Pontryagin–Thom construction for approximation of mappings by embeddings

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Dedicated to J. Keesling for his 60th birthday

Abstract

Let \( n \geq 3 \) and \( d < \frac{n-3}{2} \) be positive integers, \( f: S^n \to S^n \) be a \( C^0 \)-mapping, and \( J: S^n \subset \mathbb{R}^{2n-d} \) denote the standard embedding. As an application of the Pontryagin–Thom construction in the special case of the two-point configuration space, we construct complete algebraic obstructions \( O(f) \) and \( \bar{O}(f) \) to discrete and isotopic realizability (realizability as an embedding) of the mapping \( J \circ f \). The obstructions are described in terms of stable (equivariant) homotopy groups of neighborhoods of the singular set \( \Sigma(f) = \{(x, y) \in S^n \times S^n \mid f(x) = f(y), \ x \neq y \} \).

A standard method of solving problems in differential topology is to translate them into homotopy theory by means of bordism theory and Pontryagin–Thom construction. By this method we give a generalization of the van-Kampen–Skopenkov obstruction to discrete realizability of \( f \) and the van-Kampen–Melikhov obstruction to isotopic realizability of \( f \). The latter are complete only in the case \( d = 0 \) and are the images of our obstructions under a Hurewicz homomorphism.

We consider several examples of computation of the obstructions.

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1. Introduction

Let \( f : S^n \to \mathbb{R}^{2n-d} \) be an arbitrary continuous mapping. We assume that the mapping \( f \) is in the metastable range, i.e., \( 0 \leq d \leq \frac{n-3}{2} \), and that \( n \geq 3 \) (the latter condition is to ensure codimension three).

**Definition 1.1.** We shall call a mapping \( f : S^n \to \mathbb{R}^{2n-d} \) discretely realizable if for arbitrary \( \varepsilon > 0 \) there exists an embedding \( g : S^n \to \mathbb{R}^{2n-d} \) such that \( \text{dist}_{C^0}(f, g) < \varepsilon \).

**Definition 1.2.** We shall call a mapping \( f : S^n \to \mathbb{R}^{2n-d} \) isotopically realizable if there exists a continuous homotopy \( F : S^n \times [0; 1] \to \mathbb{R}^{2n-d} \) such that the following conditions hold:

(i) \( f \) coincides with the restriction \( F|_{S^n \times \{0\}} : S^n \times \{0\} \to \mathbb{R}^{2n-d} \);
(ii) the restriction \( F|_{S^n \times (0; 1]} : S^n \times (0; 1] \to \mathbb{R}^{2n-d} \) is a smooth isotopy (in particular, \( F|_{S^n \times \{t\}} \) is a smooth embedding for \( 0 < t \leq 1 \)).

The following result shows that discrete realizability does not imply isotopic realizability.

**Theorem** (Melikhov [17]; see also [6]). There exists a mapping \( f : S^3 \to \mathbb{R}^6 \) that is discretely realizable but not isotopically realizable.

In the present paper we describe algebraically a complete obstruction \( \tilde{O}(f) \) to discrete realizability and a complete obstruction \( O(f) \) to isotopic realizability of an arbitrary smooth mapping \( f \) in the metastable range, developing an approach of Skopenkov (see [22,6]). Our main results are stated in Section 3.

If \( f : S^n \to \mathbb{R}^{2n-d} \) is given, or assumed to lie in a given class of mappings, we call the question of discrete (isotopic) realizability of \( f \) the Discrete (Isotopic) Realization Problem. The solution of the Discrete Realization Problem is related to computation of stable homotopy groups of spheres, see [5]. Let us state the basic algebraic tasks and recall some results concerning these notions.

We introduce a non-negative integer valued function \( d(n) \) and an integer multivalued function \( N(d) \) as follows. Let us consider the space \( A(n,d) \) of mappings \( f : S^n \to \mathbb{R}^{2n-d} \) that factor into the composition of a continuous mapping \( g : S^n \to S^n \) and the standard embedding \( I_{n,d} : S^n \subset \mathbb{R}^{n+1} \subset \mathbb{R}^{2n-d} \).

**Definition 1.3.** Let \( d(n) \) denote the maximal integer such that an arbitrary mapping \( f \in A(n,d(n)) \) is discretely realizable.

**Definition 1.4.** Let \( N(d) \) denote the set of positive integers \( n \) such that an arbitrary mapping \( f \in A(n,d) \) is discretely realizable.
Problem 1.5 (Asymptotic Discrete Realization Problem). Find the set \( N(d) \) and the function \( d(n) \). In particular, compute the maximal \( d \) such that the complement to \( N(d) \) (in the set \( \mathbb{N} \) of nonnegative integers) is a finite set, and describe the set \( N(1) \).

Theorem 1.6 [5,3]. (i) \( N(0) \) contains all integers except 1, and possibly 2, 3, 7.\(^1\)
(ii) \( d(n) \) has no upper bound.

Remark. The proof of (ii) is not elementary and is based on the Cohen immersion theorem [10]: an arbitrary smooth \( m \)-manifold can be immersed in \( \mathbb{R}^{2n-\alpha(n)} \), where \( \alpha(n) \) is the number of units in the dyadic expansion of \( n \).

Definition 1.7. We say that a smooth mapping \( f : S^n \to S^n \) is generic if associated mapping \( f \times f : S^n \times S^n \to S^n \times S^n \) is transversal along the diagonal \( \Delta \subset S^n \times S^n \) and the singular set \( \Sigma_f \subset S^n \times S^n \setminus \Delta \) is homeomorphic to the interior of a smooth compact manifold with boundary. (Here \( \Delta \) denotes the diagonal \( \{(x,x) | x \in S^n\} \) and \( \Sigma_f = \{(x,y) | x \neq y, f(x) = f(y)\} \).

Remark 1.8. An arbitrary stable mapping \( f : S^n \to S^n \) is generic. (The mapping \( f \) is called stable if there exists an \( \varepsilon > 0 \) such that an arbitrary mapping \( g : S^n \to S^n \) with \( \text{dist}_{\mathcal{C}_\infty}(f,g) < \varepsilon \) is equivalent to \( g \) via some diffeomorphisms in the domain and the range. For the properties of stable \( \mathcal{C}_\infty \)-mappings see, e.g., [8].)

The following conjecture yields an affirmative (particular) answer to the problem in [4].

Conjecture 1.9. Suppose that \( d \leq \frac{n-3}{2} \), and let \( f \in A(n,d) \) be a discretely realizable generic smooth mapping. Then \( f \) can be arbitrarily \( C^1 \)-closely approximated by a smooth embedding. (That is, for arbitrary \( \varepsilon > 0 \) there exists a smooth embedding \( g : S^n \to \mathbb{R}^{2n-d} \) such that \( \text{dist}_{C_1}(f,g) < \varepsilon \) where \( \text{dist}_{C_1} \) denotes distance with respect to values and first partial derivatives.)

Moreover, the mapping \( f \) is a Prem-mapping, i.e., the embedding \( g \) can be taken “strictly above” \( f \) with respect to the projection \( \pi : \mathbb{R}^{2n-d} \setminus \{0\} \to I_{n,d}(S^n) \), more precisely, \( \pi \circ g = I_{n,d} \circ f \).

The following two conjectures can make the present paper interesting to the general audience.

Conjecture 1.10. There exists a generic immersion \( f : S^7 \to \mathbb{R}^{11} \) such that the composition of \( f \) with the inclusion \( \mathbb{R}^{11} \subset \mathbb{R}^{13} \) is discretely realizable but not \( C^1 \)-approximable by embedding.

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\(^1\) In [2, Theorem 3] examples of mappings \( S^3 \to S^3, S^7 \to S^7 \) was constructed. The author claimed that the mappings are not discretely realizable in \( \mathbb{R}^6 \) and \( \mathbb{R}^{14} \). The proof of Lemma 27 contains a gap: the subgroups \( K_1 \) and \( K_2 \), generally speaking, are different.
Conjecture 1.11. There exists a mapping \( f : S^n \to \mathbb{R}^{2n-1}, \) \( n \) is sufficiently large, that is discretely but not isotopically realizable, although the first (cohomological) obstruction \( o(f) \) to isotopic realizability is trivial. (The obstruction \( o(f) \) was defined in [6] and is the image of \( O(f) \) under a Hurewicz homomorphism; see Section 3 for more details.)

This mapping is detected by the total obstruction \( O(f) \) as well as by the equivariant analog of the first-order functional operation \( Sq^2 \) in the Steenrod–Sitnikov cohomology group. (S.A. Melikhov has previously constructed a similar example \( S^9 \to \mathbb{R}^{13} \) using different arguments [18], but his technique fails to produce such an example \( S^n \to \mathbb{R}^{2n-d} \) with \( d < 5 \).

However, in general, the case \( f \in A(n, d) \) seems to be more important. This case could be considered as a dynamical system on the standard sphere. In particular, our results give a solution of the Daverman Problem and a particular solution of the Generalized Daverman Problem.

Definition 1.12. We say a metric compactum \( X \) is \( S^n \)-like if for any \( \varepsilon > 0 \) there is an \( \varepsilon \)-mapping \( f : X \to S^n \) of the compactum into the standard sphere of dimension \( n \).

Problem 1.13 (Daverman Problem, [11, Problem E16]). Is it true that an arbitrary \( S^n \)-like compactum can be embedded in the space \( \mathbb{R}^{2n} \)?

For the Generalized Daverman Problem see [3,5].

One can define analogous obstructions \( O(f), \tilde{O}(f) \) in the case of mapping \( f : T^n \to T^n \subset \mathbb{R}^{2n-d} \). By a result of Keessling and Wilson, the obstruction \( \tilde{O}(f) \) in the case \( f : T^n \to T^n \subset \mathbb{R}^{2n}, \ n = 3, 7 \) is trivial, see [15]. These considerations give rise to a new relationship between Topological Dynamics and Stable Homotopy Theory which is a subject of investigation in a further paper.

2. Preliminary constructions

This chapter is organized as follows. In Section 2.1 we recall a definition of the framed bordism group \( \Omega_m^{fr}(X, A) \), where \( (X, A) \) is an arbitrary pair of CW-complexes. In fact, this framed bordism group is nothing but the \( m \)th stable homotopy group of the pair \( (X, A) \). In Section 2.3 we introduce an equivariant analogue \( \Omega_m^{sf(k)}(X, A; G) \) of this group, where \( k \) is a positive integer and \( G \in [X; \mathbb{R}P^\infty] \) a given mod 2 cohomology class. The abbreviation \( sf \) stands for “skew framing” by means of \( k \) copies of the (possibly non-oriented) line bundle associated with \( g \). Next, for a finite dimensional compactum \( X \) we define the “Steenrod–Sitnikov skew-framed bordism group” \( \Omega_m^{sf(k)}(X; G) \) (Section 2.6), which is an extraordinary equivariant version of the usual Steenrod–Sitnikov homology group. (Since for \( X \) a CW-complex this coincides with the previous group, we use the same notation.)

We also consider the “Čech skew-framed bordism group” \( \tilde{\Omega}_m^{sf(k)}(X; G) \) (Section 2.8) and the derived limit associated with a sequence of neighborhoods of \( X \) (Section 2.9). These three groups \( \Omega_m^{sf(k)}(X; G), \tilde{\Omega}_m^{sf(k)}(X; G), \lim_{\to} \Omega_m^{sf(k)}(X_i; G) \) are related by a short exact
sequence. In addition, we have Hurewicz homomorphisms from these framed bordism groups to homology groups with local coefficients.

We assume all manifolds to be $C^1$-smooth and all mappings between manifolds to be $C^1$-smooth.

2.1. The framed bordism group $\Omega^fr_m(X, A)$

Let $M^m$ be an oriented compact $m$-manifold (possibly not connected and with non-empty boundary $\partial M$). We denote by $\nu(M)$ the sable normal bundle over $M$, i.e., the bundle (up to the stable equivalence) determined by the regular neighbourhood of a smooth embedding $M \subset \mathbb{R}^k$, $k \gg m$. Let $\Sigma$ be a framing of the stable normal bundle over $M$, i.e., an isomorphism $\Sigma : s \epsilon \cong \nu(M)$, where $s$ is a positive integer, $s > m$, and $\epsilon$ denotes the trivial line bundle over $M$. (Clearly, existence of such an isomorphism implies that $M$ is stably parallelizable. Also we assume that the isomorphism $\Sigma$ corresponds with the orientation of the stable normal bundle and the canonical orientation of the framing.) We do not exclude the possibility $M = \emptyset$.

Let $(X, A)$ be a pair of CW-complexes, let us consider the monoid $O^fr_m(X, A)$ formed by triples $(M, \phi, \Sigma)$, where $M^m$ is as above, $\phi : (M, \partial M) \to (X, A)$ is a continuous mapping and $\Sigma$ is a framing of $M$. The monoid operation is given by disjoint union. A triple $(M, \phi, \Sigma)$ is said to be null-bordant if there exist:

- an oriented compact manifold pair $(W, V)$ such that $\partial W = M \cup_{\partial M} V$;
- a framing $\Psi$ of the normal bundle of $W$ such that $\Psi|_M = \Sigma$;
- a mapping $\chi : (W, V) \to (X, A)$ such that $\chi|_M = \phi$.

Two triples $(M_0, \phi_0, \Sigma_0)$ and $(M_1, \phi_1, \Sigma_1)$ are said to be bordant if the triple $(M_0 \cup (-M_1), \phi_0 \cup \phi_1, \Sigma_0 \cup \Sigma_1)$ is null-bordant, where $-M_1$ denotes $M_1$ with reversed orientation. The quotient of $O^fr_m(X, A)$ by the equivalence relation of bordism is an Abelian group, which we denote by $\Omega^fr_m(X, A)$. (The inverse element is determined by reversing of the orientation.)

By the Pontryagin–Thom construction we have $\Omega^fr_m(S^0, pt) = \Pi_m$, where $\Pi_m = \lim_{s \to \infty} \pi_{m+s}(S^s)$ is the $m$th stable homotopy group of spheres. This construction was presented in [20]. For the generalization of the Pontryagin–Thom construction in case of an arbitrary cobordism theory see [23]. We will use the special case of the construction in the next section. Moreover, $\Omega^fr_m(X, A) = \lim_{s \to \infty} \pi_{m+s}(E^s X, E^s A)$, where $E^s$ is the standard $s$-suspension, see [9]. In the case where $A = pt \in X$, we have $\Omega^fr_m(X, pt) = \lim_{s \to \infty} \pi_{m+s}(E^s X, pt)$. The latter is the $m$th stable homotopy group of the pointed space $X$.

In the next section we introduce a local coefficient system which will be used later to generalize the group $\Omega^fr_m$ to an equivariant setting.

2.2. Homology and bordism with local coefficients

Given a CW-complex $X$ and a cohomology class $G \in H^1(X; \mathbb{Z}/2)$, we define a left $\mathbb{Z}\pi$-module $O_G$, where $\pi = \pi_1(X)$, by the following action of $\pi$ on the group $\mathbb{Z}$.
of integers: \( f \mapsto (-)^{G,h(f)} \) where \( f \in \pi \), the bracket denotes the canonical pairing between homology and cohomology, \( h \) denotes the Hurewicz homomorphism \( h: \pi = \pi_1(X) \to H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z}/2) \), and \((-)^{}\) stands for the nontrivial automorphism of \( \mathbb{Z} \) (multiplication by \(-1\)).

This defines a local coefficient system \( \mathcal{O}_G \) on the space \( X \). We recall that there is a canonical bijection between \([X, \mathbb{R}P^\infty] \) and \( H^1(X; \mathbb{Z}/2) \), which is given by \([g] \mapsto g^*(G_{\mathbb{R}P^\infty})\), where \( g: X \to \mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) \) is a mapping and \( G_{\mathbb{R}P^\infty} \) denotes the generator of \( H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2 \). Therefore \( \mathcal{O}_G \) can be equivalently defined as the inverse image (in the sense of sheaf theory) \( g^*\mathcal{O}_{G_{\mathbb{R}P^\infty}} \) of the standard local coefficient system on \( \mathbb{R}P^\infty \), where \( g \) is any representative of the homotopy class corresponding to \( G \).

As usual, the chain complex of a pair \((X, A)\) of CW-complexes with coefficients in \( \mathcal{O}_G \) is defined by \( C_*(X, A; G) = C_*(X, A; \mathbb{Z}) \otimes_{\mathbb{Z}_G} \mathcal{O}_G \) where \((\tilde{X}, \tilde{A})\) denotes the universal cover. The homology groups of this chain complex are denoted by \( H_*(X, A; G) \). They are isomorphic to the equivariant homology groups \( H^*_G(X, A; \mathbb{Z}) \) where \((\tilde{X}, \tilde{A})\) is the 2-cover of \((X, A)\) associated with \( G \), and \( \mathbb{Z}/2 \) acts on \( \tilde{X} \) by exchanging the sheets and on the coefficients \( \mathbb{Z} \) by \( 1 \leftrightarrow (-1) \). The cover \( \tilde{X} \to X \) is well defined by the following homomorphism \( G \circ h: \pi_1(X) \to H_1(X) \to \mathbb{Z}/2 \), where \( G \in \text{Hom}(H_1(X); \mathbb{Z}/2) \simeq H^1(X; \mathbb{Z}/2) \).

If \( X \) is a connected \( m \)-manifold with boundary \( \partial X = A \), we say that \( X \) is \( G \)-orientable if \( H_m(X, A; G) = \mathbb{Z} \), in which case a choice of generator of this group is called a \( G \)-orientation of \( X \). More generally, a manifold is called \( G \)-orientable if each connected component \( M \) is \( G|M \)-orientable, and a \( G \)-orientation consists of \( G|M \)-orientations of all connected components \( M \). In other words, a \( G \)-orientable manifold is a manifold where orientation is reversed along those and only those paths whose classes in \( \pi \) act nontrivially on the \( \mathbb{Z}_G \)-module \( G \).

Given a cohomology class \( G \in H^1(X; \mathbb{Z}/2) \), one may consider the (non-framed) bordism theory of pairs \((M, \varphi)\) where \( \varphi: (M, \partial M) \to (X, A) \) is a mapping of a \( \varphi^*(G) \)-orientable manifold \( M \). In the next section this equivariant bordism theory will be enriched by an equivariant framing.

### 2.3. Skew-framed bordism group \( \Omega^s_m(k) (X, A; G) \)

Let \( M \) be a compact \( m \)-manifold, possibly non-connected and with non-empty boundary, \( k \) a nonnegative integer, and \( \kappa \) the \( 1 \)-dimensional vector bundle over \( M \) associated with a given cohomology class \( G \in H^1(M; \mathbb{Z}/2) \). (That is, \( \kappa = \tilde{g}^*\gamma \) where \( \gamma \) denotes the canonical line bundle over \( \mathbb{R}P^\infty \) and \( \tilde{g}: X \to \mathbb{R}P^\infty \) is any representative of \( G \).) We define a skew \( k \)-framing of \( M \) (with respect to \( G \)) to be a choice of \( kG \)-orientation of \( M \) together with a \( kG \)-orientation preserving isomorphism \( \xi: v(M) \simeq k\kappa \oplus s\varepsilon \) of the stable normal bundle \( v(M) \) with the Whitney sum of \( k \) copies of the line bundle \( \kappa \), up to \( s \) copies of the trivial line bundle \( \varepsilon \).

Let \((X, A)\) be a pair of CW-complexes, equipped with a cohomology class \( G \in H^1(X; \mathbb{Z}/2) \). We consider the monoid \( \Omega^s_m(k) (X, A; G) \) formed by triples \((M, \varphi, \xi)\), where \( M \) is as above, \( \varphi: (M^m, \partial M) \to (X, A) \) is a continuous mapping, and \( \xi \) is a skew \( k \)-framing of \( M \) with respect to \( G = \varphi^*(G) \). The monoid operation is given by disjoint union of mappings. We call a triple \((M, \varphi, \xi)\) null-bordant, if there exist:
– a compact manifold pair \((W, V)\) such that \(\partial W = M \cup_{\partial M = \partial V} V\);
– a mapping \(\chi : (W, V) \rightarrow (X, A)\) such that \(\chi|_M = \psi\);
– a skew \(k\)-framing \(\Psi\) of the normal bundle of \(W\) with respect to \(\chi^*(G)\) such that \(\Psi|_M = \Sigma\).

Two triples \((M_0, \psi_0, \Sigma_0)\) and \((M_1, \psi_1, \Sigma_1)\) are said to be bordant if the triple \((M_0 \sqcup M_1, \psi_0 \cup \psi_1, \Sigma_0 \sqcup (\Sigma_1))\) is null-bordant, where \(-\Sigma_1\) denotes \(\Sigma_1\) with reversed \(k\psi_1^*\)\((G)\)-orientation. The quotient of \(O^s_m(X; A; G)\) by the equivalence relation of bordism is an Abelian group, which we denote by \(\Omega^s_m(X; A; G)\). (The inverse element is determined by reversing of the \(k\psi^*(G)\)-orientation.) In the case \(A = \emptyset\) we define the absolute (non-reduced) skew-framed bordism group \(\Omega^s_m(X; G) := \Omega^s_m(X; \emptyset; G)\).

Clearly, if \(g\) is trivial, \(\Omega^s_m(X; A; G) = \Omega^s_m(X, A)\). In the case \(g \neq 0\) by the straightforward generalization we obtained the following construction. Let \(G_{\mathbb{R}P^\infty}\) denote the generator of \(H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)\), then \(\Omega^s_m(\mathbb{R}P^\infty; G_{\mathbb{R}P^\infty}) = \lim_{s \rightarrow \infty} \pi_{m+s}((E^G\mathbb{R}P^\infty_{k-1})\).

where the truncated projective space \(\mathbb{R}P^\infty_{k-1}\) is defined to be \(\mathbb{R}P^\infty / \mathbb{R}P^{k-1}\). This group plays a similar role to the role played by the bordism group of a point in non-equivariant bordism theories. Moreover,

\[
\Omega^s_m(X; A; G) = \lim_{s \rightarrow \infty} \pi_{m+s}((E^G(\mathbb{R}P^\infty_{k-1} \times G X), E^G(\mathbb{R}P^\infty_{k-1} \times G|_A A))).
\]

Here \(\mathbb{R}P^\infty_{k-1} \times_G X\) denotes \(((S^\infty/\Sigma^{k-1}) \times \tilde{X})/T\) where \(\tilde{X}\) is the double cover of \(X\) corresponding to the kernel of \(\pi_1(X) \rightarrow H_1(X; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2\) (evaluation of \(G\)), and \(T\) is the diagonal involution.

In particular, \(\Omega^s_m(E^G\mathbb{R}P^\infty_{k-1})\), because \(T(k\eta) = \mathbb{R}P^\infty_{k-1}\), where \(T(k\eta)\) is the Thom space of the \(k\)-dimensional bundle splitted into the direct sum of \(k\) isomorphic line bundles.

2.4. Equivariant Hurewicz homomorphism

It is well-known that the ordinary homology theory (with integral coefficients) of CW-complexes is isomorphic to the bordism theory of singular (non-framed) oriented pseudo-manifolds. We recall that an orientable \(m\)-pseudo-manifold \((P, \partial P)\) is a compact polyhedron that can be triangulated by a finite simplicial complex \((K, L)\) where each simplex is a face of some \(m\)-simplex, the union of the interiors of \(m\)-simplices and \((m-1)\)-simplices is connected, and the \(m\)-simplices of \(K\) can be oriented so that their algebraic sum is a \(\mathbb{Z}\)-cycle (with respect to the simplicial chain complex of the pair \((K, L)\)). An orientation of an orientable \(m\)-pseudo-manifold \((P, \partial P)\) is a choice of generator in \(H_m(P, \partial P; \mathbb{Z}) \simeq \mathbb{Z}\). Finally, a singular oriented \(m\)-pseudo-manifold, or geometric \(m\)-cycle (with integral coefficients) on a CW-complex \(X\) is a mapping \(\psi : (\sqcup P_i, \sqcup \partial P_i) \rightarrow (X, A)\) of disjoint union of a finite number of oriented \(m\)-pseudo-manifolds into \(X\). This description immediately yields the Hurewicz homomorphism

\[
\gamma : \Omega^s_m(X, A) \rightarrow H_m(X, A; \mathbb{Z})
\]

which factors through the ordinary (non-framed) bordism group.
We define an (oriented) geometric \( m \)-cycle with coefficients in a locally constant sheaf \( \mathcal{O} \) analogously, referring to \( \varphi^* \mathcal{O} \)-cycles instead of \( \mathbb{Z} \)-cycles. The corresponding bordism theory clearly yields the homology groups with coefficients in \( \mathcal{O} \). In particular, this leads to the equivariant Hurewicz homomorphism

\[
\gamma : \Omega^k_m(X, A; G) \to H_m(X, A; kG)
\]

where \( kG \in H^1(X; \mathbb{Z}/2) \) is trivial if \( k \) is even, and equals \( G \) if \( k \) is odd. This homomorphism factors through the non-framed equivariant bordism group discussed in the end of Section 2.2.

2.5. Reduced skew-framed bordism group \( \widetilde{\Omega}^k_m(X; G) \)

It turns out that the standard decomposition \( \Omega^k_m(X) \cong \Omega^k_m(pt) \oplus \Omega^k_m(X, pt) \) does not extend literally to the equivariant case, since a mapping \( g : X \to K(\mathbb{Z}/2, 1) \) does not factor through the point \( pt \) unless it is inessential. This leads to the following definition.

Let us consider the monoid \( \widetilde{\Omega}^k_m(X; G) \) formed by triples \((M, \varphi, \Xi)\) where \( \varphi : M \to X \) is a mapping of a closed \( m \)-manifold \( M \) with a skew \( k \)-framing \( \Xi \) with respect to \( \varphi^*(G) \), which is a skew \( k \)-framed boundary, i.e., we assume that there exists a compact manifold \( W^{m+1}, \partial W = M \), a class \( F \in H^1(W; \mathbb{Z}/2) \) such that \( F|_M = \varphi^*(G) \), and a skew \( k \)-framing \( \Psi \) of \( W \) with respect to \( F \) such that \( \Psi|_M = \Xi \). Defining bordism in the standard way, we arrive at the reduced skew-framed bordism group \( \widetilde{\Omega}^k_m(X; G) \), which is isomorphic to the kernel of \( g_* : \Omega^k_m(X, G) \to \Omega^k_m(\mathbb{RP}^\infty; G_{\mathbb{RP}^\infty}) \) where \( g : X \to \mathbb{RP}^\infty \) is a representative of \( G \).

Clearly, in the case of trivial \( G \), the groups \( \Omega^k_m(X, A) \) and \( \widetilde{\Omega}^k_m(X/A) \) are canonically isomorphic (using standard arguments of homotopy theory). In the case of non-trivial \( G \), this generalizes to a canonical isomorphism

\[
\Omega^k_m(X, A; G) \cong \widetilde{\Omega}^k_m(X \cup \text{cyl}(g|_A); [g \cup \pi])
\]

where \( g : X \to \mathbb{RP}^\infty \) is a representative of \( G \), and \( \pi : \text{cyl}(g|_A) \to \mathbb{RP}^\infty \) denotes the projection of the space \( \text{cyl}(g|_A) = A \times [0; 1] \cup g|_A \times \{0\} \subset \mathbb{RP}^\infty \). (Clearly, \( \mathbb{RP}^\infty \) can be replaced by \( \mathbb{RP}^{m+1} \).) This isomorphism can be defined by

\[
[(M, \varphi, \Xi)] \mapsto [(\partial(M \times I), \varphi|_{M \times 0} \cup \psi|_{\partial M \times I} \cup \psi|_{M \times 1} \times I|_{\partial M \times I})]
\]

where \( \psi \) denotes the obvious map

\[
\psi : \text{cyl}(\partial M \subset M) \to \text{cyl}(g|_A).
\]

In particular, if \( g|_A \) induces monomorphisms of homotopy groups up to dimension \( m + 1 \), there is a canonical isomorphism

\[
I : \Omega^k_m(X, A; G) \cong \widetilde{\Omega}^k_m(X; G).
\]

2.6. Steenrod–Sitnikov skew-framed bordism group

The applications of extraordinary Steenrod homology groups (at least in non-equivariant setting) have been considered long before, see [13] and references there. We will formulate the corresponded definition in terms of the Pontryagin–Thom construction.
Let \((X, A)\) be an arbitrary pair of finite dimensional compacta, \(g : X \to \mathbb{R}P^\infty\) be a mapping representing the given 1-dimensional mod 2 cohomology class \(G\). By Pontryagin–Nobeling theorem an arbitrary compactum \(X\) is embeddable into \(\mathbb{R}^{2n+1}\), see [21]. Without loss of generality (up to homotopy) we may assume that \(g\) factors as \(g = I \circ g'\); \(X \subset \mathbb{R}P^N \subset \mathbb{R}P^\infty\), where \(g'\) is an embedding, \(I : \mathbb{R}P^N \subset \mathbb{R}P^\infty\) is the standard inclusion and \(N \geq 2\dim X + 1\) is a fixed integer.

We introduce the group \(\Omega^m_{sf}(X, A; G)\) for arbitrary pair \((X, A)\) of finite dimensional compacta equipped with an embedding \(g : X \to \mathbb{R}P^\infty\) such that the image is contained in a finite dimensional projective subspace. Our definition will be independent from the choice of the embedding \(g\), since any two homotopic embeddings with images in a finite dimensional projective space are ambient isotopic in \(\mathbb{R}P^\infty\). If \((X, A)\) is homeomorphic to a pair of CW-complexes, this group coincides with the skew-framed bordism group defined in Section 2.3, and we will use the same notation for both groups.

Let us consider a triple \((M, \varphi, \Xi)\), where \(M\) is an \((m + 1)\)-manifold (possibly non-compact, non-connected and with boundary, which is possibly non-compact and non-connected), \(\varphi : M \to \mathbb{R}P^\infty \setminus g(X)\) is a continuous proper mapping such that the restriction \(\varphi|_{\partial M}\) is a proper mapping \(\partial M \to \mathbb{R}P^\infty \setminus g(A)\), and \(\Xi\) is a skew \(k\)-framing of \(M\) with respect to \([\varphi]\). Notice that in the case \(A = \emptyset\) the requirement on \(\varphi|_{\partial M}\) amounts to requiring that \(M\) have a compact boundary. These triples form a monoid with respect to disjoint union of mappings. Bordism of two triples \((M_\pm, \varphi_\pm, \Xi_\pm)\) is defined in the standard way (in particular, if \(A = \emptyset\) and \(\partial M_\pm = \emptyset\), a bordism \((W, \chi, \Psi)\) between these two triples may have \(\partial W = M_+ \cup M_- \cup N\) where \(N\) is an arbitrary compact manifold). The corresponding quotient of this monoid is denoted by \(\Omega^m_{sf}(X, A; G)\). Notice that any triple \((M, \varphi, \Xi)\) with compact \(M\) is null-bordant in the sense of this definition.

Thus, in particular, nontrivial elements of the absolute group \(\Omega^m_{sf}(X; \emptyset; G) := \Omega^m_{sf}(X, \emptyset; G)\) are represented by non-compact \((m + 1)\)-manifolds \(M\) with compact boundary, skew \(k\)-framed with respect to a proper mapping of \(M\) into \(\mathbb{R}P^\infty \setminus g(X)\). Bordism between such representatives is defined in the standard way, using an \((m + 2)\)-manifold \(W\) whose boundary consists of \(M_+\), \(M_-\) and a compact bordism between their boundaries.

It will be convenient to have the following reformulation of this definition. Let
\[
\mathbb{R}P^\infty = U_0 \subset U_1 \subset \cdots \subset \bigcap_{i=0}^{\infty} U_i = g(X)
\]
\[
\mathbb{R}P^\infty = V_0 \subset V_1 \subset \cdots \subset \bigcap_{i=0}^{\infty} V_i = g(A)
\]
be an infinite system of open neighborhoods of the pair \((g(X), g(A))\) in \(\mathbb{R}P^\infty\). We assume that each \((U_i, V_i), i > 0\) is homotopy equivalent to a finite CW-pair. Then every element of \(\Omega^m_{sf}(X, A; G)\) can be represented by a triple \((M, \varphi, \Xi)\) such that \(\varphi(M_i, M_i \cap \partial M) \subset (U_i, V_i)\) for each \(i\), where
\[
M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset \bigcap_{i=0}^{\infty} M_i = \emptyset
\]
is a sequence of closed neighborhoods of infinity that are submanifolds of \(M\) (i.e., the \(M_i\)’s are closures of complements to compact submanifolds).
Thus we can assume that each element of $\Omega_{m}^{sf}(X, A; G)$ is represented by a triple $([M_1], \varphi, \Xi)$ where $M_0$ is a (possibly non-compact, non-connected and with boundary) $(m + 1)$-manifold, each $M_{i+1}$ is a closed neighborhood of infinity that is a submanifold of $M_i$, $\varphi: M_0 \to \mathbb{RP}^\infty$ is a mapping such that $\varphi(M_i, M_i \cap \partial M_i) \subset (U_i, V_i)$ for each $i$ (we do not require that $\varphi(M_0) \cap g(X) = \emptyset$ since this can be achieved by general position, using that $g(X)$ is contained in a finite dimensional subspace), and $\Xi$ is a skew framing of $M_0$ with respect to $\varphi^*(G_{\mathbb{RP}^\infty})$. Bordism of two such triples is defined in the straightforward way.

The reduced subgroup $\tilde{\Omega}_{m}^{sf}(X; G)$ contains only those elements that can be represented by triples $(M, \varphi, \Xi)$ with $\partial M = \emptyset$, and coincides with the kernel of the homomorphism $g_* : \Omega_{m}^{sf}(X; G) \to \Omega_{m}^{sf}(\mathbb{RP}^\infty; G_{\mathbb{RP}^\infty})$, which can be geometrically described as $[(M, \varphi, \Xi)] \mapsto [(\partial M, \varphi|_M, \Xi|_M)]$.

2.7. Čech skew-framed bordism group

Let $(X, A)$ be a pair of finite dimensional compacta, $g : X \to \mathbb{RP}^\infty$ an embedding with image contained in a finite dimensional projective subspace, and assume that the complement $\mathbb{RP}^N \setminus g(X)$ of the compactum in the projective space is equipped with a stratification (1). The Čech skew-framed bordism group $\hat{\Omega}_{m}^{sf}(X, A; G)$, where $G = g^*(G_{\mathbb{RP}^\infty}) \in H^1(X; \mathbb{Z}/2)$, is defined to be the inverse limit

$$\hat{\Omega}_{m}^{sf}(X, A; G) := \lim_{\leftarrow} \Omega_{m}^{sf}(U_i, V_i; G_i)$$

where $G_i$ denotes $G_{\mathbb{RP}^\infty}|_{U_i}$, and bonding maps are induced by the inclusions $(U_{i+1}, V_{i+1}) \subset (U_i, V_i)$.

The absolute group $\Omega_{m}^{sf}(X; G) := \Omega_{m}^{sf}(X, \emptyset; G)$ can be equivalently defined as $\lim_{\leftarrow} \Omega_{m}^{sf}(U_i; G_i)$. Let us give an explicit description of this group. We may consider an element of $\hat{\Omega}_{m}^{sf}(X; G)$ as the bordism class of a triple $([L_i], \varphi_i, \Xi_i)$, obtained from a triple $([M_i], \varphi, \Xi)$ considered in Section 2.6 (with compact $\partial M_0$) by setting $L_i = \partial M_i$, $\varphi_i = \varphi|_{L_i}$, and $\Xi_i = \Xi|_{L_i}$.

The reduced subgroup $\tilde{\Omega}_{m}^{sf}(X; G)$ contains only those elements of the inverse limit that are trivialized in the first group $\Omega_{m}^{sf}(U_0; G_0)$ of the inverse spectrum.

2.8. Derived limit $\tau_{m+1}^{sf}(X; G)$ and Milnor exact sequence

Let $\tau_{m+1}^{sf}(X; G)$ denote the derived limit $\lim_{\leftarrow} \Omega_{m+1}^{sf}(U_i; G_i)$ of the inverse spectrum considered in the previous section. Let us describe this group geometrically. Represent an element of the direct product of the groups $\Omega_{m+1}^{sf}(U_i; G_i)$ by a collection of triples $(M_i, \varphi_i, \Xi_i), i \in \mathbb{N}$. Two such collections $(M_i^\pm, \varphi_i^\pm, \Xi_i^\pm)$ represent the same element of the derived limit if and only if there exists a (possibly noncompact) manifold $W$ with boundary

$$\partial W = N \cup \bigcup_{i=1}^{\infty} (M_i^+ \cup M_i^-),$$
where \( N \) is compact, a mapping \( \chi : W \to \mathbb{R}P^\infty \) that restricts to \( \varphi_i^\pm \) over each \( M_i^\pm \) and takes sufficiently small neighborhoods of infinity in \( W \) into arbitrary small neighborhoods of \( g(X) \), and a skew \( k \)-framing \( \Psi \) of \( W \) that restricts to \( \pm \Xi_i^\pm \) over each \( M_i^\pm \).

The three groups constructed above (for simplicity we assume \( A = \emptyset \)) fit into the short exact sequence

\[ 0 \to \Gamma_{m+1}^{d(k)}(X; G) \to \Omega_{m}^{d(k)}(X; G) \to \hat{\Omega}_{m}^{d(k)}(X; G) \to 0. \]

This follows directly from the definitions. The analogous sequences are well known for arbitrary (extraordinary) homology theory (see [14] and references in [17]). One can analogously define Steenrod–Sitnikov and Čech homology with coefficients in the local system \( O_G \), namely:

\[ H_m(X; G) := H^{st}_{m}(C_*(U_i; G_i)); \quad \hat{H}_m(X; G) := \lim_{\leftarrow} H_m(U_i; G_i). \]

Then we arrive at the following commutative diagram with exact rows, see [14]:

\[
\begin{array}{cccccc}
0 & \to & \Gamma_{m+1}^{d(k)}(X; G) & \to & \Omega_{m}^{d(k)}(X; G) & \to & \hat{\Omega}_{m}^{d(k)}(X; G) & \to & 0 \\
0 & \to & \lim^{1} H_{m+1}(U_i; kG_i) & \to & H_{m}(X; kG) & \to & \hat{H}_{m}(X; kG) & \to & 0 \\
\end{array}
\]

2.9. The derived limit functor \( \lim^{1} \)

Let \( p_{i+1} : \Omega_{m+1}^{d(k)}(U_{i+1}; G_{i+1}) \to \Omega_{m+1}^{d(k)}(U_i; G_i) \) denote the homomorphism induced by the inclusion \( U_{i+1} \subset U_i \). We recall that an element \( \alpha \in \Gamma_{m+1}^{d(k)}(X; G) \), represented as a collection \( \{a_i, i \in \mathbb{N}\} \) of elements \( a_i \in \Omega_{m+1}^{d(k)}(U_i; G_i) \), is trivial if and only if the infinite sequence of equations

\[ a_i = p_{i+1}(a_{i+1}) \]

where \( i \in \mathbb{N} \), has a solution \( a_i \in \Omega_{m+1}^{d(k)}(U_i; G_i), i \in \mathbb{N} \).

**Example** ([17]; see also [6]). Let us assume that \( \Omega_{m+1}^{d(k)}(U_i; G_i) = \mathbb{Z} \) and \( p_i : \mathbb{Z} \to \mathbb{Z} \) is multiplication by 3. Consider the element \( \alpha \in \Gamma_{m+1}^{d(k)}(X; G) \) given by the collection \( a_i = 1, i \in \mathbb{N} \). The equation \( x_1 = \frac{3^i - 1}{3} \pmod{3^k} \) cannot be satisfied for arbitrary \( k \). It follows that \( \alpha \neq 0 \).

**Example** [18]. In the previous example, put \( a_i = 2 \). Then a solution of the system (2) is given by the collection \( x_i = -1 \).

2.10. Example

Let \( X = \mathbb{R}P^5 \) and take \( G \) to be trivial, so that \( \Omega_{m}^{d(k)}(\mathbb{R}P^5; G) = \Omega_{m}^{ht}(\mathbb{R}P^5) \). Let us show that the Hurewicz image \( \gamma(\Omega_{2}^{ht}(\mathbb{R}P^5)) \subset H_5(\mathbb{R}P^5; \mathbb{Z}) = \mathbb{Z} \) is the subgroup of index 2. For
an arbitrary mapping \( \varphi : M \to \mathbb{R}P^5 \), where \( M \) is a stably parallelizable 5-manifold, we consider the submanifold \( N = \varphi^{-1}(\mathbb{R}P^2 \subset \mathbb{R}P^5) \) and the restriction \( f = \varphi|_N : N \to \mathbb{R}P^2 \).

Since \( v_N = f^*v_{\mathbb{R}P^2 \subset \mathbb{R}P^5} \) and \( \mathbb{R}P^5 \) is orientable, \( f^*w_1(\tau_{\mathbb{R}P^5}) = w_1(\tau_N) \). Therefore

\[
\langle w^2_1(\tau_N); [N]_{\text{mod } 2} \rangle = \langle w^2_1(\tau_{\mathbb{R}P^5}); f_*[N]_{\text{mod } 2} \rangle = \deg_{\text{mod } 2}(f) \equiv \deg(\varphi) \pmod{2}.
\]

For the total normal Stiefel–Whitney class we have

\[
w(v_N) = f^*w(v_{\mathbb{R}P^2 \subset \mathbb{R}P^5}) = f^*w(\tau_{\mathbb{R}P^5}) = f^*(1 + w_1(\tau_{\mathbb{R}P^5}))^3 = (1 + w_1(\tau_N))^3.
\]

On the other hand, since \( N \) is 2-dimensional, \( w_2(v_N) = 0 \). Consequently \( w^2_1(\tau_N) = w^2_1(v_N) = 0 \) which implies \( \deg(\varphi) \equiv 0 \pmod{2} \). Obviously the double cover \( S^3 \to \mathbb{R}P^5 \) represents the generator.

Now let us consider \( X = \lim_\leftarrow \{\mathbb{R}P^5, p_i\} \) where \( \deg(p_i) = 3 \) for each \( i \). From the previous examples we deduce that the Hurewicz homomorphism \( \gamma : \Gamma_s^5(X) \to \lim_\leftarrow H_5(\mathbb{R}P^5; \mathbb{Z}) \) is not surjective.

### 2.11. Homomorphisms \( r^k_{nm} \) and \( \rho^k_{nm} \)

We will consider an additional structure on the constructed bordism and homology groups. This structure is determined by a collection of homomorphisms

\[
\begin{align*}
\rho^k_{nm} &: \Omega^s(k-n)(X, A; G) \to \Omega^s_m(k-m)(X, A; G), \\
r^k_{nm} &: H_0(X, A; (k-n)G) \to H_m(X, A; (k-m)G).
\end{align*}
\]

We define the homomorphism \( \rho^k_{nm} \) (3) as follows. Consider a triple \((L, \varphi, \Sigma) \in \Omega^s_n(k-n)(X, A; G)\), where \( \varphi : (L^n, \partial L) \to (X, A) \) is a continuous mapping and \( \Sigma \) is a skew \((k-n)\)-framing of the stable normal bundle over \( L \) with respect to \( \varphi^*(G) \). Let us represent the homotopy class \( \tilde{G} = G \circ \varphi \in [L^n; \mathbb{R}P^\infty] \) by a mapping \( \tilde{g} : L^n \to \mathbb{R}P^p \). For any \( m < n \), let \((M^n, \partial M) \subset (L^n, \partial L)\) be the submanifold determined by \( M = \tilde{g}^{-1}(\mathbb{R}P^m) \).

Obviousely, the stable normal bundle over \( M \) is canonically isomorphic to the direct sum of \((k-n)+(n-m)\) copies of the line bundle \( \kappa \to M \subset L \), where \( \kappa \) is classified by the cohomology class \( G|_M \). This isomorphism determines a skew \((k-m)\)-framing \( \Sigma' \) of the normal bundle over \( M \). We set \( \rho^k_{nm}((L, \varphi, \Sigma)) = [(M, \varphi|_M, \Sigma')] \). The homomorphism \( r^k_{nm} \) is defined analogously: we take the cap product of a cycle \( \varphi \in C_*(X, A; (k-n)G) \) with the cocycle \( g^*([i]^*) \in H^{n-m}(X; (k-m)G) \), where \( g : X \to \mathbb{R}P^\infty \) is a representative of \( G \) and \([i]^* \) is the element of \( H^{n-m}(\mathbb{R}P^\infty; (k-m)G|_{\mathbb{R}P^\infty}) \) given by the duals to the fundamental classes of the projective subspaces \( \mathbb{R}P^{n-m}(n-m) \subset \mathbb{R}P^N \).

It is clear that \( \gamma \circ \rho^k_{nm} = r^k_{nm} \circ \gamma \). The homomorphisms \( r^k_{nm} \) and \( \rho^k_{nm} \) give rise to the analogous homomorphisms \( r^k_{nm} \) (respectively \( r^k_{nm} \)) and \( \rho^k_{nm} \) (respectively \( \rho^k_{nm} \)) for Čech (respectively Steenrod–Sitnikov) local homology and Čech (respectively Steenrod–Sitnikov) skew-framed bordism.
3. Complete obstructions $\bar{O}(f)$, $O(f)$ to discrete and isotopic realizability

Let us fix an arbitrary continuous mapping $f : S^n \to \mathbb{R}^{2n-d}$ where $0 \leq d \leq \frac{n+3}{2}$. It gives rise to $f^2 = f \times f : S^n \times S^n \to \mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d}$, which is equivariant with respect to the factor exchanging involutions $T$ and $t$. We use the following notation: $\Delta_X = \{(x, x) \in X \times X | x \in X\}$,

$$\hat{K} = S^n \setminus \Delta_{S^n}, \quad K = \hat{K} / T$$

and

$$\hat{\Sigma} = (f^2)^{-1}(\Delta_{\mathbb{R}^{2n-d}}) \setminus \Delta_{S^n}, \quad \Sigma = \hat{\Sigma} / T.$$

Let $f_t : S^n \to \mathbb{R}^{2n-d}$, $t \in (0, 1]$, be a generic smooth regular homotopy (i.e., a generic homotopy which is a smooth immersion) such that $f_1$ is the (standard) embedding and $f_t \to f$ as $t \to 0$, which exists by [2], Lemma 2, where it was called a local isotopic realization of $f$. We may assume that $f_\epsilon$ is $\epsilon$-close to $f$ (in the $C^0$ topology) for each $\epsilon > 0$. Then

$$\hat{\Sigma}_\epsilon = (f_\epsilon^2)^{-1}(\Delta_{\mathbb{R}^{2n-d}}) \setminus \Delta_{S^n} \quad \text{and} \quad \Sigma_\epsilon = \hat{\Sigma}_\epsilon / T$$

are smooth $d$-dimensional submanifolds of $\hat{K}$ and $K$ respectively, contained in the $\epsilon$-neighborhood of $\hat{\Sigma} \cup \partial \hat{K}$ and $\Sigma \cup \partial K$ respectively, where $\epsilon = \epsilon(t) \to 0$ as $\epsilon \to 0$, and $\partial K = \partial \hat{K} / T$ where $\partial \hat{K}$ denotes the corona of the evident compactification $\hat{K} \cup \partial \hat{K}$ of the open manifold $\hat{K}$, which is homeomorphic to the complement of the interior of a regular neighborhood of $\Delta_{S^n}$. Let

$$\Phi : S^n \times S^n \times (0, 1] \to \mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d}$$

be the mapping defined by $\Phi(x, y, t) = (f_t(x), f_t(y))$, then

$$\hat{\Sigma}_{(0, 1]} = \Phi^{-1}(\Delta_{\mathbb{R}^{2n-d}}) \quad \text{and} \quad \Sigma_{(0, 1]} = \hat{\Sigma}_{(0, 1]} / T$$

are proper smooth submanifolds of $\hat{K} \times I \setminus \hat{\Sigma} \times 0$ and $K \times I \setminus \Sigma \times 0$, respectively.

Let us show that the manifold $\Sigma_{(0, 1]}$ is canonically equipped with a skew $(n-d)$-framing $\mathcal{E}$ with respect to the class $G \in [\Sigma_{(0, 1)}; \mathbb{R}P^n]$ inducing the $T$-space. The canonical $t$-equivariant trivialization of the normal bundle of the diagonal $\Delta_{\mathbb{R}^{2n-d}}$ in the target space $\mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d}$ induces a $T$-equivariant trivialization $\mathcal{E}$ of the normal bundle of $\Sigma_{(0, 1)}$ inside $\hat{K} \times (0, 1]$. Therefore the normal bundle $\nu(\Sigma_{(0, 1]} \subset K \times (0, 1])$ is canonically decomposed into the Whitney sum of $2n-d$ copies of the line bundle $\kappa$, associated with $G$. By the following lemma, the normal bundle $\nu_K$ is canonically stably isomorphic to the bundle $(-n)\kappa$, which determines the required stable decomposition $\mathcal{E}$ of the normal bundle of $\Sigma_{(0, 1]}$ into $n-d$ copies of $\kappa$.

**Lemma 3.1.** The tangent bundle $\tau_K$ is canonically stably isomorphic to $n\kappa$.

**Proof.** Let $\hat{\nu} = \{(x, -x) \in S^n \times S^n\}$ denote the anti-diagonal of $S^n$, then $\nu = \hat{\nu} / T$ is homeomorphic to $\mathbb{R}P^n$. Therefore, $\tau_{\nu} \oplus \epsilon \simeq (n+1)\kappa$, where $\epsilon$ denotes the trivial line bundle [19]. The normal bundle of $\nu$ inside $K$ can be obtained from $\tau_{\nu}$ by the twist of the fibers along the generator of $H_1(\nu)$. Hence $\nu(\nu \subset K) \oplus \kappa \simeq (n+1)\epsilon$. This determines a canonical stable isomorphism $\tau_K \simeq \tau_{\nu} \oplus \nu(\nu \subset K) \simeq (n+1)\kappa \oplus (-1)\kappa$. $\quad \Box$
Remark 3.2 (not used in the sequel). Let us give an alternative definition of the skew $(n-d)$-framing of $\Sigma_{(d)}$. By general position $(d \leq \frac{n-3}{2})$, the local isotopic realization $F: S^n \times (0, 1] \to \mathbb{R}^{2n-d} \times (0, 1]$, where $F(x, t) = (f_t(x), t)$, has no triple points, in particular, the double cover $\tilde{S}_{(d)}$ of $F$ is a smooth submanifold of $\mathbb{R}^{2n-d}$. By the same reason, the projection onto the first factor of $S^n \times S^n$ yields a diffeomorphism between $\tilde{S}_{(d)}$ and the submanifold $\tilde{S}_{(d)} = \{ x \in S^n \times (0, 1] \mid \exists y \neq x: F(x) = F(y) \}$ of $S^n \times (0, 1]$. Finally, the double cover $\tilde{S}_{(d)} \to \Sigma_{(d)}$ has the same point inverses as the double cover $F: \tilde{S}_{(d)} \to S_{(d)}$ whence $S_{(d)}$ is diffeomorphic to $\Sigma_{(d)}$ by a level-preserving diffeomorphism. But $S_{(d)}$ is the self-intersection of the immersed manifold $F(S^n \times (0, 1])$, which can be canonically framed using the canonical framing of the standard embedding $F(S^n \times 1) = f_1(S^n)$, and it follows by a well-known argument (see for instance [7] or [18]) that $S_{(d)}$ (and therefore $\Sigma_{(d)}$) is canonically equipped with a skew $(n-d)$-framing $\Xi$ with respect to the class $G \in [S_{(d)}; \mathbb{R}P^\infty]$ classifying the double cover $F: \tilde{S}_{(d)} \to S_{(d)}$. The proof of equivalence of the two definitions of $\Xi$ is left to the reader.

Definition 3.3. The double cover $\tilde{K} \times I \to K \times I$ is classified by some $G' \in [K \times I; \mathbb{R}P^\infty]$, which can be represented by a smooth embedding $e: K \times I \to \mathbb{R}P^\infty$. By definition, the triple $(\Sigma_{(d)}, e|_{\Sigma_{(d)}}, \Xi)$ represents an element, which we denote by $O(f)$, of the reduced skew-framed Steenrod–Sitnikov bordism group $\overline{\Omega}^{s(n-d)}_d(\Sigma \cup \partial K)$. It is straightforward that $O(f)$ does not depend on the choice of local isotopic realization $f_t$, since any other one $f'_t$ (with $f'_t$ being an embedding) is level-preserving regularly homotopic to $f_t$ through local isotopic realizations of $f$ [2, proof of Lemma 2]. By the same reason, $O(f) = 0$ if $f$ is isotopically realizable.

Similarly, the collection of triples $(\Sigma_{(d)}, e|_{\Sigma_{(d)}}, \Xi|_{\Sigma_{(d)}})$, where $\epsilon = 1, 2, 4, \ldots$, represents an element, which we denote by $\tilde{O}(f)$, of the reduced skew-framed Čech bordism group $\overline{\Omega}^{s(n-d)}_d(\Sigma \cup \partial K)$. Clearly (see also [2], proof of Lemma 2), it does not depend on the choice of immersions $f_1, f_1/2, f_1/4, \ldots$ and hence $\tilde{O}(f) = 0$ if $f$ is discretely realizable.

The following fact provides an additional algebraic information (from stable homotopy groups if $f \in \mathcal{A}(n, d)$) on the obstructions $O(f)$ and $\tilde{O}(f)$.

Proposition 3.4. Suppose that $0 \leq d < c \leq \frac{n-3}{2}$.

(i) Let $f: S^n \to \mathbb{R}^{2n-c}$ be a mapping and $J: \mathbb{R}^{2n-c} \subset \mathbb{R}^{2n-d}$ the standard inclusion. Then

\[ O(J \circ f) = \rho_{d}^{c}(O(f)) \quad \text{and} \quad \tilde{O}(J \circ f) = \tilde{\rho}^{c}_{d}(\tilde{O}(f)). \]

(For the definition of $\rho$ and $\tilde{\rho}$ see Section 2.11.)

(ii) If $f \in \mathcal{A}(n, d)$, the obstruction $\tilde{O}(f)$ lies in the image of the composition

\[ I \circ \tilde{\rho}_{0,d}^{c}: \overline{\Omega}^{s(n-d)}_d(\Sigma \cup \partial K, \partial K) \to \overline{\Omega}^{s(n-d)}_d(\Sigma \cup \partial K, \partial K; G) \to \overline{\Omega}^{s(n-d)}_d(\Sigma \cup \partial K; G). \]

Similarly for $O(f)$. (For the definition of $I$ see Section 2.5.)
Proof. To prove (ii) let us construct an element $\tilde{O}_0(f)$ such that $\tilde{O}(f) = I \circ \tilde{O}_0(f)$. We approximate $f$ by a generic smooth map $f_\epsilon \in A(n, d)$. As $f_\epsilon$ may be not an immersion, the manifold $\Sigma_\epsilon = \Sigma(f_\epsilon)$ is no longer compact in general. In this case we may assume that $\Sigma_\epsilon \subset K$ is a submanifold with boundary on $\partial K$. Since $\tilde{\Sigma}_\epsilon$ is a regular preimage of the diagonal $\Delta_{S^n} \subset S^n \times S^n$, the normal bundle $\nu(\tilde{\Sigma}_\epsilon) \subset \tilde{K}$ is canonically equivariantly stably trivial, hence $\nu(\Sigma_\epsilon) \subset K$ is canonically stably equivalent to $nK$. Thus by Lemma 3.1, $\Sigma_\epsilon$ admits a canonical stable trivialization $\mathcal{Z}$ of the normal bundle. The collection of triples $(\Sigma_\epsilon, e|_{\Sigma_\epsilon}, \mathcal{Z})$ for $\epsilon = 1, \frac{1}{2}, \frac{1}{4}, \ldots$ represents an element $\tilde{O}_0(f)$ of the relative Čech bordism group $\Omega^R_n((\Sigma \cup \partial K, \partial K)$. Starting from a generic smooth homotopy $f_\epsilon$, $f_0 = f$, $f_1 = f$, one analogously obtains an element $O_0(f) \in \Omega^R_n((\Sigma \cup \partial K, \partial K)$. To determine $\tilde{O}(f)$ we put the composition $I^2_{n,d} \circ f^2_\epsilon : K \rightarrow (S^d)^2 \rightarrow (\mathbb{R}^{2n-c})^2$ in general position by a small equivariant deformation $I^2_{n,d} \circ f^2_\epsilon \rightarrow g_\epsilon$. The singular set $\Sigma(g_\epsilon) \subset \Sigma_\epsilon$ coincides with the inverse image of the projective subspace $\mathbb{R}P^{n-(n-c)} \subset \mathbb{R}P^n$ with respect to the canonical cocycle $G$. Using the canonical isomorphism $I$ (Section 2.5) we obtain the element $I \circ \rho^0_{0, d}(\tilde{O}_0(f))$. This element is defined geometrically as the obstruction $\tilde{O}(f)$ for discrete realization of $f$ in $\mathbb{R}^{2n-d}$. This proves the equation $\tilde{O}(f) = I \circ \rho^0_{0, d}(\tilde{O}_0(f))$. The proof of the equation $O(f) = I \circ \rho^0_{0, d}(O_0(f))$ and the part (i) is analogous. \qed

Remark 3.5. The Hurewicz images

\[
\tilde{o}(f) := \gamma(\tilde{O}(f)) \in \tilde{H}_d(\Sigma \cup \partial K; (n-d)G) \quad \text{and} \quad o(f) := \gamma(O(f)) \in H_d(\Sigma \cup \partial K; (n-d)G)
\]

were studied in [22] and [6] respectively, where they were shown to be complete obstructions to discrete and isotopic realizability (respectively) in the case $d = 0$. The simplest calculation of the obstruction $o(f)$ for Melikhov’s example $f : S^3 \rightarrow \mathbb{R}^6$ is presented in [6] (see also [18] for further examples). We call $\tilde{o}(f)$ the van Kampen–Skopenkov obstruction and $o(f)$ the van-Kampen–Melikhov obstruction.

In view of the following result, completeness of $\tilde{o}(f)$ and $o(f)$ for $d = 0$ can be regarded as a corollary of bijectivity of $\gamma : \Omega^R_0(\Sigma; G) \rightarrow H_0(\Sigma; kG)$ and surjectivity of $\gamma : \Omega^R_1(\Sigma; G) \rightarrow H_1(\Sigma; kG)$ for CW-complexes.

Remark 3.6. Let $M^n$ be a stable parallelized manifold, equipped with a cocycle $g \in H^1(M; \mathbb{Z}/2)$. By the J.F. Adams Theorem [1] the following class $h(M, g) = \langle w^n_0(g) ; [M] \rangle$ is trivial if $n \neq 1, 3, 7$. This class $h$ is determined by the composition $\Omega^{s0}_n \rightarrow \Omega^{s1}_n \rightarrow \Pi_n \rightarrow \mathbb{Z}/2$, where $\Omega^{s1}_{n-1} \rightarrow \Pi_n$ is a Khan–Pridy homomorphism, $\Pi_n \rightarrow \mathbb{Z}/2$ is the Hopf invariant, see, e.g., [12] and the references there. Therefore the homomorphism

\[
\rho^{n}_{0,0} : \Omega^{s0}_n(X; G) \rightarrow \Omega^{s0}_0(X; G)
\]

is trivial if $n \neq 1, 3, 7$. This gives a new short proof of Theorem 1.6(i) as a consequence of the following theorem.

Main Theorem. Let $f : S^n \rightarrow \mathbb{R}^{2n-d}$ be an arbitrary continuous mapping, $n \geq 3$. Then
(i) $f$ is discretely realizable iff \( \tilde{O}(f) = 0 \), provided that $d < \frac{2n-3}{2}$.

(ii) $f$ is isotopically realizable iff $O(f) = 0$, provided that $d < \frac{2n-5}{2}$.

**Proof.** Let us prove (i). We will use the notation introduced in the beginning of this section. By the assumption there exists a function $\delta = \delta(\varepsilon) > 0$ such that the triple $(\Sigma_r, e|\Sigma_r, \Sigma|\Sigma_r)$ is null-bordant via some null-bordism $(W, \chi, \Psi)$ such that $\chi(W)$ lies in the $\delta$-neighborhood of $e(\Sigma \times 0)$. By general position we may assume that $\chi(W) \subset e(K \times 0)$. It follows that $(\Sigma_r, \hat{\Sigma})$ is the boundary of some pair $(\hat{W}, \hat{\Psi})$ where $\hat{W}$ is a smoothly embedded submanifold of $\hat{K}$ with boundary $\partial \hat{W} = \Sigma_r$, and $\hat{\Psi}$ is a $T$-equivariant framing of $v(\hat{W} \subset \hat{K})$ such that $\hat{\Psi}|_{\partial \hat{W}} = \hat{\Sigma}$. By the equivariant version of a well-known folklore construction, $G_0 = f^2_n$ is equivariantly $2\delta$-homotopic to a mapping $G_1$ such that $G^{-1}(\Delta_{\mathbb{R}^{2n-d}}) = \Delta_{\mathbb{S}^n}$. (The bordism $(\hat{W}, \hat{\Psi})$ arises as the naturally framed submanifold $\bigcup \{G^{-1}(\Delta_{\mathbb{R}^{2n-d}}) \setminus \Delta_{\mathbb{S}^n} \}$, where $G_t: S^n \times S^n \to \mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d}$ is the homotopy.)

We give more details here, following the approach of [16] for the E-H-P James–Whitehead exact sequence. Let us consider $U(\hat{W}, \partial \hat{W}) = \hat{\Sigma} \subset \hat{K} \times I$ an regular equivariant neighbourhood. The canonical equivariant mapping $G: U(\hat{W}) \to \mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d}$, $\hat{W} = G^{-1}(\Delta_{\mathbb{R}^{2n-d}})$, $F|_{U(\partial \hat{W})} = f^2_n$ is well defined. We denote the projection of $\hat{W}$ on the bottom of $\hat{K} \times I$ by $\pi \hat{W}$ and by $\hat{V} \subset \hat{K} \times \{0\}$ a regular neighbourhooed of $\pi \hat{W}$.

Obviously, the obstruction to equivariant extension $G_1$ of $G$ to $\hat{V} \times I$, such that the restriction on the bottom coincides with $f^2_n$ and the inverse image $G_1^{-1}(\Delta_{\mathbb{R}^{2n-d}}) = W$ is trivial by the dimension reason (the space $\mathbb{R}^{2n-d} \times \mathbb{R}^{2n-d} \setminus \Delta_{\mathbb{R}^{2n-d}} / T$ has trivial homotopy groups in dimensions $2, \ldots, d+1$). The required homotopy $G_t$ is constructed by the standard extension of the homotopy $G_1$ from $\hat{V} \times I$ to $\hat{K} \times I$.

By Skopenkov’s Criterion for discrete realizability [22] (in the formulation of [17, Criterion 1.7a]) $f$ is discretely realizable, which completes the proof of (i). The proof of (ii) is analogous, using [17, Criterion 1.7b]. \(\square\)

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**References**


