Sets of type-(1, n) in symmetric designs for $\lambda \geq 3^\dagger$

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Abstract

A set of type-(m, n) S is a set of points of a design with the property that each block of the design meets either $m$ points or $n$ points of $S$. The notions of type and of parameters of a $k$-set (there called characters) were introduced for the first time by Tallini Scafati in [M. Tallini Scafati, \{k, n\}-archi di un piano grafico finito, con particolare riguardo a quelli con due caratteri. Note I and Note II, Rend. Accad. Naz. Lincei 40 (8) (1996) 812–818 (1020–1025)]. If $m = 1$, $S$ gives rise to a subdesign of the design. Under weaker conditions for the order of each symmetric design, the parameters of sets of type-(1, n) in projective planes were characterised by G. Tallini and the biplane case was dealt with by S. Kim, by solving the corresponding Diophantine equation for each case, separately. In this paper, we first characterise the parameters of sets of type-(1, n) in the triplane with more generalised order conditions than prime power order. Next, we generalise the result on triplanes to arbitrary symmetric designs for $\lambda \geq 3$. As results, a non-existence condition for special parameter sets and a characterisation of parameters for the existence, restricted by some derived bounds, are given.

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1. Introduction

Let $S$ be a subset of the point set of a 2-$(v, k, \lambda)$ design $D$. For given integers $m$, $n$ with $0 \leq m < n \leq k$, a set $S$ is called a set of type-(m, n) in $D$, if each block of $D$ meets $S$ in either $m$ points or $n$ points. If a set of type-(m, n) is an $s$-set (of cardinality $s$), we refer to it as an $(s; n, m)$-set in $D$. A block of $D$ which meets $S$ in $i$ points is called an $i$-secant. Let $t_j$ be the number of $j$-secants. It is easy to verify that the following linear equations hold (see Tallini

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Scafati [8,9] and Kim [4]):

(i) \( t_m + t_n = b \)

(ii) \( mt_m + nt_n = rs \)

(iii) \( m(m - 1)t_m + n(n - 1)t_n = \lambda s(s - 1) \)

where \( s \) is the number of points of \( S \), \( b \) the number of blocks of \( D \) and \( r \) the replication number of \( D \) which is the number of blocks passing through a point of \( D \). On eliminating \( t_m, t_n \) from (1), the following Diophantine equation [5–7] holds:

\[
\lambda^2 s^2 - (r(m + n - 1) + \lambda)s + bmn = 0
\]

which is called the classical equation.

Let \( \sigma_j \) be the number of \( j \)-secants passing through a point \( P \not\in S \), and \( \rho_j \) the number of \( j \)-secants passing through \( Q \) in \( S \). Then we have the following properties of \( \sigma_j \) and \( \rho_j \) (see [6]).

**Result 1.** Let \( S \) be an \((s; m, n)\)-set in a given \(2-(v, k, \lambda)\) design \( D \). Then,

(i) \( \sigma_m + \sigma_n = r \) and \( \rho_m + \rho_n = r \),

(ii) \( m\sigma_m + n\sigma_n = \lambda s \) and \( (m - 1)\rho_m + (n - 1)\rho_n = \lambda(s - 1) \).

Many combinatorial properties of sets of type\((m, n)\) in designs can be seen in de Resmini [6] when \( \lambda \geq 2 \) and in Tallini [7] when \( \lambda = 1 \).

In this paper, we deal with sets of type\((1, n)\) in symmetric designs. We suppose \( S \) to be \((s; 1, n)\)-set in a symmetric \(2-(v, k, \lambda)\) design which implies that \( m = 1, b = v = k(k - 1)/\lambda + 1 \) and \( r = k \). Then, from Result 1, it is easy to show that

\[
\begin{align*}
\sigma_1 &= \frac{nk - \lambda s}{n - 1} & \rho_1 &= \sigma_m - \frac{k - \lambda}{n - 1} \\
\sigma_n &= \frac{\lambda s - k}{n - 1} & \rho_n &= \sigma_n + \frac{k - \lambda}{n - 1}.
\end{align*}
\]

The classical equation (2) for an \((s; 1, n)\)-set in a symmetric \(2-(v, k, \lambda)\) design may be written as follows:

\[
\lambda^2 s^2 - \lambda(kn + \lambda)s + (k^2 - k + \lambda)n = 0.
\]

If we write \( k - \lambda = \alpha, n - 1 = \beta, \) and \( \lambda(s - 1) = w \), then (4) can be written as

\[
w^2 - (\alpha\beta + \alpha + \lambda\beta)w + \alpha(\beta + 1)(\alpha + \lambda - 1) = 0.
\]

From (3) and (5), we have the following lemma for the divisibility among the parameters.

**Lemma 2.** Let \( S \) be an \((s; 1, n)\)-set of a symmetric \(2-(v, k, \lambda)\). Then,

(i) \( (n - 1) \mid (k - \lambda) \) and \( (n - 1) \mid (\lambda(s - 1)) \),

(ii) \( (n - 1)^2 \mid (k - \lambda)(\lambda - 1) \).

**Proof.** Condition (i) follows directly from Result 1. If \( \lambda = 1 \), (ii) is obvious. We suppose \( \lambda \geq 2 \).

With the notation as above, Eq. (5) can be stated as

\[
w^2 - (\alpha\beta + \alpha + \lambda\beta)w + \alpha(\beta + 1)(\alpha + \lambda - 1) = -\alpha(\lambda - 1).
\]

From (i), the left hand side of this equation is divisible by \( \beta^2 \). Hence, (ii) holds. ■
An \((s; 1, n)\)-set \(S\) in a design \(D\) gives rise to a subdesign of \(D\) with set of blocks such that each block is defined as the set of points of \(S\) which are incident with a block of \(D\) and each block consists of more than one point. An \((s; 1, n)\)-set in a finite projective plane is sometimes called a blocking set. In 1966, the Diophantine equation (4) for blocking sets was solved by Tallini Scafati [8], with the hypothesis of prime power order of a projective plane. Her integral solutions of (4) for \(S\) finite projective planes of order \(p^h(n − 1)\), where \(p\) is a prime and \(h\) is a non-negative integer, so that all possible sets of type-(1, \(n\)) in the planes are completely characterised from the arithmetical point of view, stated in [7] as follows.

**Result 3.** Suppose \(S\) is an \((s; 1, n)\)-set in a projective plane of order \(q\) and \(q/(n − 1) = p^h\) where \(p\) is a prime and \(h\) is a positive integer. Then \(q = p^{2h}\), \(n = 1 + \sqrt{q}\) and \(S\) is either a Baer subplane or a unital.

According to the notation of Bose and Shrikhande [1], a 2-(\(v^*, k^*, \lambda\)) subdesign of a symmetric 2-(\(v, k, \lambda\)) design \(D\) is called a Baer subdesign of \(D\) if it is symmetric and \(k^* = 1 + \sqrt{k − \lambda}\). If \(\lambda = 1\), it is a Baer plane. A 2-(\(v^*, k^*, \lambda\)) subdesign of \(D\) is called a Hermitian subset of \(D\) if \(v^* = \sqrt{k−\lambda}(k − 1) + 1\) and \(k^* = 1 + \sqrt{k − \lambda}\). It is a unital (see [2]) when \(\lambda = 1\).

If an \((s; 1, n)\)-set \(S\) gives rise to a 2-(\(s, n, \lambda\)) subdesign of \(D\) by taking an \(n\)-secant as blocks of the subdesign, an \((s; 1, n)\)-set will be called a Baer subdesign when \(s = 1 + \sqrt{\frac{k−\lambda}{\lambda}(k−1)+1}\), \(n = 1 + \sqrt{k − \lambda}\), and a Hermitian subset when \(s = \sqrt{\frac{k−\lambda}{\lambda}(k−1)+1}\), \(n = 1 + \sqrt{k − \lambda}\), respectively.

If a set of type-(1, \(n\)) is a Baer subdesign, Eq. (4) provides another non-negative integral root which corresponds to a parameter set of another subdesign of parameters of a Hermitian set (see [6]). Hence, Tallini’s result (Result 3) shows that, if there is a set of type-(1, \(n\)) in a finite projective plane of order \(q\) where \(q/(n − 1)\) is a prime power, it is either a Baer subdesign or a Hermitian set, i.e. a Baer subplane or a unital in the projective plane, respectively.

In biplanes (i.e. symmetric 2-(\(v, k, 2\)) designs), a result analogous with Tallini’s on sets of type-(1, \(n\)) appears in Kim [3,4], which can be stated as follows.

**Result 4.** Let \(S\) be an \((s; 1, n)\)-set in a symmetric 2-(\(v, k, 2\)) design and \((k−2)/(n−1)^2 = p^h\), \(p\) a prime and \(h\) a non-negative integer. Then, either \(k−2 = (n−1)^2\), or \(k−2 = (2n−5)(n−1)^2\).

This result also implies that a set of type-(1, \(n\)) in a bplane is a Baer subplane, or a Hermitian subset, or a subdesign with parameters satisfying \(k−2 = (2n−5)(n−1)^2\).

In the results of Tallini and Kim, we notice that the order conditions \((k−1)/(n−1) = p^h\) and \((k−2)/(n−1)^2 = p^h\) are supposed, respectively, since \(n−1\) and \((n−1)^2\) divide \(k−1\) and \((2−1)(k−2)^2\), respectively, as shown in Lemma 2. These hypotheses of orders also cover known prime power orders of projective planes and biplanes, respectively.

In this paper, we deal with sets of type-(1, \(n\)) in general symmetric 2-(\(v, k, \lambda\)) designs for \(\lambda \geq 3\) under the order condition \(2(k−3) = (n−1)^2 p^h\) derived form Lemma 2. We first find all the sets of positive integral solutions of the Diophantine equation (4) in triplanes (i.e. \(\lambda = 3\)) so that they are characterised as stated in Theorem 9, which says that, with the given order condition, there does not exist a set of type-(1, \(n\)) in triplanes unless \(p = 2, 3\). Next, we generalise considering sets of type-(1, \(n\)) in a symmetric 2-(\(v, k, \lambda\)) design for \(\lambda \geq 3\), in general. Under the hypothesis that \((\lambda−1)(k−\lambda) = (n−1)^2 p^h\), we generalise Proposition 5 about the non-existence condition in triplanes to arbitrary symmetric designs, as stated in Theorem 10. Finally,
we suppose that \( p \) does not divide \( \lambda (\lambda - 1) \) (motivated from the result for the case \( p \neq 2, 3 \) in triplanes). Then, some bounds of the cardinality of a set of type-(1, \( n \)) are established in Corollary 13. Further, we eliminate non-existence cases from the bounds so that, as a conclusion, a characterisation of possible sizes of sets of type-(1, \( n \)) in a symmetric \((v, k, \lambda)\) design is given, as in Theorem 17.

2. Sets of type-(1, \( n \)) in triplanes

In this section, we characterise the parameters of \((s; 1, n)\)-sets in triplanes of order \( k - 3 \). We find all the possible types of positive integer solutions of Eq. (5) under the assumption that \( (\lambda - 1)(k - 3)/(n - 1)^2 \) is a prime power where \( \lambda = 3 \).

Let \( T \) be a symmetric 2-(v, k, 3) design, i.e. a triplane. From now on, we simplify the notation using \( k - 3 = \alpha \), \( n - 1 = \beta \) and \( w = 3(s - 1) \). With this notation, Diophantine equations (5) for \( \lambda = 3 \) can be written as

\[
w^2 - (\alpha \beta + \alpha + 3\beta)w + \alpha(\alpha + 2)(\beta + 1) = 0. \quad (6)
\]

From Lemma 2 (ii), since \( \beta^2 \mid 2\alpha \), we suppose

\[
2(k - 3)/(n - 1)^2 = 2\alpha /\beta^2 = p^h
\]

for a prime \( p \) and a non-negative integer \( h \). Let \( w_\beta = 2w/\beta \) (=6(s - 1)/\( \beta \)). Then, since \( \alpha \) is divisible by \( \beta \), (6) implies

\[
w_\beta^2 - (\beta^2 p^h + \beta p^h + 6)w_\beta + p^h(\beta^2 p^h + 4)(\beta + 1) = 0. \quad (7)
\]

Now we solve (7) with respect to the integer parameters. Firstly, we have the following proposition for the case \( h = 0 \).

**Proposition 5.** Suppose \( h = 0 \). Then, there is no positive integral solution set of parameters of Eq. (7).

**Proof.** Since we suppose \( h = 0 \) in (7), we have

\[
w_\beta^2 - (\beta^2 + \beta + 6)w_\beta + (\beta^2 + 4)(\beta + 1) = 0. \quad (8)
\]

If we put \( f_1(x) = x^2 - (\beta^2 + \beta + 6)x + (\beta^2 + 4)(\beta + 1) \), we have

\[
\begin{align*}
f_1(\beta) &= \beta^2 - 2\beta + 4 > 0 \\
f_1(\beta + 1) &= -\beta - 1 < 0
\end{align*}
\]

for all positive integer \( \beta \), which means that there is at least one non-integer root of \( f_1(x) \). Since the coefficients of \( f_1(x) \) are all integers and its leading coefficient is 1, we conclude that there is no integer root for \( f_1(x) \). \( \blacksquare \)

Next, we find all possible positive integral solutions of (6) for each value of a prime \( p \).

**Proposition 6.** Suppose \( p \neq 2 \) and \( p \neq 3 \). Let \( h \) be a positive integer. Then, there is no positive integer solution set of parameters of Eq. (7).
List 1
Evaluation of (11) when $\beta \leq c \leq \beta^2 + 6$

<table>
<thead>
<tr>
<th>$c$</th>
<th>$g_1$</th>
<th>$g_2$</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$2\beta - 4$</td>
<td>$\beta^2$</td>
<td>$g_1 &lt; g_2$</td>
</tr>
<tr>
<td>$\beta + 1$</td>
<td>$2(\beta + 1)$</td>
<td>$\beta + 1$</td>
<td>$p = 2$</td>
</tr>
<tr>
<td>$\beta + 2$</td>
<td>$2(\beta + 4)$</td>
<td>$-\beta^2 + 2\beta + 4$</td>
<td>$g_1 &gt; 0, g_2 &lt; 0$ if $\beta \geq 4$</td>
</tr>
<tr>
<td>$\beta^2 - 2$</td>
<td>$2(3\beta^2 - 2\beta - 2)$</td>
<td>$\beta^2$</td>
<td>$\frac{g_1}{g_2} = 6 - \frac{4(\beta - 1)}{\beta^2} \notin Z$ if $\beta \geq 4$</td>
</tr>
<tr>
<td>$\beta^2 - 1$</td>
<td>$2(\beta + 1)(3\beta - 5)$</td>
<td>$\beta + 1$</td>
<td>$p^h = 2(\beta - 5)$</td>
</tr>
<tr>
<td>$\beta^2$</td>
<td>$2\beta^2 - 4\beta - 16$</td>
<td>$-\beta^2 + 2\beta + 4$</td>
<td>$g_1 &gt; 0, g_2 &lt; 0$ if $\beta \geq 4$</td>
</tr>
<tr>
<td>$\beta^2 + 2$</td>
<td>$6\beta^2 - 4\beta + 8$</td>
<td>$3\beta^2 - 2\beta + 4$</td>
<td>$p^h = 2$</td>
</tr>
<tr>
<td>$\beta^2 + 3$</td>
<td>$6\beta^2 - 4\beta + 14$</td>
<td>$4\beta^2 - 3\beta + 9$</td>
<td>$1 &lt; p^h = 1 + \frac{2\beta^2 - 6\beta + 5}{4\beta^2 - 3\beta + 9} &lt; 2$ if $\beta \geq 2$</td>
</tr>
<tr>
<td>$\beta^2 + 4$</td>
<td>$6\beta^2 - 4\beta + 20$</td>
<td>$5\beta^2 - 4\beta + 16$</td>
<td>$1 &lt; p^h &lt; 2$</td>
</tr>
<tr>
<td>$\beta^2 + 5$</td>
<td>$6\beta^2 - 4\beta + 26$</td>
<td>$6\beta^2 - 5\beta + 25$</td>
<td>$1 &lt; p^h &lt; 2$</td>
</tr>
<tr>
<td>$\beta^2 + 6$</td>
<td>$6\beta^2 - 4\beta + 32$</td>
<td>$7\beta^2 - 6\beta + 36$</td>
<td>$p^h &lt; 1$</td>
</tr>
</tbody>
</table>

Proof. If (7) has integer solutions and two integer roots of (7) with respect to $w_\beta$ are written as $x_1$ and $x_2$, then we have

\[
\begin{align*}
x_1 + x_2 &= \beta^2 p^h + \beta p^h + 6 \\
x_1x_2 &= p^h(\beta^2 p^h + 4)(\beta + 1).
\end{align*}
\]

(9)

Since $p \neq 2, 3$, we know that one root, say $x_1$, and $p$ are coprime and so we can put the other root $x_2 = cp^h$ for some positive integer $c$. From (9), we have

\[
f(c) := p^h c^2 - (\beta^2 p^h + \beta p^h + 6)c + (\beta^2 p^h + 4)(\beta + 1) = 0
\]

(10)

which implies the following ratio:

\[
p^h = \frac{6c - 4(\beta + 1)}{c^2 - (\beta^2 + \beta)c + \beta^2(\beta + 1)}
\]

(11)

if the denominator is not 0.

Let $g_1(c)$ and $g_2(c)$ be the numerator and the denominator of (11), respectively. List 1 shows the evaluation of $g_1$ and $g_2$ for the integer values of $c$ such that $\beta \leq c \leq \beta^2 + 6$. If we suppose $\beta \geq 4$, then List 1 implies the following.

- If $c \leq \beta$ or $c \geq \beta^2 + 6$, we have $g_1 < g_2$ which implies $p^h < 1$, a contradiction.
- If $\beta + 2 \leq c \leq \beta^2 - 2$, we have $g_1 > 0$ and $g_2 < 0$ and then $p^h < 0$ which implies a contradiction.
- For the rest of the values of $c$, each value of $c$ causes a contradiction.

Hence we conclude that if $p \neq 2, 3$ and $\beta \geq 4$, there is no positive integer solution of (6). Now consider the case $1 \leq \beta \leq 3$.

(i) $\beta = 1$; From (11), we have the ratio

\[
p^h = \frac{6c - 8}{c^2 - 2c + 2}.
\]

If $c \leq 1$ or $c \geq 7$, we have $g_1 < g_2$ which implies $p^h < 1$, which is a contradiction. If $c = 2$ or $3$, then $p^h = 2$ so that $p = 2$, which is a contradiction since we suppose $p \neq 2$. 

...
List 2
Evaluation of (12) for $3 \leq c \leq 9$

<table>
<thead>
<tr>
<th>$c$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^h$</td>
<td>2</td>
<td>3</td>
<td>18/7</td>
<td>2</td>
<td>30/19</td>
<td>36/28</td>
<td>42/39</td>
</tr>
</tbody>
</table>

List 3
Evaluation of (13) for $4 \leq c \leq 14$

<table>
<thead>
<tr>
<th>$c$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^h$</td>
<td>10/4</td>
<td>14</td>
<td>–</td>
<td>26</td>
<td>$2^3$</td>
<td>38/9</td>
<td>44/16</td>
<td>2</td>
<td>56/36</td>
<td>62/49</td>
<td>68/64</td>
</tr>
</tbody>
</table>

When $c = 4, 5, 6$, we have $p^h = \frac{16}{10}, \frac{22}{17}, \frac{28}{26}$, respectively, which are contradictions. Hence, for all integer $c$, there is no integer solution when $p \neq 2, 3$ and $\beta = 1$.

(ii) $\beta = 2$; From (11), we have the ratio

$$p^h = \frac{6c - 12}{c^2 - 6c + 12}.$$  \hspace{1cm} (12)

If $c = 1$, then $p^h < 0$. If $c = 2$, then $p^h = 0$. If $c \geq 10$, then $0 < p^h < 1$. We evaluate the ratio for $3 \leq c \leq 9$, as stated in List 2, which shows that no integral value of $c$ satisfies (12).

(iii) $\beta = 3$; From (11), we have the ratio

$$p^h = \frac{6c - 16}{c^2 - 12c + 36}.$$  \hspace{1cm} (13)

If $c < 9 - \sqrt{29}$ (i.e. $c \leq 3$) or $c > 9 + \sqrt{29}$ (i.e. $c \geq 15$), then $g_1 < g_2$, implying $p^h < 1$, a contradiction. In List 3, we evaluate the ratio for the rest values of $c$ such that $4 \leq c \leq 14$, which shows that no value of $c$ satisfies (13), except $c = 6$. However, if $c = 6$ and $\beta = 3$, then (10) does not hold. Hence, we conclude that there is no solution of (13) when $p \neq 2, 3$ and $\beta = 3$.

Therefore, we have completed the proof. ■

Now, we solve Diophantine equation (7) when $p = 2$ and $p = 3$.

**Proposition 7.** Suppose $p = 3$. Then, the only positive integer solution set for (6) is \{s = 5, $\beta = 2, \alpha = 6$\}.

**Proof.** If we fix $p = 3$, (7) is written as

$$w_\beta^2 - (3^h \beta^2 + 3^h \beta + 6)w_\beta + 3^h (3^h \beta^2 + 4)(\beta + 1) = 0.$$  \hspace{1cm} (14)

Since Proposition 5 shows that there is no positive integral solution of (6) when $h = 0$, we divide the proof into three parts: $h = 1$, $h = 2$ and $h \geq 3$.

(i) $h = 1$;

If $h = 1$, since $w_\beta^2 - (3^2 \beta^2 + 3^2 \beta + 6)w_\beta + 3(3^2 \beta^2 + 4)(\beta + 1) = 0$ from (14), we note that $w_\beta$ is divisible by 3. Thus, (14) can be written as

$$f_2 (x) := 3x^2 - 3(\beta^2 + \beta + 2)x + (3\beta^2 + 4)(\beta + 1) = 0$$  \hspace{1cm} (15)

where $3x = w_\beta$. Note that

$$\begin{align*}
&f_2(\beta + 1) = \beta + 1 > 0 \\
&f_2(\beta + 2) = -3\beta^2 + 4\beta + 4 < 0 \quad \text{if } \beta \geq 3
\end{align*}$$
which implies that there is a non-integer root of \( f_2(x) \) between \( \beta + 1 \) and \( \beta + 2 \). Since the turning point of \( f_2(x) \) is at \( x_0 = \frac{\beta(\beta+1)}{2} + 1 \) which is an integer, we conclude that \( f_2(x) \) has no integer root when \( \beta \geq 3 \).

Next, we consider the cases when \( \beta = 1 \) and \( \beta = 2 \), respectively. If \( \beta = 1 \), then from (15) we have \( 3x^2 + 12x + 14 = 0 \) which does not have real roots. Finally, if \( \beta = 2 \), (15) implies \( 3x^2 - 24x + 48 = 0 \) which has an integer root \( x = 4 \), which makes the solution set \( \{s = 5, \beta = 2, \alpha = 6\} \).

(ii) \( h = 2 \);

Eq. (14) is written as \( w_\beta^2 - (9\beta^2 + 9\beta + 6)w_\beta + 9(9\beta^2 + 4)(\beta + 1) = 0 \) and it implies that \( w_\beta \) is divisible by 3. If we put \( w_\beta = 3x \), (14) can be simplified as

\[
 f_3(x) := x^2 - (3\beta^2 + 3\beta + 2)x + (9\beta^2 + 4)(\beta + 1) = 0.
\]

Note that

\[
\begin{align*}
 f_3(3\beta + 3) &= 7\beta - 1 > 0 \\
 f_3(3\beta + 4) &= -3\beta^2 + 10\beta + 12 < 0 \quad \text{if } \beta \geq 5
\end{align*}
\]

which implies that a non-integer root of \( f_3(x) \) exists between \( 3\beta + 3 \) and \( 3\beta + 4 \) if \( \beta \geq 5 \). Since the turning point of \( f_3(x) \) is at an integral value of \( x \), we conclude that the other root is not an integer.

Now we evaluate \( f_3(x) \) for \( \beta = 1, 2, 3 \) and 4. If \( \beta = 1 \) and 2, we have \( x^2 - 8x + 26 = 0 \) and \( x^2 - 20x + 120 = 0 \), respectively, and they do not have any real roots. If \( \beta = 3 \) and 4, the equations are \( x^2 - 38x + 340 = 0 \) and \( x^2 - 62x + 740 = 0 \), respectively, and both have no integer roots. Hence we conclude that there is no integer root \( x \) of \( f_3(x) \).

(iii) \( h \geq 3 \);

From (14), we have

\[
 w_\beta^2 - (3^h\beta^2 + 3^h\beta + 6)w_\beta + 3^h(3^h\beta^2 + 4)(\beta + 1) = 0
\]

which implies that \( w_\beta \) is divisible by 3. Thus, if we put \( 3x = w_\beta \), then we have

\[
 x^2 - (3^{h-1}\beta^2 + 3^{h-1}\beta + 2)x + 3^{h-2}(3^h\beta^2 + 4)(\beta + 1) = 0. \quad (16)
\]

Let two integer roots of (16) be denoted by \( x_1 \) and \( x_2 \). Then

\[
\begin{align*}
 x_1 + x_2 &= 3^{h-1}\beta^2 + 3^{h-1}\beta + 2 \\
 x_1x_2 &= 3^{h-2}(3^h\beta^2 + 4)(\beta + 1)
\end{align*} \quad (17)
\]

Notice that \( h \geq 3 \). From (17), since \( x_1x_2 \) is divisible by 3 while \( x_1 + x_2 \) is not divisible by 3, we notice that one root is a multiple of 3 and the other is coprime to 3. Without loss of generality, we put \( x_2 = 3^hc \) for some positive integer \( c \). By eliminating \( x_1 \) from (17), we have the following equation:

\[
3^{h-2}c^2 - (3^{h-1}\beta^2 + 3^{h-1}\beta + 2)c + (3^h\beta^2 + 4)(\beta + 1) = 0. \quad (18)
\]

Then we have the following ratio:

\[
3^{h-2} = \frac{2c - 4(\beta + 1)}{c^2 - 3(\beta^2 + \beta)c + 9\beta^2(\beta + 1)}.
\]

Let \( g_1(c) \) and \( g_2(c) \) be the numerator and the denominator of (19), respectively. In List 4, we evaluate \( g_1(c) \) and \( g_2(c) \). From List 4, it follows that, if \( \beta \geq 6 \), the only cases in which there could exist integer solutions satisfying (19) occur when \( c = 3\beta^2 - 3 \) and \( c = 3\beta^2 - 2 \).
List 4
Evaluation of Eq. (19)

<table>
<thead>
<tr>
<th>c</th>
<th>$g_1 (c)$</th>
<th>$g_2 (c)$</th>
<th>Note</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\beta + 3$</td>
<td>$2\beta + 2$</td>
<td>$9\beta + 9$</td>
<td>$3^{\beta-2} = 2/9$</td>
</tr>
<tr>
<td>$3\beta + 4$</td>
<td>$2\beta + 4$</td>
<td>$-3\beta^2 + 12\beta + 16$</td>
<td>$g_2 &lt; 0$ if $\beta \geq 6$</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>$g_1 &gt; 0$, $g_2 &lt; 0$ if $\beta \geq 6$</td>
</tr>
<tr>
<td>$3\beta - 4$</td>
<td>$6\beta^2 - 4\beta - 12$</td>
<td>$-3\beta^2 + 12\beta + 16$</td>
<td>$g_1 &gt; 0$, $g_2 &lt; 0$ if $\beta \geq 6$</td>
</tr>
<tr>
<td>$3\beta^2 - 3$</td>
<td>$6\beta^2 - 4\beta - 10$</td>
<td>$9\beta + 9$</td>
<td>$g_1 &gt; g_2 &gt; 0$ if $\beta \geq 4$</td>
</tr>
<tr>
<td>$3\beta^2 - 2$</td>
<td>$6\beta^2 - 4\beta - 8$</td>
<td>$3\beta^2 + 6\beta + 4$</td>
<td>$g_1 &gt; g_2 &gt; 0$ if $\beta \geq 4$</td>
</tr>
<tr>
<td>$3\beta^2 - 1$</td>
<td>$6\beta^2 - 4\beta - 6$</td>
<td>$6\beta^2 + 3\beta + 1$</td>
<td>$g_2 &gt; g_1 &gt; 0$ for all $\beta$</td>
</tr>
</tbody>
</table>

List 5
Evaluation of $3^{\beta-2}$ when $\beta = 3$, $15 \leq c \leq 23$, $c \neq 18$

<table>
<thead>
<tr>
<th>$c = \cdots$</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^h = \cdots$</td>
<td>14/9</td>
<td>4</td>
<td>18</td>
<td>22</td>
<td>6</td>
<td>26/9</td>
<td>28/16</td>
<td>30/25</td>
</tr>
</tbody>
</table>

If $c = 3\beta^2 - 3$, from (19) we have

$$3^{\beta-2} = \frac{6\beta^2 - 4\beta - 10}{9\beta + 9} = \frac{2(3\beta - 5)}{9}$$

which is a contradiction since $2(3\beta - 5)$ is even while $3^{\beta-2}$ is odd. Next, if $c = 3\beta^2 - 2$, we have

$$3^{\beta-2} = \frac{6\beta^2 - 4\beta - 8}{3\beta^2 + 6\beta + 4} = 2 + \frac{-16(\beta + 1)}{3\beta^2 + 6\beta + 4}.$$  

Since $3\beta^2 + 6\beta + 4 > 16(\beta + 1)$ if $\beta \geq 6$, we have that $3^{\beta-2}$ is not an integer which is a contradiction. Therefore, we conclude that if $\beta \geq 6$, we do not have any integer solution of (16).

Finally, we consider the cases when $\beta \leq 5$. From (19), we have

$\beta = 1$; $3^{\beta-2} = \frac{2c - 8}{c^2 - 6c + 18}$,

$\beta = 2$; $3^{\beta-2} = \frac{2c - 12}{c^2 - 18c + 108}$,

$\beta = 3$; $3^{\beta-2} = \frac{2c - 16}{c^2 - 36c + 324}$,

$\beta = 4$; $3^{\beta-2} = \frac{2c - 20}{c^2 - 60c + 720}$ and

$\beta = 5$; $3^{\beta-2} = \frac{2c - 24}{c^2 - 90c + 1350}$.

When $\beta = 1, 2$, then each ratio is greater then 0 and less then 1 and so is not an integer.

Suppose $\beta = 3$. If $c \leq 14$ or $c \geq 24$, then $2c - 16 \leq c^2 - 36c + 324$, which implies $p^h < 1$, a contradiction. If $c \neq 18$ and $15 \leq c \leq 23$, then each evaluation of $3^{\beta-2}$ shown in List 5 implies a contradiction. If $c = 18$, the parameters do not satisfy (18).

Suppose $\beta = 4$ and let $g_3 (c) = 2c - 20$ and $g_4 (c) = c^2 - 60c + 720$. Note that $g_4 (c) < 0$ if $17 \leq c \leq 43$, and $g_3 (c) = g_4 (c)$ when $16 < c < 17$ or $45 < c < 46$. 


Thus, the possible integer $c$ satisfying $3^{h-2} = g_3(c)/g_4(c)$ can be $c = 44$ or $45$. However, if $c = 44$, then $3^{h-2} = 68/16 \notin \mathbb{Z}$, a contradiction, and if $c = 45$, then $3^{h-2} = 70/45 \notin \mathbb{Z}$, a contradiction.

Suppose $\beta = 5$ and $g_5(c) = 2c - 24$ and $g_6(c) = c^2 - 90c + 1350$. We notice that $g_6(c) < 0$ if $20 \leq c \leq 70$, and $g_5(c) = g_6(c)$ when $18 < c < 20$ or $71 < c < 74$. Thus, the possible integer $c$ satisfying $3^{h-2} = g_5(c)/g_6(c)$ can be $c = 19$, $71$, $72$, or $73$. If $c = 19$, then $3^{h-2} = 14$, a contradiction, and if $c = 71$, $72$, $73$, then $3^{h-2} = 118$, $120/54$, $122/109$, respectively, so that they are contradictions. Hence, we complete the proof when $h \geq 3$.

By (i), (ii), (iii) and (iv), therefore, the proposition holds. $\blacksquare$

**Proposition 8.** Suppose $p = 2$. The positive integral parameters satisfying (7) imply one of the following cases.

(i) $\alpha = \beta^2$ when $h = 1$.

(ii) $\beta^2 (3\beta - 5) = 2\alpha$ where $\beta \geq 4$, $2^h = 3\beta - 5$ and $h \geq 3$.

(iii) $\alpha = 36$ and $\beta = 3$ when $h = 3$.

**Proof.** We take the same method with the one in the previous proposition. From (7), if $p = 2$, we have

$$w^2_\beta - (2^h \beta^2 + 2^h \beta + 6)w_\beta + 2^h (2^h \beta^2 + 4)(\beta + 1) = 0. \tag{20}$$

Since Proposition 5 shows that there is no positive integral solution of (6) when $h = 0$, we evaluate (20) with three cases that $h = 1$, $h = 2$ and $h \geq 3$.

(I) $h = 1$;

Eq. (20) implies that

$$w^2_\beta - 2(\beta^2 + \beta + 3)w_\beta + 2(2\beta^2 + 4)(\beta + 1) = 0$$

which implies that $w_\beta$ is divisible by 2. Let $2x = w_\beta$. Then we have

$$x^2 - (\beta^2 + \beta + 3)x + (\beta^2 + 2)(\beta + 1) = 0$$

which implies that $x = \beta^2 + 2$ or $x = \beta + 1$, as required.

(II) $h = 2$;

From (20) we have

$$w^2_\beta - (2^2 \beta^2 + 2^2 \beta + 6)w_\beta + 2^2 (2^2 \beta^2 + 4)(\beta + 1) = 0$$

which implies that $w_\beta$ is divisible by 2. Let $2x = w_\beta$. Then we have

$$f_1 (x) := x^2 - (2\beta^2 + 2\beta + 3)x + (4\beta^2 + 4)(\beta + 1) = 0.$$

Note that

$$f_1 (2\beta + 2) = 2\beta + 2 = f_1 (2\beta^2 + 1) > 0$$

and

$$f_1 (2\beta + 3) = -2\beta^2 + 4\beta + 4 = f_1 (2\beta^2) < 0$$

if $\beta \geq 3$. Thus, there is a non-integer root between $2\beta + 2$ and $2\beta + 3$ and there is another one between $2\beta^2$ and $2\beta^2 + 1$. Hence there is no integer root of $f_1 (x)$ if $\beta \geq 3$. If $\beta = 1$, then $f_1 (x) = x^2 - 7x + 16 > 0$ which has no real roots. If $\beta = 2$, then $f_1 (x) = x^2 - 15x + 60 > 0$ which has no real roots. Therefore, there is no integer root of $f_1 (x)$. 

(III) \( h \geq 3 \):

From (20), we have

\[
w_\beta^2 - (2^h \beta^2 + 2^h \beta + 6)w_\beta + 2^h (2^h \beta^2 + 4)(\beta + 1) = 0
\]

which implies 2 divides \( w_\beta \). If we define \( w_\beta = 2x \), we have

\[
f_2(x) := x^2 - (2^{h-1} \beta^2 + 2^{h-1} \beta + 3)x + 2^{h-1}(2^{h-1} \beta^2 + 2)(\beta + 1) = 0.
\]

If there are two integer roots (say \( x_1 \) and \( x_2 \)), since 2 divides \( x_1 + x_2 \), without loss of generality, we can assume \( x_1 \) is coprime to 2 and \( x_2 = 2^{h-1}c \) for some positive integer \( c \). Then we have \( x_1 + 2^{h-1}c = 2^{h-1}\beta^2 + 2^{h-1}\beta + 3 \) and \( 2^{h-1}c \cdot x_1 = 2^{h-1}(2^{h-1} \beta^2 + 2)(\beta + 1) \).

If we eliminate \( x_1 \) from these two equations, we have the following equation:

\[
2^{h-1}c^2 - (2^{h-1} \beta^2 + 2^{h-1}\beta + 3)c + (2^{h-1} \beta^2 + 2)(\beta + 1) = 0. \tag{21}
\]

Then we have the following ratio:

\[
2^{h-1} = \frac{3c - 2(\beta + 1)}{c^2 - (\beta^2 + \beta)c + \beta^2(\beta + 1)}. \tag{22}
\]

Let \( g_1(c) = 3c - 2(\beta + 1) \) and \( g_2(c) = c^2 - (\beta^2 + \beta)c + \beta^2(\beta + 1) \). We observe the following.

- If \( c \leq \beta \) then \( 0 < g_1(c)/g_2(c) < 1 \) or \( g_1(c)/g_2(c) < 0 \).
- \( g_1(\beta + 1) = g_2(\beta + 1) = \beta + 1 \).
- If \( \beta + 2 \leq c \leq \beta^2 - 2 \) and \( \beta \geq 4 \), then \( g_1(c)/g_2(c) < 0 \) since \( g_2(\beta + 2) = -\beta^2 + 2\beta + 4 = g_2(\beta^2 - 2) < 0 \) (when \( \beta \geq 4 \)) and \( g_2(\beta + 1) = \beta + 1 = g_2(\beta^2 - 1) > 0 \).
- \( g_2(\beta^2 + 2) = g_1(\beta^2 + 2) = 3\beta^2 - 2\beta + 4 \).
- If \( \beta^2 + 3 \), then \( 0 < g_1(c)/g_2(c) < 1 \) since \( g_2(\beta^2 + 3) - g_1(\beta^2 + 3) = \beta^2 - \beta + 2 > 0 \) if \( \beta \geq 1 \).

Suppose \( \beta \geq 4 \). Then, from the above arguments, we notice that the possible integer \( c \) satisfying (21) is \( c = \beta + 1, \beta^2 - 1, \beta^2, \beta^2 + 1 \) or \( \beta^2 + 2 \).

(i) If \( c = \beta + 1 \) or \( \beta^2 + 2 \), we have \( 2^{h-1} = 1 \) which is a contradiction since we suppose \( h \geq 3 \).

(ii) If \( c = \beta^2 - 1 \), then from (22) we have

\[
2^{h-1} = \frac{(\beta + 1)(3\beta - 5)}{\beta + 1} = 3\beta - 5
\]

as required.

(iii) If \( c = \beta^2 \), then we have

\[
2^{h-1} = \frac{3\beta^2 - 2\beta - 2}{\beta^2} = 3 - \frac{2\beta + 2}{\beta^2} \notin \mathbb{Z} \text{ if } \beta \geq 2
\]

which is a contradiction.

(iv) If \( c = \beta^2 + 1 \), then we have

\[
2^{h-1} = \frac{3\beta^2 - 2\beta + 1}{2\beta^2 - \beta + 1} = 1 + \frac{\beta^2 - \beta}{2\beta^2 - \beta + 1}.
\]

Note that \( 2\beta^2 - \beta + 1 > \beta^2 - \beta \) for all \( \beta \) and so \( (\beta^2 - \beta)/(2\beta^2 - \beta + 1) \) can be a non-negative integer only if \( \beta = 1 \), while we suppose \( \beta \geq 4 \).

On the other hand, the evaluations for the excluded values \( \beta = 1, 2, 3 \) are as follows.
If $\beta = 1$, from (22), we have
\[ 2^{h-1} = \frac{3c - 4}{c^2 - 2c + 2}. \]
If $c = 1$ then $2^{h-1} = -1$ which is a contradiction. If $c = 2$, 3 then $2^{h-1} = 1$, which is a contradiction, since $h \geq 3$. If $c \geq 4$ then $2^{h-1} < 1$ which is a contradiction.

(ii) If $\beta = 2$, then we have
\[ 2^{h-1} = \frac{3c - 6}{c^2 - 6c + 12}. \]
Since $3c - 6 = c^2 - 6c + 12$ when $c = 3$ or 6, we have the following.
If $c \leq 2$ or $c \geq 7$ then $2^{h-1} < 1$. If $c = 3, 6$ then $2^{h-1} = 1$. If $c = 4$ then $2^{h-1} = 6/4$. If $c = 5$ then $2^{h-1} = 9/7$. Hence, these cases do not occur.

(iii) If $\beta = 3$, then we have
\[ 2^{h-1} = \frac{3c - 8}{c^2 - 12c + 36}. \]
Since $3c - 8 = c^2 - 12c + 36$ when $c = 4$ or 11, we have the following.
If $c \leq 3$ or $c \geq 12$ then $2^{h-1} < 1$. If $c = 4, 11$ then $2^{h-1} = 1$. If $c = 5$ then $2^{h-1} = 7/1$. From (21), if $c = 6$ then $2^{h-1} = 8/3$. If $c = 7$ then $2^{h-1} = 13/1$. If $c = 8$ then $2^{h-1} = 16/4 = 2^2$, so that $h = 3$ which means $\alpha = 36$. If $c = 9$ then $2^{h-1} = 19/9$. If $c = 10$ then $2^{h-1} = 22/16$. Notice that we have the unique solution when $\beta = 3$ and $c = 8$.

By (I), (II) and (III), we complete the proof.

With Propositions 6–8, we find all possible positive integer solutions of Eq. (7) which is the special case of Eq. (6) under the hypothesis $2\alpha = \beta^2 p^h$ that is suggested from Lemma 2 about the divisibility of parameters of $(s; 1, n)$-set. Now we establish the following theorem which characterises the parameters of an $(s; 1, n)$-set in a triplane.

**Theorem 9.** Let $S$ be an $(s; 1, n)$-set in a symmetric $2-(v, k, 3)$ design. Suppose the parameters satisfy the condition $2(k - 3)/(n - 1)^2 = p^h$ where $p$ is a prime and $h$ is a positive integer. Then, firstly, there does not exist an $(s; 1, n)$-set when $p \neq 2$. Secondly, if $p = 2$, then either $k - 3 = (n - 1)^2$, 2, $k - 3 = (n - 1)^2 (3n - 8)$ or $(k - 3 = 36, n - 1 = 3, p^h = 2^3)$. Lastly, if $p = 3$, then the only possible existence of a $(s; 1, n)$-set is a $(5; 1, 3)$-set as a $2-(5, 3, 3)$ design in a symmetric $2-(25, 9, 3)$ design.

**Proof.** It is clear from Propositions 6–8. The statement for $p = 3$ follows immediately, since $\alpha = 6$ and $\beta = 2$, so that $n = 3$ and $k = 9$.

3. Generalisations in symmetric designs for $\lambda \geq 4$

Let $D$ be any symmetric $2-(v, k, \lambda)$ design for some $\lambda \geq 4$ and let $S$ be an $(s; 1, n)$-set in $D$. Recall that $(k - \lambda)(\lambda - 1)$ is divisible by $(n - 1)^2$ from Lemma 2 (ii). We suppose that
\[ \frac{(\lambda - 1)(k - \lambda)}{p^h(n - 1)^2} \]
for a prime $p$ and a non-negative integer $h$. Then, Propositions 5 and 6 give motivations for having Theorem 10 on a non-existence condition of parameters and Theorem 17 on characterising the size of $(s; 1, n)$-sets when $p$ and $\lambda(\lambda - 1)$ are coprime, respectively.

As we have done in the previous section, we use the notation $\alpha = k - \lambda$, $\beta = n - 1$ and let
\[ w_\beta = \frac{\lambda(\lambda - 1)(s - 1)}{\beta}. \]
Then, the classical equation (5)
\[ \lambda^2 s^2 - \lambda (kn + \lambda)s + (k(k - 1) + \lambda)n = 0 \]
may be written as
\[ w_\beta^2 - (\beta^2 p^h + \beta p^h + \lambda(\lambda - 1))w_\beta + p^h(\beta^2 p^h + (\lambda - 1)^2)(\beta + 1) = 0. \] (24)

Proposition 5 is generalised for all \( \lambda \geq 3 \), as follows.

**Theorem 10.** In a symmetric 2-(v, k, \lambda) design with \( \lambda \geq 3 \), there does not exist an \( (s; 1, n) \)-set whose parameters satisfy \( (\lambda - 1)(k - \lambda) = (n - 1)^2 \).

**Proof.** Since \( h = 0 \), (24) is written as
\[ w_\beta^2 - (\beta^2 + \beta + \lambda(\lambda - 1))w_\beta + (\beta^2 + (\lambda - 1)^2)(\beta + 1) = 0. \]
If we say \( f(x) := x^2 - (\beta^2 + \beta + \lambda(\lambda - 1))x + (\beta^2 + (\lambda - 1)^2)(\beta + 1) \), we have
\[ f(\beta) = \beta^2 - (\lambda - 1)\beta + (\lambda - 1)^2 > 0 \quad \text{for all } \beta, \]
since the discriminant of \( f(\beta) \) is less than 0, and we note that
\[ f(\beta + 1) = -(\beta + 1)(\lambda - 2) < 0 \quad \text{for all } \beta. \]
It implies that there is at least one non-integer root of \( f(x) \) between \( x = \beta \) and \( x = \beta + 1 \). Since the coefficients of \( f(x) \) are all integers so that the sum of two roots and the multiple of two roots are all integers, we conclude that there is no integer root of \( f(x) \). \[ \square \]

**Example 11.** From Theorem 10, there is no set of type-(1, 13) in a symmetric 2-(66, 26, 10) design whose construction is known in [10, p. 83], since the parameters satisfy \( (\lambda - 1)(k - \lambda) = (10 - 1)(26 - 10) = 12^2 = (13 - 1)^2 = (n - 1)^2 \).

From now on, suppose \( p \) is coprime to \( \lambda \) and \( \lambda - 1 \). Let \( h \) be a positive integer. Note that if we put \( x_1 \) and \( x_2 \) as two roots of (24) with respect to \( w_\beta \), we have
\[ \begin{align*}
    x_1 + x_2 &= \beta^2 p^h + \beta p^h + \lambda(\lambda - 1) \\
    x_1 x_2 &= p^h(\beta^2 p^h + (\lambda - 1)^2)(\beta + 1).
\end{align*} \] (25)
Since we suppose that \( p \) does not divide \( \lambda(\lambda - 1) \) and \( h > 0 \), we have that \( x_1 x_2 \) is divisible by \( p^h \) and \( x_1 + x_2 \) is not divisible by \( p^h \), so that one of the roots (say \( x_2 \)) is written as \( x_2 = p^h c \) for some positive integer \( c \). If we eliminate \( x_1 \) from (25), we have the following quadratic equation with respect to \( c \):
\[ p^h c^2 - (\beta p^h + \beta p^h + \lambda(\lambda - 1))c + (\beta^2 p^h + (\lambda - 1)^2)(\beta + 1) = 0 \]
which implies a ratio as follows:
\[ p^h = \frac{\lambda(\lambda - 1)c - (\lambda - 1)^2(\beta + 1)}{c^2 - (\beta^2 + \beta)c + \beta^2(\beta + 1)}. \] (26)
Let \( g_1(c) \) and \( g_2(c) \) be the numerator and the denominator of (26), respectively. Note that \( g_1(c) \) is a linear function and \( g_2(c) \) is a quadratic function, with respect to \( c \). We evaluate the ratio (26) for all positive integers \( c \) as follows.
1. If $0 < c \leq \beta$, then $p^h < 1$.
2. If $c = \beta + 1$, then $p^h = \lambda - 1$ which contradicts the hypothesis that $\lambda - 1$ is not divisible by $p$ unless $h = 0$. If $h = 0$, we have $\lambda = 2$, while we suppose $\lambda > 3$.
3. If $\beta + 2 \leq c \leq \beta^2 - 2$, then $g_1(c) > 0$ and $g_2(c) < 0$ so that $p^h < 0$.
4. If $c = \beta^2 - 1$, then $p^h = (\lambda - 1) (\lambda (\beta - 2) - 1)$ which contradicts the hypothesis that $\lambda - 1$ is not divisible by $p$ unless $h = 0$. If $h = 0$, we have $\lambda = 2$, while we suppose $\lambda > 3$.
5. If $\beta^2 \leq c \leq \beta^2 + \lambda (\lambda - 1) - 1$, then $g_1(c) > g_2(c) > 0$.
6. If $c = \beta^2 + \lambda (\lambda - 1)$, then $p^h = \lambda - 1$ which contradicts the hypothesis that $\lambda - 1$ is not divisible by $p$ unless $h = 0$. If $h = 0$, we have $\lambda = 2$, while we suppose $\lambda > 3$.
7. If $c > \beta^2 + \lambda (\lambda - 1)$, then we have $p^h < 1$.

Besides case 5, all the cases have no positive integer $c$ which satisfies (26). This implies the following proposition.

**Proposition 12.** Let $p$ be a prime such that $p$ does not divide $\lambda (\lambda - 1)$. If there is a positive integer $c$ satisfying ratio (26), then $\beta^2 \leq c \leq \beta^2 + \lambda (\lambda - 1) - 1$.

Proposition 12 can be restated as the following theorem which implies some bounds on the size of $(s; 1, n)$-set in a symmetric design with the given conditions on the parameter set.

**Corollary 13.** Let $S$ be an $(s; 1, n)$-set in a symmetric $2-(v, k, \lambda)$ design. Let $p$ be a prime and $h$ be a non-negative integer. Let $\lambda \geq 4$. Suppose the parameters satisfy the conditions that $(\lambda - 1)(k - \lambda) = (n - 1)^2 p^h$ and $p$ is coprime to $\lambda$ and $\lambda - 1$. Then, we have bounds of $s$ given by

$$
\frac{(n - 1) (k - \lambda)}{\lambda} + 1 \leq s \leq \frac{(n - 1) (k - \lambda)}{\lambda} + \frac{(k - \lambda) (\lambda^2 - \lambda - 1)}{\lambda(n - 1)} + 1
$$

and

$$
\frac{k - \lambda}{\lambda} + n - \frac{(k - \lambda) (\lambda^2 - \lambda - 1)}{(n - 1)^2} \leq s \leq \frac{k - \lambda}{\lambda} + n.
$$

**Proof.** Let $c$ be a positive integer satisfying (26). From Theorem 10, if $h = 0$, there is no positive integral parameter set of solutions of (24). Let $h \geq 1$. Note that two roots $x_1, x_2$ of (24) are derived from $c$ which satisfies (26). Recall $x_2 = p^h c$ and $x_1 = \beta^2 p^h + \beta p^h + \lambda (\lambda - 1) - x_2$ from (25). Two positive integral roots of (5) (say $s_1$ and $s_2$) are derived from $x_1$ and $x_2$, respectively, such that for each $i = 1, 2$

$$
 s_i = \frac{\beta x_i}{\lambda (\lambda - 1)} + 1.
$$

by (23). From Proposition 12, a positive integer $c$ satisfying (26) can exist only if $\beta^2 \leq c \leq \beta^2 + \lambda (\lambda - 1) - 1$. Since $x_2 = p^h c$, we have

$$
\frac{\beta^3 p^h}{\lambda (\lambda - 1)} + 1 \leq s_2 \leq \frac{\beta^3 p^h + \beta p^h (\lambda (\lambda - 1) - 1)}{\lambda (\lambda - 1)} + 1
$$

which implies

$$
\frac{(n - 1) (k - \lambda)}{\lambda} + 1 \leq s_2 \leq \frac{(n - 1) (k - \lambda)}{\lambda} + \frac{(k - \lambda)(\lambda^2 - \lambda - 1)}{\lambda(n - 1)} + 1
$$

since $(\lambda - 1)(k - \lambda) = (n - 1)^2 p^h$, i.e. $(\lambda - 1)\alpha = \beta^2 p^h$. 

Similarly, since \( x_1 = \beta^2 p^h + \beta p^h + \lambda (\lambda - 1) - x_2 \), the bounds \( \beta^2 \leq c \leq \beta^2 + \lambda (\lambda - 1) - 1 \) imply

\[
\frac{k - \lambda}{\lambda} + n - \frac{(k - \lambda)(\lambda^2 - \lambda - 1)}{(n-1)^2} \leq s_1 \leq \frac{k - \lambda}{\lambda} + n
\]

which are the second bounds of \( s \). ■

The following lemma on the divisibility between the parameters implies more reduced bounds on \( c \) than the bounds given in Proposition 12.

**Lemma 14.** Suppose that \( p \) does not divide \( \lambda (\lambda - 1) \). Then the following divisibility for the parameters in (26) holds.

(i) \((\lambda - 1) | \beta^2\),

(ii) \((\lambda - 1) | c\).

**Proof.** Since \((\lambda - 1) \alpha = p^h \beta^2 \) and \( p^h \) is coprime to \( \lambda - 1 \), (i) immediately follows.

Note that \( s = \frac{\beta p^h c}{\lambda (\lambda - 1)} + 1 \), since \( w_\beta = \frac{\lambda (\lambda - 1)(s - 1)}{\beta} = p^h c \). Since \( \lambda (s - 1) = \frac{\beta p^h c}{\lambda - 1} \) is an integer and \( \lambda (s - 1) \) is divisible by \( \beta \) (from Result 1(i)), so \( p^h c \) is divisible by \( \lambda - 1 \). Since \( p \) is coprime to \( \lambda - 1 \), we conclude that \( c \) is divisible by \( \lambda - 1 \), which completes (ii). ■

**Proposition 15.** If a positive integer \( c \) satisfies (26), it follows that \( c = \beta^2 + j (\lambda - 1) \).

for some \( j = 0, 1, \ldots, \lambda - 1 \).

**Proof.** From Lemma 14 and Proposition 12, \( c \) must be a multiple of \( \lambda - 1 \) such that \( \beta^2 \leq c \leq \beta^2 + \lambda (\lambda - 1) - 1 \). ■

Moreover, the following lemma implies that if \( j = 0, 1 \) in Proposition 15, then such a \( c \) does not satisfy (26).

**Lemma 16.** If \( c = \beta^2 \) or \( \beta^2 + (\lambda - 1) \), then \( c \) does not satisfy (26).

**Proof.** Assume that \( c = \beta^2 \) satisfies (26). Then,

\[
\begin{align*}
g_1(\beta^2) &= \lambda (\lambda - 1) \beta^2 - (\lambda - 1)^2 (\beta + 1) = \lambda (\lambda - 1) \beta^2 - (\lambda - 1)^2 \beta - (\lambda - 1)^2 \\
g_2(\beta^2) &= \beta^2 - (\beta^2 + \beta) \beta + \beta^2 (\beta + 1) = \beta^2.
\end{align*}
\]

Thus, we have

\[
p^h = \frac{g_1(\beta^2)}{g_2(\beta^2)} = \frac{\lambda (\lambda - 1) - (\lambda - 1)^2 (\beta + 1)}{\beta^2}
\]

so that \( (\lambda - 1)^2 (\beta + 1)/\beta^2 \) must be an integer. Hence, \( (\lambda - 1)^2 \) must be divisible by \( \beta \), since \( \beta \) and \( \beta + 1 \) are coprime. However, this is impossible, since \( \lambda < n \) so that \( (\lambda - 1)^2 < (n - 1)^2 = \beta^2 \).

Assume that \( c = \beta^2 + \lambda - 1 \) satisfies (26). Then,

\[
\begin{align*}
g_1(\beta^2 + \lambda - 1) &= (\lambda - 1) (\lambda \beta^2 - (\lambda - 1) \beta + (\lambda - 1)^2) \\
g_2(\beta^2 + \lambda - 1) &= \lambda \beta^2 - (\lambda - 1) \beta + (\lambda - 1)^2
\end{align*}
\]
so that
\[ ph = \frac{g_1(\beta^2 + \lambda - 1)}{g_2(\beta^2 + \lambda - 1)} = \lambda - 1 \]
which contradicts that \( ph \) and \( \lambda - 1 \) are coprime. ■

As a conclusion, we close this section with the following theorem which characterises the size of an \((s; 1, n)\)-set.

**Theorem 17.** Let \( S \) be an \((s; 1, n)\)-set in a symmetric 2-(\( v, k, \lambda \)) design. Let \( p \) be a prime and \( h \) be a non-negative integer. Let \( \lambda \geq 4 \). Suppose the parameters satisfy the conditions that 
\( (\lambda - 1)(k - \lambda) = (n - 1)^2 ph \) and \( p \) is coprime to \( \lambda \) and \( \lambda - 1 \). Then, for some integer \( j = 2, \ldots, \lambda - 1 \), we have either
\[
s = \frac{k - \lambda}{\lambda} \left( n - 1 - j \left( \frac{\lambda - 1}{n - 1} \right) \right)
\]
or
\[
s = \frac{k - \lambda}{\lambda} \left( 1 - j \left( \frac{\lambda - 1}{n - 1} \right) \right).
\]

**Proof.** Let \( S \) be an \((s; 1, n)\)-set in a symmetric 2-(\( v, k, \lambda \)) design. Let \( p \) be a prime satisfying 
\( (\lambda - 1)(k - \lambda) = (n - 1)^2 ph \) and let \( p \) be coprime to \( \lambda \) and \( \lambda - 1 \). Let \( \lambda \geq 4 \). Proposition 5 implies that, if \( h = 0 \), there is no solution for the classical equation. Let \( h \geq 1 \) and let \( c \) and \( \beta \) be positive integers satisfying Eq. (26), which is derived from Eq. (5). Then, by Proposition 15 and Lemma 16, for some \( j = 2, \ldots, \lambda - 1 \) we have 
\[
c = \beta^2 + j (\lambda - 1).
\]
As mentioned in the proof of Corollary 13, note that \( x_2 = ph c, x_1 = \beta^2 ph + \beta ph + \lambda (\lambda - 1) - x_2 \), and that for \( i = 1, 2 \), we have 
\[
s_i = \frac{\beta x_i}{(\lambda - 1)} + 1 \]
which are two positive integral roots of (5). Hence, For some \( j = 2, \ldots, \lambda - 1 \), we have
\[
s_2 = \frac{\beta ph (\beta^2 + j (\lambda - 1))}{\lambda (\lambda - 1)} + 1 = \frac{k - \lambda}{\lambda} \left( n - 1 - j \left( \frac{\lambda - 1}{n - 1} \right) \right)
\]
and
\[
s_1 = \frac{\beta ph (\beta^2 ph + \beta ph + \lambda (\lambda - 1) - x_2)}{\lambda (\lambda - 1)} + 1 = \frac{k - \lambda}{\lambda} \left( 1 - j \left( \frac{\lambda - 1}{n - 1} \right) \right).
\]
Therefore, a positive integral solution \( s \) of (5) (with the given conditions of parameters) is either \( s_1 \) or \( s_2 \) for some \( j = 2, \ldots, \lambda - 1 \). ■

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**References**