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A Note on Hermite-Fejér Interpolation for the Unit Circle

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Abstract—In this note, an extension to the unit circle of the classical Hermite-Fejér Theorem is given. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let X be an arbitrary triangular matrix

$$-1 \leq x_{nn} < x_{n-1n} < \cdots < x_{1n} \leq 1, \qquad n = 1, 2, \ldots,$$

in the interval $[-1, 1]$ and let f be a function defined on this interval. Then, the interpolatory Lagrange polynomial is given by

$$L_n(f, X, x) = \sum_{k=1}^{n} l_{k,n}(x) f(x_{kn}),$$

where $l_{k,n}$ are the well-known fundamental polynomials of Lagrange. The interest of interpolation of functions is that quadrature formulas are often constructed from interpolating polynomials. Indeed, if we want to approximate the integral

$$\int_{-1}^{1} f(x)\sigma(x)\, dx,$$

where $\sigma(x)$ is a weight function on $[-1, 1]$, we can replace the function $f(x)$ in the integral by $L_n(f, X, x)$ and one obtains a quadrature formula. Furthermore, the uniform convergence of

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such interpolating polynomials to the function will give us the convergence of the quadrature formula. But, it is known (see [1]), that for any matrix X, the Lagrange interpolation can never be convergent for all continuous functions. For this reason the Hermite-Fejér interpolants were introduced. Let us recall the definition of the Hermite-Fejér interpolation problem. Indeed, it consists of finding a polynomial $P_{2n-1}(f, X, x)$ of degree at most $2n-1$ satisfying the conditions

$$\begin{aligned} &P_{2n-1}(f, X, x_{kn}) = f(x_{kn}), \\ &P'_{2n-1}(f, X, x_{kn}) = 0, \qquad k = 1, \ldots, n. \end{aligned} \tag{1.1}$$

If X is the matrix of Chebyshev nodes, i.e., $x_{kn} = (\cos(2k-1)\pi)/2n$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$, in contrast to Lagrange interpolation, Fejér proved the following.

THEOREM 1.1. *(See [2, p. 118].) In the above conditions, the sequence* $\{P_{2n-1}(f, X, \cdot)\}$ *converges uniformly for all continuous functions* f *on* $[-1, 1]$.

In this paper, we shall be concerned with the interpolation of functions defined not on $[-1, 1]$ but on the unit circle denoted by $\mathbb{T} = \{z : |z| = 1\}$. In this respect, some notations are required. Thus, for every pair (p, q) of integers, where $p \le q$, we denote by $\Lambda_{p,q}$ the linear space of all Laurent polynomials (L-polynomials)

$$L(z) = \sum_{j=p}^{q} c_j z^j, \qquad c_j \in \mathbb{C}.$$

We write Λ for the linear space of all L-polynomials, Π for the space of all polynomials and $\Pi_n (= \Lambda_{0,n})$ for the space of all polynomials of degree at most n. We shall also write $\mathbb{D} = \{z : |z| < 1\}$ for the open unit disk. Let us remark that as for interpolation on $\mathbb{T}$, L-polynomials play the same role as the usual polynomials when interpolation over an interval is considered. This is basically motivated by the fact that any continuous function on a Jordan curve C of the finite z-plane can be uniformly approximated on C by the sum of a polynomial in z and $\bar{z}$ (see [3]). In particular, if $C = \mathbb{T}$, then any continuous function on the unit circle can be uniformly approximated on $\mathbb{T}$ by L-polynomials. Finally, we will choose as interpolation nodes rotations of roots of unity. In the real case, the Chebyshev nodes defined above are the zeros of the so-called Chebyshev polynomials of the first kind $T_n(x) = \cos(n \arccos x)$, which are orthogonal on $[-1, 1]$ with respect to the positive measure $d\mu(x) = dx/\sqrt{1-x^2}$. In order to obtain nodes on $\mathbb{T}$ in a similar way, we could proceed as follows. Let $d\omega(\theta)$ be a positive measure on $[0, 2\pi]$ and consider the following Hermitian product over Π:

$$(f, g)_\omega = \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)} \, d\omega(\theta).$$

By applying the Gram-Schmidt orthogonalization process to $\{1, x, \ldots, x^n\}$ an orthogonal basis $\{\rho_k\}_0^n$ of monic polynomials can be deduced. The sequence $\{\rho_k\}_0^n$ represents the system of monic orthogonal polynomials (or Szegö polynomials) with respect to $d\omega(\theta)$. It is well known that for each n, the zeros of ρ_n lie inside $\mathbb{D}$ (see, e.g., [4, p. 184]). So they cannot be taken as interpolation nodes. In order to overcome this drawback, Jones *et al.* in [5] introduced the so-called *para-orthogonal* polynomials given by

$$B_n(z, \lambda) = \rho_n(z) + \lambda \rho_n^*(z),$$

λ being an arbitrary unimodular complex number and $\rho_n^*(z) = z^n \rho_n(1/z)$. It can be shown that B_n has exactly n distinct zeros on $\mathbb{T}$ (see [5]).

If μ is a probability measure on the interval $[-1, 1]$, then we can define a measure ω on $[0, 2\pi]$ by $d\omega(\theta) = \mu(\cos\theta)|\sin\theta|\, d\theta/2$. If $d\mu(x) = (1/\pi\sqrt{1-x^2})\, dx$, then the corresponding measure on $[0, 2\pi]$ is the *Lebesgue* measure which is given by $d\omega(\theta) = (1/2\pi)\, d\theta$. The Szegö polynomials with respect to this measure are $\rho_n(z) = z^n$ and the para-orthogonal polynomials in this case are $B_n(z, \lambda) = z^n + \lambda$, $|\lambda| = 1$ $\forall n$. Thus, we see that the zeros of the corresponding para-orthogonal polynomials are rotations of unity.

2. PRELIMINARY RESULTS

Let f be a differentiable function on $\mathbb{T}$ and $z_k \in \mathbb{T}$, $k = 1, \dots n$ with $z_k \neq z_j$, for $k \neq j$. Then we know that there exists a unique polynomial $P \in \Pi_{2n-1}$ such that

$$\begin{aligned} P(z_k) &= f(z_k), \\ P'(z_k) &= f'(z_k), \qquad k = 1, \dots n, \end{aligned}$$

and we can write $P(z)$ as (Hermite's interpolation formula [6, pp. 52–53])

$$P(z) = \sum_{k=1}^{n} A_k(z) f(z_k) + \sum_{k=1}^{n} B_k(z) f'(z_k), \tag{2.1}$$

where

$$A_k(z) = \left(1 - 2(z - z_k) l_k'(z_k)\right) l_k^2(z), \qquad k = 1, \dots n, \tag{2.2}$$

and

$$B_k(z) = (z - z_k)\, l_k^2(z), \qquad k = 1, \dots n, \tag{2.3}$$

l_k being the fundamental Lagrange polynomials given by $l_k(z) = W_n(z)/W_n'(z_k)(z - z_k)$, $k = 1, \dots n$ with $W_n(z) = \prod_{k=1}^{n}(z - z_k)$.

Let p and q be two nondecreasing sequences of nonnegative integers such that

$$p + q = 2n - 1, \qquad n = 1, 2, \dots. \tag{2.4}$$

Then, we have the following.

PROPOSITION 2.1. *Let f be a differentiable function on $\mathbb{T}$. Then there exists a unique $L \in \Lambda_{-p,q}$, with p and q satisfying (2.4), such that*

$$\begin{aligned} L(z_k) &= f(z_k), \\ L'(z_k) &= f'(z_k), \qquad k = 1, \dots n, \end{aligned} \tag{2.5}$$

where $\{z_k\}_{k=1}^{n}$ are the roots of $z^n + \lambda = 0$. The L-polynomial can be written in an explicit formula as

$$L(z) = \sum_{k=1}^{n} A_k^*(z) f(z_k) + \sum_{k=1}^{n} B_k^*(z) f'(z_k) \tag{2.6}$$

with

$$A_k^*(z) = \frac{z_k^{p+2}\left(z^n + \lambda\right)^2}{z^p n^2 \lambda^2 (z - z_k)^2} + \frac{(p - n + 1) z_k^{p+1}\left(z^n + \lambda\right)^2}{z^p n^2 \lambda^2 (z - z_k)} \tag{2.7}$$

and

$$B_k^*(z) = \frac{z_k^{p+2}\left(z^n + \lambda\right)^2}{z^p n^2 \lambda^2 (z - z_k)}. \tag{2.8}$$

PROOF. The existence and unicity of the L-polynomial satisfying (2.5) is a consequence of the fact that $\Lambda_{-p,q}$ is a Chebyshev system on $\mathbb{T}$ (see [2, p. 31]). Thus, let $L \in \Lambda_{-p,q}$ be such this solution. We can write $L(z) = P(z)/z^p$ where $P \in \Pi_q$ and

$$L'(z) = \frac{P'(z) z^p - P(z) p z^{p-1}}{z^{2p}}.$$

Since $L(z_k) = f(z_k)$, $k = 1, \dots n$, then

$$P(z_k) = z_k^p f(z_k).$$

On the other hand, since $L'(z_k) = f'(z_k)$, $k = 1, \dots n$, then

$$P'(z_k) = pz_k^{p-1}f(z_k) + z_k^p f'(z_k).$$

Let $g(z) = z^p f(z)$, then g is differentiable on $\mathbb{T}$ and $g'(z) = pz^{p-1}f(z) + z^p f'(z)$. Therefore, one has $P \in \Pi_q \subset \Pi_{2n-1}$, satisfying

$$\begin{aligned} P(z_k) &= g(z_k), \\ P'(z_k) &= g'(z_k), \qquad k = 1, \dots n. \end{aligned} \tag{2.9}$$

By (2.1), we have $P(z) = \sum_{k=1}^{n} A_k(z)g(z_k) + \sum_{k=1}^{n} B_k(z)g'(z_k)$, where $A_k(z)$ and $B_k(z)$ are given as in formulas (2.2) and (2.3), respectively. In this case, since $W_n(z) = z^n + \lambda$,

$$l_k(z) = -\frac{z_k(z^n + \lambda)}{n\lambda(z - z_k)}$$

and

$$l'_k(z) = -\frac{z_k}{n\lambda}\left(\frac{(n-1)z^n - nz_k z^{n-1} - \lambda}{(z - z_k)^2}\right).$$

Therefore, by applying the L'Hopital rule two times, we have

$$\begin{aligned} l'_k(z_k) &= -\frac{z_k}{n\lambda}\lim_{z\to z_k}\left(\frac{n(n-1)z^{n-1} - n(n-1)z_k z^{n-2}}{2(z - z_k)}\right) \\ &= -\frac{z_k}{2n\lambda}\lim_{z\to z_k}\left(n(n-1)^2 z^{n-2} - n(n-1)(n-2)z_k z^{n-3}\right) \\ &= -\frac{z_k}{2n\lambda}\left(n(n-1)^2 z_k^{n-2} - n(n-1)(n-2)z_k^{n-2}\right) \\ &= -\frac{z_k}{2n\lambda}\left(n(n-1)z_k^{n-2}\right) \\ &= -\frac{(n-1)z_k^{n-1}}{2\lambda} = \frac{(n-1)\lambda}{2z_k\lambda} = \frac{n-1}{2z_k}, \end{aligned}$$

and one has

$$\begin{aligned} A_k(z) &= \frac{z_k^2(z^n + \lambda)^2}{n^2\lambda^2(z - z_k)^2} - \frac{(n-1)z_k(z^n + \lambda)^2}{n^2\lambda^2(z - z_k)}, \\ B_k(z) &= \frac{z_k^2(z^n + \lambda)^2}{n^2\lambda^2(z - z_k)}. \end{aligned} \tag{2.10}$$

Thus,

$$\begin{aligned} L(z) = \frac{P(z)}{z^p} &= \sum_{k=1}^{n}\frac{A_k(z)z_k^p}{z^p}f(z_k) + \sum_{k=1}^{n}\frac{B_k(z)}{z^p}\left(pz_k^{p-1}f(z_k) + z_k^p f'(z_k)\right) \\ &= \sum_{k=1}^{n}\frac{A_k(z)z_k^p + B_k(z)pz_k^{p-1}}{z^p}f(z_k) + \sum_{k=1}^{n}\frac{B_k(z)z_k^p}{z^p}f'(z_k) \\ &= \sum_{k=1}^{n}A_k^*(z)f(z_k) + \sum_{k=1}^{n}B_k^*(z)f'(z_k), \end{aligned}$$

where

$$\begin{aligned} A_k^*(z) &= \left(\frac{z_k^{p+2}(z^n+\lambda)^2}{n^2\lambda^2(z - z_k)^2} - \frac{(n-1)z_k^{p+1}(z^n+\lambda)^2}{n^2\lambda^2(z - z_k)} + \frac{pz_k^{p+1}(z^n+\lambda)^2}{n^2\lambda^2(z - z_k)}\right)\Big/ z^p \\ &= \frac{z_k^{p+2}(z^n+\lambda)^2}{z^p n^2\lambda^2(z - z_k)^2} + \frac{(p-n+1)z_k^{p+1}(z^n+\lambda)^2}{z^p n^2\lambda^2(z - z_k)}, \\ B_k^*(z) &= \frac{z_k^{p+2}(z^n+\lambda)^2}{z^p n^2\lambda^2(z - z_k)}, \end{aligned}$$

and the proof follows.

LEMMA 2.2. *Let $W_n(z) = z^n + \lambda$ and let z_k, $k = 1, \ldots, n$ be the n roots of $W_n(z) = 0$. Then,*

$$\frac{|W_n(z)|^2}{n^2} \sum_{k=1}^{n} \frac{1}{|z - z_k|^2} = 1, \qquad \forall\, z \in \mathbb{T}.$$

PROOF. For all $z \in \mathbb{T}$,

$$|z - z_k|^2 = (z - z_k)\overline{(z - z_k)} = -\frac{(z - z_k)^2}{z z_k}.$$

So,

$$\sum_{k=1}^{n} \frac{1}{|z - z_k|^2} = -z \sum_{k=1}^{n} \frac{z_k}{(z - z_k)^2} = z \sum_{k=1}^{n} \frac{(z - z_k) - z}{(z - z_k)^2}$$
$$= z \left(\sum_{k=1}^{n} \frac{1}{(z - z_k)} - z \sum_{k=1}^{n} \frac{1}{(z - z_k)^2} \right).$$

On the other hand,

$$\sum_{k=1}^{n} \frac{1}{(z - z_k)} = \frac{W_n'(z)}{W_n(z)} = \frac{n z^{n-1}}{z^n + \lambda}$$

and

$$-\sum_{k=1}^{n} \frac{1}{(z - z_k)^2} = n \frac{(n-1) z^{n-2} (z^n + \lambda) - n z^{2(n-1)}}{(z^n + \lambda)^2}$$
$$= -\frac{n z^{n-2} (z^n - (n-1)\lambda)}{(z^n + \lambda)^2}.$$

Thus, one obtains

$$\begin{aligned} 0 < \sum_{k=1}^{n} \frac{1}{|z - z_k|^2} &= z \left(\frac{n z^{n-1}}{z^n + \lambda} - z \frac{n z^{n-2} (z^n - (n-1)\lambda)}{(z^n + \lambda)^2} \right) \\ &= -\frac{\lambda n^2 z^n}{(z^n + \lambda)^2} = \frac{n^2}{|z^n + \lambda|^2} = \frac{n^2}{|W_n(z)|^2} \end{aligned} \tag{2.11}$$

and the proof follows.

3. MAIN RESULT

Throughout this section, we will assume that $p(n)$ and $q(n)$ are two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = 2n - 1$, $n = 1, 2, \ldots$ with $\lim_{n \to \infty} p(n) = \lim_{n \to \infty} q(n) = \infty$ and $|p(n) - n + 1|$ bounded. Then we have the following.

THEOREM 3.1. *Let f be a continuous function on $\mathbb{T}$, i.e., $f \in C(\mathbb{T})$ and let L_n be the unique L-polynomial in $\Lambda_{-p(n),q(n)}$ satisfying*

$$\begin{aligned} L_n(z_k) &= f(z_k), \\ L_n'(z_k) &= 0, \qquad k = 1, \ldots n, \end{aligned} \tag{3.1}$$

where $\{z_k\}_{k=1}^{n}$ are the roots of $z^n + \lambda_n = 0$, $\{\lambda_n\}$ being an arbitrary sequence on $\mathbb{T}$. Then the sequence L_n converges to f uniformly on $\mathbb{T}$.

PROOF. By Proposition 2.1, we can write

$$L_n(z) = \sum_{k=1}^{n} A_k^*(z) f(z_k),$$

with A_k^* as in formula (2.7). By unicity, we have that $\sum_{k=1}^n A_k^*(z) = 1$, and therefore,

$$\sum_{k=1}^n A_k^*(z) f(z) = f(z).$$

Thus,

$$\begin{aligned}\left| f(z) - \sum_{k=1}^n A_k^*(z) f(z_k) \right| &= \left| \sum_{k=1}^n A_k^*(z) f(z) - \sum_{k=1}^n A_k^*(z) f(z_k) \right| \\ &\leq \sum_{k=1}^n |A_k^*(z)| \, |f(z) - f(z_k)|.\end{aligned}$$

Since $f \in C(\mathbb{T})$, because of the uniform continuity, for a given arbitrary positive number ϵ, there exists $\delta > 0$ such that if $|z - z_k| < \delta$, then $|f(z) - f(z_k)| < \epsilon$. So

$$\begin{aligned}\left| f(z) - \sum_{k=1}^n A_k^*(z) f(z_k) \right| &\leq \epsilon \sum_{k=1, |z-z_k|<\delta}^n |A_k^*(z)| + \sum_{k=1, |z-z_k|\geq\delta}^n |A_k^*(z)| \, |f(z) - f(z_k)| \\ &= I_{1,n} + I_{2,n}.\end{aligned} \tag{3.2}$$

If $|z - z_k| \geq \delta$, then by formula (2.7) and since z, $z_k \in \mathbb{T}$, $k = 1, \dots, n$,

$$|A_k^*(z)| \leq \frac{|z^n + \lambda_n|^2}{n^2 \delta^2} + \frac{|p(n) - n + 1| \, |z^n + \lambda_n|^2}{n^2 \delta}.$$

Now, $f \in C(\mathbb{T})$ and $\mathbb{T}$ is a compact set, hence, there exists $M > 0$ such that $|f(z)| \leq M$, $\forall\, z \in \mathbb{T}$. So,

$$|f(z) - f(z_k)| \leq 2M, \qquad \forall\, z \in \mathbb{T}, \quad \forall\, k = 1, \dots, n,$$

and since it holds on $\mathbb{T}$, $|z^n + \lambda_n|^2 \leq 4$, we obtain

$$I_{2,n} \leq \frac{8M}{n\delta^2} + \frac{8M}{\delta} \frac{|p(n) - n + 1|}{n}.$$

Since $|p(n) - n + 1|$ is bounded, $I_{2,n} < \epsilon$ for sufficiently large n.

On the other hand,

$$\begin{aligned}I_{1,n} &= \epsilon \sum_{k=1 |z-z_k|<\delta}^n |A_k^*(z)| \\ &= \epsilon \left(\frac{|z^n + \lambda_n|^2}{n^2} \sum_{k=1}^n \frac{1}{|z - z_k|^2} \right) + \epsilon(|p(n) - n + 1|) \frac{|z^n + \lambda_n|^2}{n^2} \sum_{k=1}^n \frac{|z - z_k|}{|z - z_k|^2} \\ &\leq \epsilon \left(\frac{|z^n + \lambda_n|^2}{n^2} \sum_{k=1}^n \frac{1}{|z - z_k|^2} \right) + \epsilon\delta(|p(n) - n + 1|) \left(\frac{|z^n + \lambda_n|^2}{n^2} \sum_{k=1}^n \frac{1}{|z - z_k|^2} \right).\end{aligned}$$

By Lemma 2.2,

$$I_{1,n} < \epsilon + \epsilon\delta|p(n) - n + 1| = (1 + \delta|p(n) - n + 1|)\epsilon.$$

Again, since $|p(n) - n + 1|$ is bounded, $I_{1,n} < \epsilon$ and the proof follows.

EXAMPLE. If we choose $p(n) = n - 1$, then $L \in \Lambda_{-(n-1),n}$ and

$$L_n(z) = \sum_{k=1}^n A_k^*(z) f(z_k)$$

where now $A_k^*(z)$ takes the simple form

$$A_k^*(z) = \frac{z_k^{n+1} (z^n + \lambda_n)^2}{z^{n-1} n^2 \lambda_n^2 (z - z_k)^2} = -\frac{z_k (z^n + \lambda_n)^2}{z^{n-1} n^2 \lambda_n (z - z_k)^2}.$$

4. CONCLUSION

Numerical estimation of integrals on the unit circle of the form $\int_0^{2\pi} f(e^{i\theta})\, d\omega(\theta)$, $\omega(\theta)$ being a positive measure on $[0, 2\pi)$, has become an important topic because of its connection with the rapidly growing field of digital processing (see [5] and reference therein found) and giving rise to the so-called Szegö quadrature formulas [5]. In these formulas, the integrand $f(e^{i\theta})$ is replaced by a certain interpolating Laurent polynomial. Thus, as in the real case, studying the uniform convergence of sequences of interpolating Laurent polynomials reveals as an interesting problem.

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