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A simulation of cellular automata on hexagons by cellular automata on rings

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Abstract

We consider cellular automata on Cayley graphs and compare their computational power according to their topology. We prove that cellular automata defined over a hexagonal grid can be simulated by cellular automata over a ring. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Cellular automata; Cayley graphs; Simulation

1. Introduction

Cellular automata on Cayley graphs have been introduced to prove a generalization of the Moore and Myhill theorem in [3] and serve to the general definition in [2]. This model permits to simplify and generalize some proofs.

We extend Róka's simulations results [7, 9]; we prove that cellular automata defined over a hexagonal grid can be simulated by cellular automata over a ring. We use as intermediate steps, the simulation of a cellular automaton over a hexagonal grid by a cellular automaton over a torus and the simulation of a cellular automaton over a torus by a cellular automaton over a ring.

2. Cellular automata on Cayley graphs

Let us recall some definitions. Let Γ be a finitely presented group $\Gamma = \langle \mathscr{G} | R \rangle$ with $\mathscr{G} = \{g_0, g_1, \dots, g_{d-1}\}$ the set of generators not containing the identity and R the set of relators [1]. The associated *Cayley (di)graph* $G = (V, E)$ is the graph whose vertices are the elements of Γ and whose arcs are the pairs $(x, g_i.x)$ for $x \in \Gamma$ and $g_i \in \mathscr{G}$. A *cellular automaton* (CA) over a Cayley digraph $G = (V, E)$ is a 4-tuple $\mathscr{A} = (Q, G, N, \delta)$ where Q denotes the finite set of the states, G the Cayley digraph of a finitely presented group

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Γ, N the neighborhood such that $N=\{w_1,\dots,w_k\}$ where the w_i are words over $\mathscr{G}$ and $\delta: Q^{|N|}\to Q$ is the local transition function. We associate a cell of the CA to each vertex of the digraph. The local transition function δ gives a new state to cell i at time t according to the states of its neighbors at time $t-1$. We only consider the von Neumann neighborhood defined as the set of vertices at distance one from every cell, also including the cell itself. More formally, the words which define N are all the letters over the alphabet $\mathscr{G}$ plus the identity.

Róka [9] defines CAs over different Cayley graphs as hexagons, two- and three-dimensional torus (resp. $\Gamma_H^{r,p,q}=\langle a,b,c : ab=ba, ac=ca, bc=cb, abc=1, a^r c^{1-q}=1, b^{p-1}c^{-q}=1\rangle$, $\Gamma_T^{n,m}=\langle u,v : uv=vu, u^n=1, v^m=1\rangle$ and $\Gamma_T^{n,m,k}=\langle a,b,c : ab=ba, ac=ca, bc=cb, abc=1, a^n=1, b^m=1, c^k=1\rangle$). We add to these definitions the CAs over rings whose definition is $\Gamma_R^l=\langle w : ww^{-1}=1, w^l=1\rangle$.

3. First simulation results

Theorem 1 (Róka [8]). *Any CA over $\Gamma_H^{r,p,q}$ can be simulated by a CA over $\Gamma_T^{n,m,k}$ with $n=|\alpha_1(p+r-1)-\beta_1 p|$, $m=|\alpha_2(p-1)-\beta_2(p+q-1)|$, $k=|\alpha_3(1-p-r)+\beta_3 p|$ and $(p-1)\alpha_1=\operatorname{lcm}(p-1,p+q-1)$, $(p+r-1)\alpha_2=\operatorname{lcm}(p+r-1,p)$, $(-r)\alpha_3=\operatorname{lcm}(r,q-1)$, $(p+q-1)\beta_1=\operatorname{lcm}(p-1,p+q-1)$ and $p\beta_2=\operatorname{lcm}(p+r-1,p)$, $(q-1)\beta_3=\operatorname{lcm}(r,q-1)$.*

Proof. Let τ_H be a CA over $\Gamma_H^{r,p,q}$ whose initial configuration is $c^0_{\tau_H}$. To prove that there exists τ_{3D} a CA over $\Gamma_T^{n,m,k}$ which simulate τ_H, it suffices to prove that $c^0_{\tau_H}$ is periodical in all the directions given by the generators. That is, there exists n,m and k three naturals such that $a^n=1, b^m=1$ and $c^k=1$.

Let $v_h\in\Gamma_H^{r,p,q}$ and $v_{3D}\in\Gamma_T^{n,m,k}$. The initial configuration of τ_{3D} is defined as $c^0_{\tau_{3D}}(v_{3D})=c^0_{\tau_H}(v_h)$, $c^0_{\tau_{3D}}(v_{3D}a^i)=c^0_{\tau_H}(v_h a^i)$ for $1\leqslant i<n, c^0_{\tau_{3D}}(v_{3D}b^j)=c^0_{\tau_H}(v_h b^j)$ for $1\leqslant j<m$, $c^0_{\tau_{3D}}(v_{3D}c^l)=c^0_{\tau_H}(v_h c^l)$ for $1\leqslant l<k$.

Neighbors of $c^0_{\tau_H}$ are also neighbors for $c^0_{\tau_{3D}}$. Thus, we can define τ_{3D} as $Q_{3D}=Q_H$ and $\delta_{3D}=\delta_H$ which simulates τ_H without loss of time.

We show that c_{τ_H} is periodical in all the directions given by the generators. We present the proof only for generator a, the construction is similar for the others. We have the set of relations $R=\{c=a^{-1}b^{-1}; a^r c^{1-p}=1; b^{q-1}c^{-p}=1\}$ which defines a system of two equations $\{a^{p+r-1}b^{p-1}=1, a^p b^{p+q-1}=1\}$ to be solved in variable a.

Let

$$\alpha_1=\frac{\operatorname{lcm}(p-1,q+p-1)}{p-1}\quad\text{and}\quad\beta_1=\frac{\operatorname{lcm}(p-1,q+p-1)}{q+p-1};$$

we obtain $a^{\alpha_1(p+r-1)-\beta_1 p}=1$. The other values of the theorem are obtained similarly by solving R in b and c and by defining $\alpha_2,\alpha_3,\beta_2$ and β_3. $\square$

Corollary 2 (Róka [8]). Any CA over $\Gamma_H^{r,p,q}$ can be simulated by a CA over $\Gamma_T^{n,m}$ with $n=|\alpha_1(p+r-1)-\beta_1 p|$, $m=|\alpha_2(p-1)-\beta_2(p+q-1)|$, $(p-1)\alpha_1=\operatorname{lcm}(p-1,$

$p+q-1$), $(p+r-1)\alpha_2 = \mathrm{lcm}(p+r-1, p)$, $(p+q-1)\beta_1 = \mathrm{lcm}(p-1, p+q-1)$ and $p\beta_2 = \mathrm{lcm}(p+r-1, p)$.

This is proved by observing that the third direction depends on the others.

Theorem 3 ([5]). *Any CA over $\Gamma_T^{n,m}$ can be simulated by a CA over Γ_R^{nm}.*

Proof. Let $\tau_T = (Q_T, G_T^{n,m}, N, \delta_T)$ with $G_T^{n,m}$ the graph of the group $\langle u, u^{-1}, v, v^{-1} : u^m = 1, uu^{-1} = 1, v^n = 1, vv^{-1} = 1, uv = vu\rangle$ and $N = \{1, u, u^{-1}, v, v^{-1}\}$.

We consider the CA over a ring of nm cells: $\tau_R = (Q_R, G_R^{mn}, N, \delta_R)$ with G_R^{mn} the graph of the group Γ_R^{nm}, $N = \{1, w, w^{-1}\}$ and $\delta_R : (Q_R)^3 \to Q_R$. Q_R has four layers: $Q_R = (Q_T)^3 \times Q_{\text{Sync}} = (C, U, D, \text{Sync})$ where C is the 'central' layer (where the simulation takes place), U a copy of the upper neighbor, D a copy of the bottom neighbor and Sync the synchronization layer.

The embedding of a configuration of τ_T into a configuration of τ_R (and conversely) is as follows:

- Let $\lambda = u^x v^y$ be the cell of τ_T of coordinates (x, y) and $\mu = w^k$ be the kth cell of τ_R. Let $T(\lambda)$ denote the contents of cell λ of τ_T. $U(\mu)$, $C(\mu)$ and $D(\mu)$ denote respectively the contents of layer U, C and D of cell μ of τ_R. $T(\lambda)$ is mapped on $U(w^{m(x+1)+y})$, $C(w^{mx+y})$ and $D(w^{m(x-1)+y})$.
  Conversely, the τ_T's cell corresponding to the τ_R's cell μ is $T\left(u^{(k \bmod n)} v^{(k|n)}\right) = C(\mu)$ where $j|k$ denotes the quotient of euclidian division of j by k, mod the modulo. The contents of the other layers is out of interest since they contain a copy of the states of τ_T at the previous time.
- The synchronization layer is as follows: there is a general on cells $0, (m-1), m, (2m-1), 2m, \ldots, (nm-1)$. The local transition function which rules this layer is obtained from the solution to the two-general Firing Squad Synchronization Problem given in [6].

The simulation's sketch is as follows:

(a) the central layer of Q_R's states are moved rightwards on layer U as long as no firing state appears on the synchronization layer;
(b) the central layer of Q_R's states are moved leftwards on layer D as long as no firing state appears on the synchronization layer;
(c) when the firing states appear, the moving information is where it must be and on the central layers and τ_R can simulate one step of τ_T. Afterwards, the moving process (a) and (b) is again performed.

The way to send information synchronously is detailed in [4].

4. A simulation of a CA over hexagons by a CA over a ring

Theorem 4. *Any CA over $\Gamma_H^{r,p,q}$ can be simulated by a CA over Γ_R^{nm} where n and m are those of Corollary 2.*

This is a straightforward combination of the previous results. Observe that the converse simulations are not possible since it has been proved in [8] that there exists a cellular automaton over a three-dimensional torus which cannot be simulated by any cellular automaton over a hexagon.

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