\pi$ and $1/\pi^2$, Quart. J. Pure Appl. Math. 37 (1905) 173–198.
[21] J. Guillera, Hypergeometric identities for 10 extended Ramanujan-type series, Ramanujan J. 15 (2008) 219–234.
[22] J. Guillera, A new Ramanujan-like series for $1/\pi^2$, http://arxive.org/abs/1003.1915.
[23] P. Levrie, Using Fourier–Legendre expansions to derive series for $1/\pi$ and $1/\pi^2$, Ramanujan J. 22 (2010) 221–230.
[24] Z.-G. Liu, Gauss summation and Ramanujan type series for $1/\pi$, Int. J. Number Theory, doi:10.1142/S1793042112500169, in press.
[25] S. Ramanujan, Modular equations and approximations to π, Quart. J. Math. Oxford Ser. (2) 45 (1914) 350–372.
[26] M. Rogers, New ${}_5F_4$ hypergeometric transformations, three-variable Mahler measures, and formulas for $1/\pi$, Ramanujan J. 18 (2009) 327–340.
[27] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.
[28] W. Zudilin, More Ramanujan-type formulae for $1/\pi^2$, Russian Math. Surveys 62 (3) (2007) 634–636.