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Location Change in Marginal Distributions of Linear Functions of Random Vectors

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Suppose X and Y are $n \times 1$ random vectors such that $l'X + f(l)$ and $l'Y$ have the same marginal distribution for all $n \times 1$ real vectors l and some real valued function $f(l)$, and the existence of expectations of X and Y is not necessary. Under these conditions it is proven that there exists a vector M such that $f(l) = l'M$ and $X + M$ and Y have the same joint distribution. This result is extended to Banach-space valued random vectors.

1. INTRODUCTION AND SUMMARY

Our main result, Theorem 3, may be stated as follows. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are $n \times 1$ random vectors such that $\mathbf{t}'\mathbf{X} + f(\mathbf{t})$ and $\mathbf{t}'\mathbf{Y}$ have the same distribution for all $n \times 1$ real vectors $\mathbf{t}$ and some real valued function $f(\mathbf{t})$, then there exists an $n \times 1$ vector $\mathbf{X}_0$ such that $f(\mathbf{t}) = \mathbf{t}'\mathbf{X}_0$ and $\mathbf{X} + \mathbf{X}_0$ and $\mathbf{Y}$ have the same distribution. Of course this is nontrivial only when $E(\mathbf{X})$ does not exist. Theorem 3 is extended to random vectors taking values in a Banach space which is either separable or reflexive. To prove these we need some results on random vectors linear functions of which are symmetrically distributed. These auxiliary results, which have some interest of their own, constitute Theorem 1 and 2. The last section contains a number of open problem.

2. MAIN RESULTS

We begin by stating

THEOREM 1. *Let* $\mathbf{X}$ *be an* $n \times 1$ *random vector such that* $\mathbf{t}'\mathbf{X}$ *is symmetrically distributed around* $f(\mathbf{t})$ *for all* $n \times 1$ *real vectors* $\mathbf{t}$ *and some real valued function*

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$f(\mathbf{t})$. *Then* $\mathbf{X}$ *has a symmetrical distribution around* $\mathbf{M}$ *and* $f(\mathbf{t}) = \mathbf{t}'\mathbf{M}$, *where* $\mathbf{M} = (M_1, M_2, ..., M_n)$ *and* M_i *is the point of symmetry of the marginal distribution of* X_i.

Note that Theorem 1 becomes trivial when $E(X_i)$ is finite, $i = 1, 2, ..., n$. For then $f(\mathbf{t}) = E(\mathbf{t}'\mathbf{X}) = \sum t_i E(X_i) = \sum t_i M_i$. To deal with the case of infinite expectations, we have been unable to use the method of truncating the random vector $\mathbf{X}$. We have used instead a certain "wrapping up" technique below. We begin by proving a simple lemma.

LEMMA 1. *Under the hypotheses of Theorem* 1, $f(\mathbf{t})$ *is a continuous homogeneous function of* $\mathbf{t}$.

Proof. If $\mathbf{t}'\mathbf{X}$ is symmetrical around $f(\mathbf{t})$, then $c\mathbf{t}'\mathbf{X}$ is symmetrical around $cf(\mathbf{t})$ where c is a scalar. So $f(c\mathbf{t}) = cf(\mathbf{t})$ proving homogeneity. To prove continuity, let $\mathbf{t}_n \to \mathbf{t}$. Then $\mathbf{t}_n'\mathbf{X} \to \mathbf{t}'\mathbf{X}$ everywhere, which implies $\mathbf{t}_n'\mathbf{X} \to \mathbf{t}'\mathbf{X}$ in law. Continuity of $f(\mathbf{t})$ is an easy consequence of this.

We now explain the 'wrapping up' technique. Let $a > 0$. If b, c are two real numbers, then we say $b = c \pmod{2a}$ if $b = c + 2am$ where m is an integer. Let $R(b)$, the residue of b mod $(2a)$, be defined as a real number c such that $-a \leqslant c \leqslant a$ and $b = c \pmod{2a}$. If $\mathbf{b}$ is a real vector, then $R(\mathbf{b})$ is the vector of residues of the components mod $(2a)$. The wrapping up technique is to use the random vector $R(\mathbf{X})$.

We now prove Theorem 1. By convolution with a n-dimensional normal random vector if necessary, we may assume without any loss of generality that $\mathbf{X}$ has an absolutely continuous distribution so that for any real a, c:

$$P(X_i = c + ma) = 0 \qquad \text{for} \quad m = 0, \pm 1, \pm 2, ... \tag{1}$$

Take $a > 0$ and $\mathbf{t}$ to be an $n \times 1$ vector of integers $t_1, t_2, ..., t_n$. By hypothesis $\mathbf{t}'\mathbf{X} - f(\mathbf{t})$ is symmetrical around zero. This and (1) imply that $R(\mathbf{t}'\mathbf{X} - f(\mathbf{t}))$ is symmetrical around zero. Now,

$$\begin{aligned} R(\mathbf{t}'\mathbf{X} - f(\mathbf{t})) &= (R(\mathbf{t}'\mathbf{X}) - f(\mathbf{t})) \bmod (2\mathrm{a}) \\ &= (\mathbf{t}'R(\mathbf{X}) - f(\mathbf{t})) \bmod (2\mathrm{a}) \end{aligned} \tag{2}$$

since $t_1, t_2, ..., t_n$ are integers.

Since $R(\mathbf{t}'\mathbf{X} - f(\mathbf{t}))$ is symmetrical around zero and has finite expectation, being bounded,

$$E(R(\mathbf{t}'\mathbf{X} - f(\mathbf{t})) = 0. \tag{3}$$

By (2) and (3):

$$(\mathbf{t}'E(R(\mathbf{X})) - f(\mathbf{t})) = 0 \qquad \bmod(2a). \tag{4}$$

Without any loss of generality we can assume that $\mathbf{M} = \mathbf{0}$, i.e., X_i is symmetrical around zero. This and (1) imply that $R(X_i)$ is symmetrical around zero, and hence

$$E(R(X_i)) = 0. \tag{5}$$

By (4) and (5),

$$f(\mathbf{t}) = 0 \qquad \mathrm{mod}(2a). \tag{6}$$

Since this is true for all $a > 0$, $f(\mathbf{t}) = 0$ for all integral $\mathbf{t}$. Then by homogeneity $f(\mathbf{t}) = 0$ if $\mathbf{t}$ is rational. But $f(\mathbf{t})$ is continuous, and hence, is zero for all $\mathbf{t}$. Thus, $f(\mathbf{t}) = \mathbf{t}'\mathbf{M}$, since $\mathbf{M}$ has been assumed to be zero. This proves the second part of the theorem. To prove the first part note that $E(e^{i\mathbf{t}'\mathbf{X}})$ is real since $\mathbf{t}'\mathbf{X}$ is symmetrical around $f(\mathbf{t}) = 0$, and hence

$$E[\exp(i\mathbf{t}'(-\mathbf{X}))] = \overline{E[\exp(i\mathbf{t}'\mathbf{X})]} = E[\exp(i\mathbf{t}'\mathbf{X})]$$

which implies that $-\mathbf{X}$ and $\mathbf{X}$ have the same distribution. This completes the proof.

We extend this result to random vectors taking values in a Banach space. Now suppose that $\mathbf{X}$ and $\mathbf{Y}$ take values in B, where B is a Banach space equipped with a σ-algebra $\mathbf{B}$. Let B^* be the topological dual of B, i.e., the set of all continuous linear functionals on B. By the weak star topology of B^* we mean as usual the weak topology induced by B [1, p. 462].

Let $\mathbf{B}_0$ be the Borel σ-algebra, i.e., $\mathbf{B}_0$ is the smallest σ-algebra containing all open sets of B. Let $\mathbf{B}_1$ be the smallest σ-algebra with respect to which all $\mathbf{t}(\ )$, $\mathbf{t} \in B^*$, are measurable. We shall always assume below that $\mathbf{B} = \mathbf{B}_1$. It is well-known that if B is separable, then $\mathbf{B}_0 = \mathbf{B}_1$, (see, e.g., [3]). We will say that $\mathbf{X}$ has a symmetric distribution around $\mathbf{M}$ if

$$P(\mathbf{X} - \mathbf{M} \in C) = P(-(\mathbf{X} - \mathbf{M}) \in C) \qquad \text{for all} \quad C \in \mathbf{B}_1.$$

PROPOSITION 1. *If* $\mathbf{M} \in B$ *is such that* $\mathbf{t}(\mathbf{X})$ *is symmetrically distributed about* $\mathbf{t}(\mathbf{M})$ *for all* $\mathbf{t} \in B^*$, *then* $\mathbf{X}$ *is symmetrically distributed about* $\mathbf{M}$.

Proof. By symmetry of $\mathbf{t}(\mathbf{X} - \mathbf{M})$ about zero,

$$E[\exp(i\mathbf{t}(\mathbf{X} - \mathbf{M}))] = E[\exp(-i\mathbf{t}(\mathbf{X} - \mathbf{M}))]$$

i.e., $\mathbf{X}$ and $-\mathbf{X}$ have the same characteristic functional. The result now follows from [3, Property III, p. 235] for characteristic functionals.

LEMMA 2. *Let* B *be a separable Banach space and* $f(\ )$ *be a linear functional on* B^*, *the topological duel of* B. *Suppose that* $f(\ )$ *is sequentially continuous in* *the weak star topology of B^*. Then there exists $\mathbf{M} \in B$ such that $f(\mathbf{t}) = \mathbf{t}(\mathbf{M})$ for all $t \in B^*$.*

Proof. By [1, Theorem 1, p. 426], the closed unit ball in B^* is metrizable. Hence, f is weak star-continuous when restricted to the closed unit-ball, and thus, f is continuous in the bounded B-topology of B^* [1, p. 427]. The result now follows from the proof of [1, Theorem 6, p. 428].

THEOREM 2. *If for all $\mathbf{t} \in B^*$, $\mathbf{t}(\mathbf{X})$ is symmetrical around $f(\mathbf{t})$, then f is a sequentially continuous linear functional on B^*, where B^* is given its weak star topology. Furthermore, if B is reflexive or separable, then there exists $\mathbf{M} \in B$ such that $f(\mathbf{t}) = \mathbf{t}(\mathbf{M})$ and $\mathbf{X} - \mathbf{M}$ has a symmetric distribution.*

Proof. Let c_1, c_2 be real numbers and $\mathbf{t}_1, \mathbf{t}_2 \in B^*$. Let $Y_i = \mathbf{t}_i(\mathbf{X})$ for $i = 1, 2$ and let $\mathbf{Y}' = (Y_1, Y_2)$. By Theorem 1 applied to $\mathbf{Y}$ we get

$$f(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) = c_1 f(\mathbf{t}_1) + c_2 f(\mathbf{t}_2)$$

proving that f is a linear functional on B^*. Let $\mathbf{t}_i \to \mathbf{t}$ weakly, i.e., $\mathbf{t}_i(\mathbf{X}) \to \mathbf{t}(\mathbf{X})$ for all $\mathbf{X} \in B$. Then $\mathbf{t}_i(\mathbf{X}) \to \mathbf{t}(\mathbf{X})$ everywhere, which implies that $f(\mathbf{t}_i) \to f(\mathbf{t})$, i.e., f is weakly sequentially continuous.

If B is reflexive, the existence of $\mathbf{M}$ follows from the fact that weak star sequential continuity implies that f is continuous when B^* is given its strong topology. An application of Proposition 1 now shows that $\mathbf{X}$ is symmetrical around $\mathbf{M}$. If B is separable, we first apply Lemma 2 to derive the existence of $\mathbf{M}$ and then apply Proposition 1. This completes the proof.

Let B^{**} be the strong dual of B^*. Since weak star sequential continuity of f implies its strong continuity, there exists $\mathbf{X}^{**} \in B^{**}$ such that $f(\mathbf{t}) = \mathbf{X}^{**}(\mathbf{t})$. Let B be embedded as a closed linear subspace of B^{**} and suppose $\mathbf{B}^{**}$ is a σ-algebra such that $B \cap \mathbf{B}^{**} = \mathbf{B}_1$. Then $\mathbf{X}$ can be thought of as a random vector taking values in B^{**} and $(\mathbf{X} - \mathbf{X}^{**})(\mathbf{t})$ has a symmetric distribution about zero but in general $X^{**} \notin B_x$ and so will not be symmetrical.

Having established the necessary auxiliary results, we now come to the main results.

THEOREM 3. *Suppose $\mathbf{X}$ and $\mathbf{Y}$ are $n \times 1$ random vectors such that $\mathbf{t}'\mathbf{Y}$ has the same distribution as $\mathbf{t}'\mathbf{X} + f(\mathbf{t})$ for all $n \times 1$ real vectors $\mathbf{t}$. Then there exists an $n \times 1$ vector $\mathbf{M}$ such that $\mathbf{X} + \mathbf{M}$ and $\mathbf{Y}$ have the same distribution.*

Proof. Let $\mathbf{Z}$ have the same distribution as $-\mathbf{X}$ and be independent of both $\mathbf{X}$ and $\mathbf{Y}$. Then $\mathbf{X} + \mathbf{Z}$ is a random vector symmetric about zero. Hence, $\mathbf{t}'(\mathbf{X} + \mathbf{Z})$ is symmetric around zero. On the other hand, by our hypothesis on $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{t}'(\mathbf{X} + \mathbf{Z}) + f(\mathbf{t})$ has same distribution as $\mathbf{t}'(\mathbf{X} + \mathbf{Z})$. Hence, $\mathbf{t}'(\mathbf{X} + \mathbf{Z})$

is symmetrical around $f(\mathbf{t})$. By Theorem 1 there exists $\mathbf{M}$ such that $f(\mathbf{t}) = \mathbf{t}'\mathbf{M}$. Hence,

$$E[\exp(i\mathbf{t}'\mathbf{Y})] = E[\exp(i\mathbf{t}'\mathbf{X} + if(\mathbf{t}))] = E[\exp(i\mathbf{t}'\mathbf{X} + i\mathbf{t}'\mathbf{M})]$$

which proves Theorem 3.

Using Theorem 2 instead of Theorem 1 we get the following:

THEOREM 4. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are random vectors taking values in a Banach space* B *and for all* $\mathbf{t} \in B^*$, $\mathbf{t}(\mathbf{Y})$ *has same distribution as* $\mathbf{t}(\mathbf{X}) + f(\mathbf{t})$, *then* $f(\mathbf{t})$ *is a linear functional on* B^* *and sequentially continuous in the weak star topology. If* B *is separable or reflexive, then there exists* $\mathbf{M} \in B$ *such that* $\mathbf{Y}$ *and* $\mathbf{X} + \mathbf{M}$ *have the same distribution.*

3. SOME UNSOLVED PROBLEMS

Suppose $\mathbf{X}$ and $\mathbf{Y}$ are $n \times 1$ random vectors such that $\mathbf{t}'\mathbf{X}$ and $(\mathbf{t}'\mathbf{Y})\ c(\mathbf{t})$ have the same distribution where $c(\mathbf{t}) > 0$. In the spirit of the previous section one can ask if this implies that $c(\mathbf{t}) = c$ and that $\mathbf{X}$ and $c\mathbf{Y}$ have the same distribution. The following example shows that in general the answer is no: Let $n = 2$. Let $\mathbf{X}$ have Cauchy density with independent components and let $\mathbf{Y}$ have a bivariate Cauchy density. Then calculation of the characteristic function shows that $c(\mathbf{t}) = (|\,t_1\,| + |\,t_2\,|)/(t_1^2 + t_2^2)^{1/2}$ [2, p. 497].

It would be interesting to characterize all random vectors $\mathbf{X}$ for which $\mathbf{t}'\mathbf{X}$ has the same distribution as $(\mathbf{t}'\mathbf{Y})\ c(\mathbf{t})$ for a given random vector $\mathbf{Y}$. Some other problems of the same type are discussed below.

Suppose $\mathbf{X}$ is an $n \times 1$ random vector such that the type of distribution of $\mathbf{t}'\mathbf{X}$ does not depend on $\mathbf{t}$, i.e., there exist $f(\mathbf{t})$ and $c(\mathbf{t}) > 0$ such that $(\mathbf{t}'\mathbf{X} - f(\mathbf{t}))/c(\mathbf{t})$ has same distribution as $\mathbf{t}_0'\mathbf{X}$ where $\mathbf{t}_0$ is a fixed vector. Let us take $\mathbf{t}_0'\mathbf{X} = X_1$, the first component of $\mathbf{X}$. What can we say about the distribution of $\mathbf{X}$? In the case where $\mathbf{X}$ has a finite dispersion matrix D, the answer is provided below:

Without any loss of generality assume that D is positive definite. Let $\mathbf{Y} = P\mathbf{X}$ where P is a nonsingular matrix. Then there exist $f(\mathbf{t})$ and $d(\mathbf{t}) > 0$ such that $(\mathbf{t}'\mathbf{Y} - f(\mathbf{t}))/d(\mathbf{t})$ has same distribution as $Y_1 - M_1$. Choose P such that $\mathbf{Y}$ has dispersion matrix equal to the identity matrix. Clearly $f(\mathbf{t}) = \mathbf{t}'\mathbf{M}$ where $\mathbf{M}' = (M_1, M_2, ..., M_n)$, $M_i = E(Y_i)$ and $d(\mathbf{t}) = (\mathrm{var}(\mathbf{t}'\mathbf{Y}))^{1/2} = (t_1^2 + t_2^2 + \cdots + t_n^2)^{1/2}$. Then

$$\begin{aligned} E[\exp(i\mathbf{t}'(\mathbf{Y} - \mathbf{M}))] &= E[\exp(i(Y_1 - M_1)(t_1^2 + t_2^2 + \cdots + t_n^2)^{1/2})] \\ &= f(t_1^2 + t_2^2 + \cdots + t_n^2). \end{aligned}$$

Hence $\mathbf{Y} - \mathbf{M}$ has a spherically symmetric distribution. Conversely if $\mathbf{Y} = P\mathbf{X}$ has a spherically symmetric distribution, then $\mathbf{t}'\mathbf{X}$ and X_1 have the same type of distribution.

We now give an example where $\mathbf{X}$ does not possess a finite dispersion matrix. Take $\mathbf{X}$ to be a vector of n independent stable random variables $X_1, X_2, \ldots, X_n$ of the same type. The bivariate Cauchy considered earlier is another example. Can we get examples where $\mathbf{X}$ is neither stable nor has finite dispersion matrix?

Generalizing this problem suppose m is fixed and $1 \leqslant m \leqslant n$. Can one find $\mathbf{X}$ such that for all $m \times n$ matrices L of rank m the distribution of $L\mathbf{X}$ is of the same type as that of $(X_1, X_2, \ldots, X_m)$? We conjecture that the answer is no. It is easy to prove this when $\mathbf{X}$ has finite dispersion matrix.

We propose another problem: Suppose $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed random vectors such that for any pair of $m \times n$ matrices L_1, L_2 of rank m, there exist an $m \times n$ matrix L_3 of rank m such that the distributions of $L_1\mathbf{X} + L_2\mathbf{Y}$ and $L_3\mathbf{X}$ are of the same type. Can one characterize the distribution of $\mathbf{X}$? For $m = n$, Parthasarathy [4] has shown that $\mathbf{X}$ must be multivariate normal.

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