A characterization of the Dedekind completion of a totally ordered group of infinite rank

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ABSTRACT

In non-Archimedean functional analysis the Dedekind completion of a linearly ordered group of infinite rank is an important object, being the natural home for the norms of vectors as well as of linear operators. However the standard construction by cuts does not give the much needed actual description of the elements obtained. In this paper we consider a class of Hahn products, called \( \Gamma_\alpha \) (\( \alpha \) an ordinal), whose rank is the order-type of \( \alpha \). We give an operational representation of every element of the Dedekind completion of such a group in terms of the supremum and infimum of its convex subgroups.

1. INTRODUCTION

In the last years a theory of Banach spaces over fields with a Krull valuation of infinite rank has been in constant development (see [5,8]). It started with the construction by Keller in [4] of what was called at that time a non-classical Hilbertian space (nowadays a Form Hilbert space). The crucial point was that the scalar field admitted a valuation with a value group \( G \) of infinite rank.

As the theory progressed, \( G^\# \), the Dedekind completion of \( G \) became an important object. In order to obtain a satisfactory operator theory it is necessary to define their norm and the natural way is, for an operator \( A \) on the space \( E \), that \( \| A \| \) should be the supremum of the set \( \{ g \in G: \| Ax \| \leq g\| x \| \text{ for all } x \in E \} \) and

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therefore (the semigroup) $G^\#$ has to be considered. In addition, for any space $E$ a non-Archimedean norm $\| \cdot \| : E \to G^\#$ can be defined; see [5] for an example in which the norm of every vector of an orthogonal base of $E$ is the supremum of a convex subgroup. Interesting properties of $G^\#$-normed spaces are obtained in [10].

Hahn products are specially suited as value groups for these non-Archimedean Banach spaces, thus a subclass of them was built in [7]. Let $\alpha$ be an ordinal, $\{G_\beta\}_{\beta < \alpha}$ be a family of subgroups of the multiplicative group of the positive real numbers. A group $\Gamma_\alpha$ is a Hahn product with skeleton $(\alpha, G_\beta)$, multiplicatively written and antilexicographically ordered. In that paper it was proved that the ordinal $\alpha$ gives information about the (ultra)metrizability of the field and the behaviour of its absolutely convex subsets. These properties were obtained from the special characteristics of the groups $\Gamma_\alpha$ and its convex subgroups, for instance, $\Gamma_\alpha$ is the union of a countable chain of convex subgroups if and only if the ordinal $\alpha$ has cofinality $\omega$.

Several completions of totally, partially or lattice ordered groups have been described in the literature [1–3]. But in every case, even when the completion is by the process of Dedekind cuts, the final object considered is again a group, extension of the first one. We stress the fact that if the rank of $G$ is greater than 1, then $G^\#$ is only a semigroup since the supremum (infimum) of a convex subgroup cannot be invertible. And, as we have seen, for the Banach space theory, it is crucial that these elements should appear in the completion.

In this paper we obtain an explicit operational description of the elements of the Dedekind completion $\Gamma_\alpha^\#$ of the Hahn product $\Gamma_\alpha$. After the Preliminaries in Section 2, we deal in Section 3 with the case in which every group of the family $\{G_\beta\}_{\beta < \alpha}$ is complete. In the fourth section we generalize to a description of the completion $\Gamma_\alpha^\#$ in the case the groups $\{G_\beta\}_{\beta < \alpha}$ are arbitrary subgroups of $(\mathbb{R}^+, \cdot)$.

An unexpected result is obtained in the case when not every group $G_\beta$ is complete. Let $\Gamma'_\alpha$ be the group associated to the family $\{G_\beta\}_{\beta < \alpha}$, where $G_\beta$ is the completion of $G_\beta$ for each $\beta < \alpha$, and $(\Gamma'_\alpha)^\#$ its completion. We prove the surprising fact that $\Gamma_\alpha^\#$ is not dense in $(\Gamma'_\alpha)^\#$ with respect to the order topology. Moreover, there are intervals in $(\Gamma'_\alpha)^\#$ which do not contain any element of $\Gamma_\alpha^\#$ (see Example 4.2).

2. PRELIMINARIES

Let $G$ be a nontrivial totally ordered multiplicatively written group with unit 1. We denote the (Dedekind) completion of $G$ by $G^\#$. Two associative and commutative operations can be defined over $G^\#$ by

\[
\begin{align*}
x \ast y & := \inf_{G^\#} \{g_1g_2 : g_1, g_2 \in G, g_1 \geq x, g_2 \geq y\}, \\
x \cdot y & := \sup_{G^\#} \{g_1g_2 : g_1, g_2 \in G, g_1 \leq x, g_2 \leq y\}
\end{align*}
\]

for every $x, y \in G^\#$.

Notice that these operations coincide over $G$. Moreover, for all $g \in G$ and $x \in G^\#$, $g \ast x = g \cdot x$. We will denote this common value by $gx$. 634
A unique extension of the inversion, called the antipode, can be defined over $G^\#$. It is a map $\omega : G^\# \rightarrow G^\#$ such that $\omega \circ \omega$ is the identity, $\omega(t) = s$ whenever $s, t$ are the supremum and infimum respectively of some convex subgroup $H$ and $\omega(gx) = g^{-1}\omega(x)$ for all $g \in G, x \in G^\#$. (See [6], Theorem 1.4.8 for details.)

The convex hull of a subset $A \subseteq G$ is

$$\text{conv}_{G^\#} A = \{x \in G^\#: \exists g_1, g_2 \in A (g_1 \preceq x \preceq g_2)\}.$$ 

In this paper we will work with a class of Hahn products, defined as $\Gamma_\alpha$ in [7]. Their structure is the following.

Let $\alpha$ be an ordinal. For each $\beta < \alpha$, let $G_\beta$ be a totally ordered group of rank 1. The group $\Gamma_\alpha$ associated to the family $\{G_\beta\}_{\beta < \alpha}$ is defined by:

$$\Gamma_\alpha := \left\{ f : \alpha \rightarrow \bigcup_{\beta < \alpha} G_\beta : f(\beta) \in G_\beta \text{ and } \supp(f) = \{ \beta < \alpha : f(\beta) \neq 1_{G_\beta} \text{ is finite} \} \right\}$$

with componentwise multiplication and antilexicographical ordering.

If $f \in \Gamma_\alpha$, then the degree of $f$ is $\deg(f) := \max \supp(f)$.

To every element $b$ of $G_\beta$ we associate its characteristic function $\chi(\beta, b) \in \Gamma_\alpha$, defined as

$$\chi(\beta, b)(\gamma) := \begin{cases} b & \text{if } \gamma = \beta, \\ 1_{G_\gamma} & \text{if } \gamma \neq \beta. \end{cases}$$

This function will be used several times in the proofs of the main theorems of this paper.

The convex subgroups of $\Gamma_\alpha$ are easily described. Following Ribenboim's notation (see [9]), we define for each $\beta < \alpha$, the sets $H_\beta^* := \{ f \in \Gamma_\alpha : \deg(f) < \beta \}$ and $H_\beta := \{ f \in \Gamma_\alpha : \deg(f) \leq \beta \}$, both are convex subgroups. If $\beta = \gamma + 1$ for an ordinal $\gamma$, $H_\beta^* = H_\gamma$ and it is a principal convex subgroup (there exists an element $g \in G$ such that $H_\gamma$ is the smallest convex subgroup which contains it); otherwise, $H_\beta^*$ is a limit convex subgroup (union of a chain of principal convex subgroups) properly contained in $H_\beta$. Note that $H_0^* = \{1_{\Gamma_\alpha}\}$ and if $\beta$ is a limit ordinal, the group $H_\beta^*$ contains all the subgroups $H_\gamma$ for $\gamma < \beta$ and it is properly contained in $H_\beta$. In the other hand, if $H$ is a convex subgroup of $\Gamma_\alpha$ then there exists an ordinal $\beta < \alpha$ such that either $H = H_\beta$ or $H = H_\beta^*$ (see [7], Proposition 2.1). We denote by $s_\beta^*$ and $t_\beta^*$, respectively, the supremum and infimum of $H_\beta^*$ in $\Gamma_\alpha^\#$. Whenever we use the notation $H_\beta$ for a principal convex subgroup, then its supremum and infimum will be written as $s_\beta$ and $t_\beta$.

3. THE COMPLETION OF $\Gamma_\alpha$: FIRST CASE

In this section we characterize the completion $\Gamma_\alpha^\#$ in the case that the family $\{G_\beta\}_{\beta < \alpha}$ contains only complete groups, that is to say, cyclic groups or isomorphic
copies of the multiplicative group \((0, \infty)\). We establish the main result of this section in Theorem 3.2. First, we will prove the following lemma which shows that for every totally ordered group, the orbits in \(G^\#\) of the supremum and infimum of convex subgroups are disjoint.

**Lemma 3.1.** Let \(G\) be a totally ordered and multiplicatively written nontrivial group and let \(\{H_i\}_{i \in I}\) be the collection of all its proper convex subgroups, where \(I\) is a totally ordered set with a least element, say \(-\infty\), such that \(H_{-\infty} = \{1_G\}\) and \(H_i \subset H_j\) whenever \(i < j\). Let \(s_i = \sup_{G^\#} H_i\), \(t_i = \inf_{G^\#} H_i\). Then

1. \(G s_i \cap G s_j = \emptyset\), whenever \(i \neq j\).
2. \(G t_i \cap G t_j = \emptyset\), whenever \(i \neq j\).
3. \(G t_i \cap G s_j = \emptyset\), whenever \(i \neq j\).

**Proof.** We will use the following equalities derived from the definition of the operations \(\cdot\) and \(*\) (see [6], Proposition 1.4.11). Assume \(i < j\).

\[
\begin{align*}
s_i \cdot t_j &= t_j, \quad s_i \cdot t_j = t_j, \\
t_i \cdot t_j &= t_j, \quad s_i \cdot s_j = s_j, \\
s_j \cdot t_j &= t_j, \quad s_j \cdot s_j = s_j.
\end{align*}
\]

(1) We proceed by contradiction. Suppose that there are \(g_1, g_2 \in G\) such that \(g_1 s_i = g_2 s_j\) and \(i < j\). If it were the case that \(i = -\infty\), then \(g_1 = g_2 s_j\) which implies that \(s_j \in G\). This cannot be since \(j > -\infty\) which implies \(s_j \notin G\).

Therefore \(i > -\infty\). But now, using associativity we have \(g_1(s_i \cdot t_j) = g_2(s_j \cdot t_j)\) as well as \(g_1(s_i \cdot t_j) = g_2(s_j \cdot t_j)\). By the equalities at the start of the proof we have \(g_1 t_j = g_2 t_j\) and \(g_1 t_j = g_2 s_j\), so \(t_j = s_j\) a contradiction. Thus, \(G s_i \cap G s_j = \emptyset\), whenever \(i \neq j\).

(2) Let \(g_1 t_i = g_2 t_j\). By applying the antipode \(\omega\), we obtain \(g_1^{-1} s_i = g_2^{-1} s_j\), and the result follows from (1).

(3) Again we derive a contradiction. Suppose there exist \(g_1, g_2 \in G\) such that \(g_1 t_i = g_2 s_j\) with \(i \neq j\). As in case (1), neither index \(i\) nor \(j\) can be equal to \(-\infty\). Hence there are two cases to consider. If \(-\infty < i < j\), using commutative and associativity of the operation \(*\), we have \(g_1(t_i \cdot s_i) = g_2(s_j \cdot s_i)\), which implies \(g_1 s_i = g_2 s_j\), false by (1). In the other hand, if \(-\infty < j < i\), then \(g_1(t_i \cdot t_j) = g_2(s_j \cdot t_j)\), and \(g_1 t_i = g_2 t_j\), false by (2).

\(\square\)

**Remark.** The only remaining case, \(G t_i = G s_i\), is well known. It is equivalent to the fact that \(G/H_i\) is quasi-discrete (i.e. the minimum of \(\{g \in G/H_i: g > 1\}\) exists) (see [6], Proposition 1.4.12). When \(G\) is a group \(\Gamma_\alpha\), this implies that \(\Gamma_\alpha s_\beta^* = \Gamma_\alpha t_\beta^*\) if and only if \(G_\beta\) is a cyclic group (see [7], Proposition 2.6).

The next theorem shows the principal result of this section, the characterization of the completion of the group \(\Gamma_\alpha\) when the family \(\{\overline{G_\beta}\}_{\beta < \alpha}\) contains only complete groups.

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Theorem 3.2. Let $\alpha$ be an ordinal, $\{G_\beta\}_{\beta<\alpha}$ a family of totally ordered multiplicatively written complete groups of rank 1. Then each element $x$ in the Dedekind completion $\Gamma_\alpha^#$ of the group $\Gamma_\alpha$ can be written as $x = fu$ for some $f \in \Gamma_\alpha$ and $u$ the supremum or the infimum of some convex subgroup.

Proof. We have to prove that if $x \in \Gamma_\alpha^#$, then there exists an element $f \in \Gamma_\alpha$ and an ordinal $\lambda < \alpha$ such that $x = fs^*_\lambda$ or $x = ft^*_\lambda$. (Note that $x \in \Gamma_\alpha$ if and only if $x = xs^*_0 = xt^*_0$.)

The first case, $\alpha$ is a limit ordinal, bears the weight of the proof. We shall fix an ordinal $\xi$ and consider the set of the "$\xi$-coordinates" of the functions $f \in \Gamma_\alpha$ such that $f \leq x$. There are three possibilities to discuss, and it will shown that in any of them $x$ has the prescribed form. The second case, when $\alpha$ is a successor ordinal, can then readily be reduced to one of the alternatives just discussed. This completes the proof.

Let us then assume that $\alpha$ is a limit ordinal, so that $\Gamma_\alpha$ is an strictly increasing union of proper convex subgroups. For each $x \in \Gamma_\alpha^#$, there are possibly infinitely many $\delta < \alpha$ such that $x \in \text{conv}_{\Gamma_\alpha^#} H_\delta \cup \{s_\delta\}$, where $H_\delta$ is a principal convex subgroup of $\Gamma_\alpha$. Let $\delta_x$ be the smallest of them and consider the interval $F_x = (t_{\delta_x}, x] = \{f \in \Gamma_\alpha: t_{\delta_x} < f \leq x \leq s_{\delta_x}\}$ contained in $\Gamma_\alpha$. It is clear that $\deg(f) \leq \delta_x$ for every $f \in F_x$ and strict inequality holds when $f \in H_{\delta_x}^*$.

We define the set $F_x[\delta_x] := \{f(\delta_x): f \in F_x\} \subseteq G_{\delta_x}$. We have three cases to consider.

Case 1. If $F_x[\delta_x]$ is not bounded above, we will prove that $x = s_{\delta_x}$. Indeed, it is clear that $s_{\delta_x}$ is an upper bound of $F_x$. Now, if $1 \leq w < s_{\delta_x}$, then there exists $g \in \Gamma_\alpha$ such that $w \leq g < s_{\delta_x}$ (see [5], Proposition 1.1.4). Then $\deg(g) \leq \delta_x$ and by hypothesis, there exists $f \in F_x$ such that $g(\delta_x) < f(\delta_x)$. Thus $f < g$ and $w$ is not an upper bound of $F_x$.

Case 2. Now, let $F_x[\delta_x]$ be bounded above and suppose that $a := \sup_{G_{\delta_x}} F_x[\delta_x]$ does not belong to $F_x[\delta_x]$. This means that $G_{\delta_x} \cong (0, \infty)$. We will prove that $x = \chi(\delta_x, a)t_{\delta_x}^*$, with $\chi(\delta_x, a)$ the characteristic function defined in Preliminaries.

That $\chi(\delta_x, a)t_{\delta_x}^*$ is an upper bound of $F_x$ is straightforward. First, note that the set $F'_x = \{f \in F_x: \deg f = \delta_x\}$ is nonempty and $\sup_{\Gamma_\alpha^#} F'_x = \sup_{\Gamma_\alpha^#} F_x$. Let $f$ be an arbitrary element of $F'_x$. Then $\deg f = \delta_x$ and $f(\delta_x) < \alpha$, therefore $f < \chi(\delta_x, a)g$ for every $g \in H_{\delta_x}^*$. Hence $f \leq \chi(\delta_x, a)t_{\delta_x}^*$ and $\chi(\delta_x, a)t_{\delta_x}^*$ is an upper bound of $F_x$ as claimed.

Now consider $h \in \Gamma_\alpha$ such that $t_{\delta_x} < h < \chi(\delta_x, a)t_{\delta_x}^*$. It is not possible that $h(\delta_x) > a$ because of the last inequality. Neither can it be that $h(\delta_x) = a$ since in that case $h\chi(\delta_x, a^{-1}) > t_{\delta_x}^*$, a contradiction. Therefore $h(\delta_x) < a$. But then, by hypothesis there exists $f \in F_x$ such that $h(\delta_x) < f(\delta_x) < a$. It follows that $h < f < x$. As no element of $\Gamma_\alpha$ lies between $x$ and $\chi(\delta_x, a)t_{\delta_x}^*$, this two elements are equal as it was claimed.
Case 3. If $F_x[\delta_x]$ is bounded above and it has a maximum $a := \max F_x[\delta_x]$ in $F_x[\delta_x]$, we will obtain the result by transfinite induction over $\delta_x$.

If $\delta_x = 0$ then $x \in \Gamma_\alpha$ and $x = \max F_x$.

Let $\delta_x = \gamma + 1$ be a successor ordinal and define $A_x := \{ f \in F_x : f(\delta_x) = a \}$. For every $f \in A_x$, let $f' = f \chi(\delta_x,a-1)$. This means that $f'$ coincides with $f$ except for the coordinates $\delta_x$ where $f'(\delta_x) = I_{\delta_x}$.

Now let $A^- := \{ f' : f \in A_x \}$. It is clear that $\sup_{\Gamma_\alpha} A_x = \chi(\delta_x,a) \sup_{\Gamma_\alpha} A^-$. Then $A^- \subseteq \conv_{\Gamma_\alpha} H_\gamma \cup \{ s_\gamma \}$ and, by induction hypothesis, $\sup A^- = fu$, for some $f \in \Gamma_\alpha$ and $u$ the supremum or the infimum of some (principal or limit) convex subgroup. So, $x = \chi(\delta_x,a)fu$.

For the limit case, let $\delta_x = \bigcup_{\beta < \delta_x} \beta$ be a limit ordinal.

As before let $A_x := \{ f \in F_x : f(\delta_x) = a \}$ and $A^- = \{ f' = f \chi(\delta_x,a-1) : f \in A_x \}$. We have two cases. If there exists an ordinal $\lambda < \delta_x$ such that $A^- \subseteq \conv_{\Gamma_\alpha} H_\lambda \cup \{ s_\lambda \}$, we use the induction hypothesis and then $x = \chi(\delta_x,a) \sup_{\Gamma_\alpha} A^-$. In the other case, the set $\deg(A^-) = \{ \deg(f') : f' \in A^- \}$ tends to $\delta_x$, hence either $\sup_{\Gamma_\alpha} A^- = s^*_\delta_x$ which implies $x = \chi(\delta_x,a)s^*_\delta_x$ or $\inf_{\Gamma_\alpha} A^- = t^*_\delta_x$ which implies $x = \chi(\delta_x,a)t^*_\delta_x$.

We finish the proof considering the case which $\Gamma_\alpha$ has a maximal proper convex subgroup $H$. That implies that $\alpha$ is a successor ordinal, say $\alpha = \gamma + 1$. Given $x \in \Gamma_\alpha$, if there exists a convex subgroup $H$ such that $x \in \conv_{\Gamma_\alpha} H$ we proceed as the above proof. Otherwise, we define $\delta_x := \gamma$ and necessarily $F_x[\delta_x]$ is bounded above. Therefore, with exactly the same reasoning as in Cases 2 and 3 above, we obtain the statement of the theorem. □

Theorem 3.2 shows that if all groups of the family $\{G_\beta\}_{\beta < \alpha}$ are complete then every element of the completion of the associated group $\Gamma_\alpha$ belongs to some orbit $\Gamma_\alpha u$, where $u$ is the supremum or infimum of some convex subgroup $H$. We will use this result in the next section to describe the completion for an arbitrary $\Gamma_\alpha$. Let us examine the following example.

Example 3.3. In order to illustrate the case when $\alpha$ is a limit ordinal, let us consider $\alpha = \omega + \omega$. Assume that every $G_\beta$ is a subgroup of the multiplicative group $\mathbb{R}^+$, let $A = \{ f \in \Gamma_\alpha : f^2 \leq \chi(\omega,2) \}$. If $G_\omega$ is the cyclic group generated by 2, we are in case 3 and the supremum of $A$ is $s^*_\omega$. In fact, the degree of any $g \in \Gamma_\alpha$ such that $g > s^*_\omega$ is at least $\omega$, hence $g^2 > \chi(\omega,2)$. If, on the other hand, $G_\omega = \mathbb{R}^+$ then it is clear that $\sup_{\Gamma_\alpha} A = \chi(\omega,\sqrt{2})s^*_\omega$, an element of the group times the supremum of a limit convex subgroup.

4. THE COMPLETION OF $\Gamma_\alpha$. GENERAL CASE

In this section we characterize the completion $\Gamma_\alpha^*$ when the family $\{G_\beta\}_{\beta < \alpha}$ is an arbitrary family of totally ordered groups of rank 1.

Let $\alpha$ be an ordinal, $\{G_\beta\}_{\beta < \alpha}$ and $\{G'_\beta\}_{\beta < \alpha}$ be two families of totally ordered groups of rank 1 such that $G_\beta \subseteq G'_\beta$ for all $\beta < \alpha$ and let $\Gamma_\alpha$ and $\Gamma'_\alpha$ be the groups associated with each family.
Note that there is a natural correspondence between the set of convex subgroups of $\Gamma_\alpha$ and $\Gamma'_\alpha$, and since $\Gamma_\alpha$ can be canonically embedded in $\Gamma'_\alpha$, we will assume that $\Gamma_\alpha \subseteq \Gamma'_\alpha$.

We call $H'_\beta$ $(H_\beta)$ the convex subgroups of $G_\beta$ and $H'_\beta^*$ $(H_\beta^*)$ the convex subgroups of $\Gamma'_\alpha$. It is easy to see that $G_\beta$ is cofinal (or equal to) $G'_\beta$, so each $H'_\beta^*$ is cofinal in $H'_\beta$. Let $s'_\beta^* = \sup_{(\Gamma'_\alpha)^*} H'_\beta^*$ and $t'_\beta^* = \inf_{(\Gamma'_\alpha)^*} H'_\beta^*$. Then $\Gamma'^*_\alpha$ can be embedded in $(\Gamma'_\alpha)^*$ through the mapping $\tau : \Gamma'^*_\alpha \rightarrow (\Gamma'_\alpha)^*$ defined for each $x \in \Gamma'_\alpha$ by

$$\tau(x) = \sup \{ g \in \Gamma_\alpha : g \leq x \}.$$

Clearly $\tau$ is strictly increasing and $\tau(g) = g$ for all $g \in \Gamma_\alpha$. Furthermore, a straightforward proof shows that $\tau(s'_\beta^*) = s'_\beta^*$ and $\tau(t'_\beta^*) = t'_\beta^*$. Therefore, $\Gamma'^*_\alpha$ can be considered as a subset of $(\Gamma'_\alpha)^*$ and we can identify the supremum and infimum of all the convex subgroups. Nevertheless the next proposition shows that $\Gamma'^*_\alpha$ is not dense in $(\Gamma'_\alpha)^*$.

**Remark.** We also could have used the embedding $\sigma(x) = \inf_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g \geq x \}$. These two embeddings are different in general and this fact is the clue to show that $\Gamma'^*_\alpha$ is not dense in $(\Gamma'_\alpha)^*$. First we show the following proposition.

**Proposition 4.1.** Let $\{G_\beta\}_{\beta < \alpha}$ be an arbitrary family of totally ordered groups of rank 1. For each $\beta < \alpha$, let $G'_\beta$ be the (Dedekind) completion of $G_\beta$ and let $\Gamma'_\alpha$ be the group associated to the family $\{G'_\beta\}_{\beta < \alpha}$. Let $\beta > 0$, such that $G_\beta$ is not complete, and let $b \in G'_\beta \setminus G_\beta$. Then for each $f \in \Gamma'_\alpha$ with $\deg(f) = \beta$ and $f(\beta) = b$, we have

1. $\sup_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g \leq f \} = \chi(\beta,b)t'_\beta^*$
2. $\inf_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g \geq f \} = \chi(\beta,b)s'_\beta^*$.

**Proof.** We shall only consider the first statement, the proof of the second one follows the same lines. Let $\beta < \alpha$ be such that $G_\beta \neq G'_\beta$, and let $b \in G'_\beta \setminus G_\beta$.

We consider the set of all elements in $\Gamma'_\alpha$ that satisfy the hypothesis of the theorem $F = \{ f \in \Gamma'_\alpha : \deg(f) = \beta$ and $f(\beta) = b \}$.

Notice that for all $g \in \Gamma_\alpha$ we have that $g \leq \chi(\beta,b)$ if and only if $g$ is a lower bound of $F$. In other words $\sup_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g$ is a lower bound of $F \} = \sup_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g \leq \chi(\beta,b) \}$.

Since the construction of $\Gamma'_\alpha$ includes only complete groups, by Theorem 3.2 we have that $\inf_{(\Gamma'_\alpha)^*} F = \chi(\beta,b)t'_\beta^*$. Hence $\sup_{(\Gamma'_\alpha)^*} \{ g \in \Gamma_\alpha : g \leq \chi(\beta,b) \} \leq \chi(\beta,b)t'_\beta^*$.

Equality follows from the fact that $\chi(\beta,b)t'_\beta^* < \chi(\beta,b)$ in the order of $(\Gamma'_\alpha)^*$. \qed

Recall that a set $A$ is dense in a set $X$ if $a = \sup_X \{ x \in A : x \leq a \} = \inf_X \{ x \in A : x \geq a \}$ for each $a \in X$. Since the supremum of the set $\{ g \in \Gamma_\alpha : g \leq f \}$ is not equal to $f$ we obtain the following corollary.

**Corollary 4.1.** Let $\Gamma_\alpha, \Gamma'_\alpha$ as in the above proposition. If for some $\beta > 0$ the group $G_\beta$ is not complete, then $\Gamma'^*_\alpha$ is not dense in $(\Gamma'_\alpha)^*$.
Example 4.2. Consider a group $\Gamma_\omega$ with $G_1 \cong (\mathbb{Q}^+, \cdot)$, and let $f = (1, \sqrt{2}, 1, \ldots) \in \Gamma_\omega'$. We contend that the interval $(f t_0, f s_0)$ in $\Gamma_\omega'$ does not contain any element of $\Gamma_\omega$. In fact, such an element should have the form $g = (a, b, 1, \ldots)$ for some $a \in G_0$ and $b \in \mathbb{Q}^+$. But clearly such a $g$ does not belong to the interval.

The next theorem gives a complete description of the elements of $(\Gamma'_\omega)^\#$.

Theorem 4.3. Under the hypothesis of Proposition 4.1, each element $x \in \Gamma_\omega^\# \setminus \Gamma_\omega$ satisfies one of the following sentences:

1. $x = gu$, with $g \in \Gamma_\alpha$ and $u$ the supremum or the infimum of a convex subgroup.
2. $x = g \chi(\beta, b) t_\beta^*$, for some $g \in \Gamma_\alpha$ and $\beta < \alpha$ such that $G_\beta \neq G_\beta'$ and $b \notin G_\beta$.

Proof. Let $x \in \Gamma_\omega^\#$. Using the mapping $\tau$ from $\Gamma_\omega^\#$ in $(\Gamma'_\omega)^\#$ previously defined, we may assume $\Gamma_\omega \subset (\Gamma'_\omega)^\#$. Then $x = f u$, with $f \in \Gamma'_\omega$ and $u$ the supremum or infimum of some (principal or limit) convex subgroup. This also includes the case $u = 1$, the supremum (and infimum) of the trivial subgroup $H_0^*$.

Note first that if $f \in \Gamma'_\omega \setminus \Gamma_\omega$ there exist ordinals $\beta < \alpha$ such that $f(\beta) \in G_\beta' \setminus G_\beta$. Hence, each $f \in \Gamma'_\alpha$ can be written as $f = g f'$, where $g \in \Gamma_\alpha$, $f' \in \Gamma'_\alpha$ and $f'(\deg(f')) \in G_{\deg(f')} \setminus G_{\deg(f')}$. Let $\delta_0 := \deg(f')$. By the definition of $\tau$, we have $fu = \sup_{(\Gamma'_\omega)^\#}(h \in \Gamma_\omega; h \leq fu) = g \sup_{(\Gamma'_\omega)^\#}(h \in \Gamma_\omega; h \leq f'u)$. If $f' \in \text{Stab}(u)$, then $x = gu$, as we want. If that is not so, then by the above proposition, $\sup_{(\Gamma'_\omega)^\#}(h \in \Gamma_\omega; h \leq f'u) = \chi(\delta_0, f'(\delta_0)) t_{\delta_0}^*$ and therefore $\tau(x) = g \chi(\delta_0, f'(\delta_0)) t_{\delta_0}^*$ and we are done. \qed

Therefore we have proven that every element $x$ in the completion of $\Gamma_\alpha$ either belongs to some orbit $\Gamma_\alpha u$, where $u$ is the supremum of some convex subgroup of $\Gamma_\alpha$, or $u$ is the infimum of some convex subgroup $H$ where $\Gamma_\alpha / H$ is quasidense, or $x$ belongs to some orbit $\Gamma_\alpha \chi(\beta, b) t_\beta^*$, where $G_\beta$ is not a complete group (hence, $G_\beta$ is a countable dense subgroup of the multiplicative group $(0, \infty)$), and $b \notin G_\beta$.

REFERENCES


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