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Intersections of circuits and cocircuits in binary matroids

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Abstract

Oxley has shown that if, for some $k \geqslant 4$, a matroid M has a k-element set that is the intersection of a circuit and a cocircuit, then M has a 4-element set that is the intersection of a circuit and a cocircuit. We prove that, under the above hypothesis, for $k \geqslant 6$, a binary matroid will also have a 6-element set that is the intersection of a circuit and a cocircuit. In addition, we determine explicitly the regular matroids which do not have a 6-element set that is the intersection of a circuit and cocircuit. Finally, we prove that in the case of graphs, if for some $k \geqslant 4$, a circuit and a cocircuit intersect in k elements, then there must be a circuit and a cocircuit that intersect in $(k-2)$ elements.

1. Introduction

Several matroid results are concerned with the cardinalities of the intersections of circuits and cocircuits. For example, it is well known that a circuit and a cocircuit in a matroid cannot have exactly one common element; every pair of elements in a connected matroid is the intersection of a circuit and a cocircuit; and a matroid is binary if and only if every set which is the intersection of a circuit and a cocircuit has even cardinality. In fact, Seymour [6,8] proved that a matroid is binary if and only if it does not have a triad, that is, a 3-element set which is the intersection of a circuit and a cocircuit, and that if a connected matroid has a triad then every pair of elements is in a triad. Note that, triad is now commonly used to denote a 3-element cocircuit. Oxley [4] proved that a connected matroid, having at least three elements, is a series–parallel network if and only if it does not have a quad, that is, a 4-element set which is the intersection of a circuit and a cocircuit, and that if a 3-connected matroid has a quad then every pair of elements is in a quad. Furthermore, he showed that if, for some $k \geqslant 4$, a matroid M has a k-element set which is the intersection of a circuit and a cocircuit, then M has a quad. In this paper, we will investigate further sets which are the intersection of a circuit and a cocircuit. In particular, we will concentrate

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on such sets having 6 elements. In Section 2 we will determine precisely when a binary matroid has a 6-element set that is the intersection of a circuit and a cocircuit. The main result in this section shows that, if for some $k \geqslant 6$, a binary matroid M has a k-element set that is the intersection of a circuit and a cocircuit, then M has a 6-element set that is the intersection of a circuit and a cocircuit. This result lends credence to the following conjecture by Oxley [3, 14.8.3]: if for some $k \geqslant 4$, a binary matroid M has a k-element set that is the intersection of a circuit and a cocircuit, then M has a $(k-2)$-element set that is the intersection of a circuit and a cocircuit. We also give an example to show that there is a 3-connected binary (in fact, graphic) matroid that has a 6-element set that is the intersection of a circuit and a cocircuit, yet not every pair of elements is in such a set. In Section 3, we determine explicitly the regular matroids which do not have a 6-element set that is the intersection of a circuit and cocircuit. Finally, in Section 4 Oxley's conjecture is proved for graphs.

The matroid terminology will, in general, follow Oxley [3]. For convenience, a k-element set that is the intersection of a circuit and a cocircuit is called a *special k-set*. The *ground set*, *rank*, and *corank* of the matroid M are denoted by $E(M), r(M)$, and $r^*(M)$, respectively. If $T \subseteq E(M)$, then the *deletion* and *contraction* of T from M are denoted as $M \backslash T$ and M/T, respectively. The *dual* of a matroid will be denoted by M^*. The *fundamental circuit* of the element e with respect to the basis B is denoted by $C(e,B)$. A matroid is *binary* if it can be represented by a matrix over the field of two elements. A standard form for a matrix representing a matroid is $[I_r \mid D]$, where I_r is the $r \times r$ identity matrix. The column vector corresponding to the ith column is denoted by $\bar{i}$.

We will assume familiarity with the concepts of matroid connection and with the operations of series and parallel connection, direct sums, 2-sums, and 3-sums. For matroids M_1 and M_2 such that $E(M_1) \cap E(M_2) = \{p\}$, we denote the parallel connection of M_1 and M_2 with respect to the basepoint p as $P(M_1, M_2)$. The following fundamental link between 3-connection and parallel connection was proved by Seymour [7].

Theorem 1.1. *A connected matroid M is not 3-connected if and only if there are matroids M_1 and M_2 each having at least three elements such that $M = P(M_1, M_2) \backslash p$, where p is not a loop or a coloop of M_1 or M_2.*

When M decomposes as in this theorem, we call M the *2-sum* of M_1 and M_2 and denote it as $M_1 \oplus_2 M_2$. If $\{x, y\}$ is a circuit of the matroid M, we say that x and y are *in parallel* in M. If instead $\{x, y\}$ is a cocircuit of M, then x and y are *in series* in M. A *parallel class* of M is a maximal subset A of $E(M)$ such that if a and b are distinct elements of A, then a and b are in parallel. *Series classes* are defined analogously. The matroid N is a *series extension* of M if $M = N/T$ and every element of T is in series with some element of M not in T. *Parallel extensions* are defined analogously. We call N a *series–parallel extension* of M if N can be obtained from M by a sequence of operations each of which is either a series or parallel extension.

A series–parallel extension of a single-element matroid is called a *series–parallel network*.

A detailed explanation of the next result and notation may be found in Seymour [7]. The matroid R_{10} is the unique splitter for the class of regular matroids. Denote by $K_5\backslash e$ the graph that is obtained from the complete graph on five vertices K_5 by deleting an edge. The matroid R_{12} is the 3-sum of $M(K_5\backslash e)$ and $M^*(K_{3,3})$, where the distinguished triangle in $K_5\backslash e$ is the one that is vertex-disjoint from e.

Theorem 1.2. *Let M be a 3-connected regular matroid. Then either $M \cong R_{10}$, or M is graphic or cographic or has a minor isomorphic to R_{12}.*

The following results on the intersection of circuits and cocircuits may be found in Oxley [4].

Proposition 1.3. *Let N be a minor of a matroid M, and suppose that X is the intersection of a circuit and a cocircuit in N. Then X is the intersection of a circuit and a cocircuit in M.*

Proposition 1.4. *Let M be a matroid containing a k-element set X which is the intersection of a circuit and a cocircuit. Then M has a minor N in which X is both a circuit and a cocircuit and $r(N)=r^*(N)=k-1$.*

Proposition 1.5. *Let M have a special k-set X. Then for some $t \in \{\lceil k/2 \rceil, \lceil k/2 \rceil + 1, \ldots, k-1\}$, M has a special t-set.*

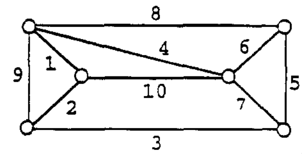
We will make frequent use of the following result on binary matroids [3], which describes the behaviour of circuits. The *symmetric difference*, $A \triangle B$, of two sets A and B equals $(A-B) \cup (B-A)$.

Proposition 1.6. *If C_1 and C_2 are circuits in a binary matroid, then $C_1 \triangle C_2$ is a disjoint union of circuits.*

Finally, in Section 3 we will use the following result (see [2]) which is a generalization of a result by Dirac [1]. W_r is the wheel with r-spokes. The graph $K_{3,p}$ is the complete bipartite graph with three vertices in one class and p vertices in the other class. The graphs $K'_{3,p}, K''_{3,p}$, and $K'''_{3,p}$ are the simple graphs obtained from $K_{3,p}$ by adding one, two, and three edges, respectively, joining vertices in the class containing three vertices.

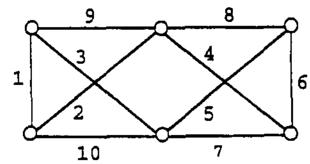
Theorem 1.7. *M is a 3-connected regular matroid with no minor isomorphic to $M^*(K_5\backslash e)$ if and only if M is isomorphic to $M(K_5), M(K_5\backslash e), M^*(K_{3,3}), M(W_r)$ for some $r \geqslant 3$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$, or $M(K'''_{3,p})$ for some $p \geqslant 3$, or R_{10}.*

$$\begin{array}{cc} \begin{matrix}1&2&3&4&5\end{matrix} & \begin{matrix}6&7&8&9&10\end{matrix} \\ \multicolumn{2}{c}{\left(\begin{array}{c|ccccc} & 1&1&1&1&1 \\ & 1&1&1&1&0 \\ I_5 & 1&1&1&0&0 \\ & 1&1&0&0&1 \\ & 1&0&1&0&0 \end{array}\right)} \end{array}$$



$M(G_{10})$

$$\begin{array}{cc} \begin{matrix}1&2&3&4&5\end{matrix} & \begin{matrix}6&7&8&9&10\end{matrix} \\ \multicolumn{2}{c}{\left(\begin{array}{c|ccccc} & 1&1&1&1&1 \\ & 1&1&1&1&0 \\ I_5 & 1&1&1&0&1 \\ & 1&1&0&0&0 \\ & 1&0&1&0&0 \end{array}\right)} \end{array}$$



$M(K_4) \oplus_2 M(K_4)$

$$\begin{array}{cc} \begin{matrix}1&2&3&4&5\end{matrix} & \begin{matrix}6&7&8&9&10\end{matrix} \\ \multicolumn{2}{c}{\left(\begin{array}{c|ccccc} & 1&1&1&1&1 \\ & 1&1&1&1&0 \\ I_5 & 1&1&1&0&1 \\ & 1&1&0&1&0 \\ & 1&0&1&1&0 \end{array}\right)} \end{array}$$

M_1

$$\begin{array}{cc} \begin{matrix}1&2&3&4&5\end{matrix} & \begin{matrix}6&7&8&9&10\end{matrix} \\ \multicolumn{2}{c}{\left(\begin{array}{c|ccccc} & 1&0&1&1&1 \\ & 1&1&0&1&1 \\ I_5 & 1&1&1&0&1 \\ & 1&1&1&1&0 \\ & 1&1&1&1&1 \end{array}\right)} \end{array}$$

M_2

Fig. 1.

2. The binary matroids with special 6-sets

Oxley [4] proved that a binary matroid has a special 4-set if and only if it has an $M(K_4)$-minor. We will first determine precisely when a binary matroid has a special 6-set. The main result of this section states that if, for some $k \geqslant 6$, a binary matroid M has a special k-set, then M has a special 6-set. Fig. 1 gives the binary matrices representing each of $M(G_{10}), M(K_4) \oplus_2 M(K_4)$, M_1, and M_2, together with the graphs G_{10} and $K_4 \oplus_2 K_4$. Observe that G_{10} is the prism graph $(K_5 \backslash e)^*$ with an edge added. Up to isomorphism there is exactly one such simple graph. The graph $K_4 \oplus_2 K_4$ is the 2-sum of K_4 with itself. The matroids M_1 and M_2 are rank-5, binary matroids. Observe that each of $M_1/\{1,2\}\backslash\{10\}$ and $M_2/\{4,5\}\backslash\{10\}$ is isomorphic to the Fano matroid. Therefore, M_1 and M_2 are non regular matroids.

Proposition 2.1. *A binary matroid M has a special 6-set if and only if M has a minor isomorphic to $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, M_1, or M_2.*

Proof. Observe that each of the matroids $M(G_{10})$, $M(K_4) \oplus_2 M(K_4)$, M_1, and M_2 has a special 6-set, namely, $\{1,2,3,4,5,6\}$. Therefore, if M has a minor isomorphic to one of the above matroids, then Proposition 1.3 implies that M must have a special 6-set. Conversely, let M be a binary matroid with a special 6-set X. Then Proposition 1.4

$$\begin{array}{cc} & \begin{array}{ccccc|ccccc} 1&2&3&4&5&6&7&8&9&10\end{array} \\ A = & \left(\begin{array}{ccccc|ccccc} &&&&&1&&&& \\ &&&&&1&&&& \\ &&I_5&&&1&&&& \\ &&&&&1&&&& \\ &&&&&1&&&& \end{array}\right)\end{array}$$

Fig. 2.

implies that M has a minor N in which X is a spanning circuit and a cospanning cocircuit, and such that $r(N)=r^*(N)=5$. Therefore, N is a 10-element, rank-5, simple, cosimple matroid. A partial binary matrix representation A, for N, of the form $[I_5 \mid D]$ is shown in Fig. 2. Without loss of generality, we may assume that $X=\{1,2,3,4,5,6\}$, so the first five columns form a basis. Then 6 must be a column of ones. Since X is a cospanning set, the set of column vectors $\{\bar{7},\bar{8},\bar{9},\overline{10}\}$ is independent. Moreover, since each $\bar{i}$ in this set is the incidence vector of the set $C(i,\{1,2,3,4,5\})-i$, and $C(i,\{1,2,3,4,5\})$ must meet the cocircuit X in a set of even cardinality, $\bar{i}$ must have exactly two ones or exactly four ones.

We will first assume that each of the column vectors $\bar{7},\bar{8},\bar{9},\overline{10}$ has four ones. Then $N\cong M_2$. Next, suppose that exactly three column vectors in $\{\bar{7},\bar{8},\bar{9},\overline{10}\}$ have four ones, say $\bar{7}=(11110)^{\mathrm{T}}$, $\bar{8}=(11101)^{\mathrm{T}}$, and $\bar{9}=(11011)^{\mathrm{T}}$. If the first two entries of $\overline{10}$ are both one or both zero, then the first two rows of A would be identical, so $\overline{10}$ may be $(10100)^{\mathrm{T}},(10010)^{\mathrm{T}}$, or $(10001)^{\mathrm{T}}$. If $\overline{10}=(10100)^{\mathrm{T}}$ then $N=M_1$. If $\overline{10}=(10010)^{\mathrm{T}}$ then swapping the third and fourth rows in the matrix representing N gives the matrix representing M_1; hence $N\cong M_1$. Similarly, if $\overline{10}=(10001)^{\mathrm{T}}$, then $N\cong M_1$. Next, suppose that two column vectors in $\{\bar{7},\bar{8},\bar{9},\overline{10}\}$ have four ones, say $\bar{7}=(11110)^{\mathrm{T}}$ and $\bar{8}=(11101)^{\mathrm{T}}$, while $\bar{9}$ and $\overline{10}$ have two ones each. Based on the symmetry of the existing columns, and the fact that no two rows in D are identical, there are four choices for the pair $\{\bar{9},\overline{10}\}$, namely $\{(11000)^{\mathrm{T}},(10100)^{\mathrm{T}}\},\{(11000)^{\mathrm{T}},(10010)^{\mathrm{T}}\},\{(10010)^{\mathrm{T}},(01010)^{\mathrm{T}}\}$, and $\{(10010)^{\mathrm{T}},(01001)^{\mathrm{T}}\}$. In the first case, $N=M(K_4)\oplus_2 M(K_4)$ and in the second case $N=M(G_{10})$. In the third case, the linear transformation $(x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}}\mapsto(x_5+x_4,\ x_3+x_4,\ x_4,\ x_2+x_4,\ x_1+x_4)^{\mathrm{T}}$ maps the matrix representing N to the matrix representing M_1; hence $N\cong M_1$. In the fourth case, the linear transformation $(x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}}\mapsto(x_5+x_4,\ x_2+x_4,\ x_3+x_4,\ x_4,\ x_1+x_4)^{\mathrm{T}}$ maps the matrix representing N to the matrix representing M_1; hence $N\cong M(G_{10})$.

Next, suppose that one column vector in $\{\bar{7},\bar{8},\bar{9},\overline{10}\}$ has four ones, say $\bar{7}=(11110)^{\mathrm{T}}$. Then since M^* has no parallel elements, each row in D must have at least two ones. In particular, row 5 must have at least two ones. Pivoting on element $[a_{5,6}]$ gives a matrix in which at least two of the last four columns have four ones, and this case is already done. Finally, suppose that none of the column vectors in $\{\bar{7},\bar{8},\bar{9},\overline{10}\}$ has four ones. Then once again using an argument similar to the previous one we can get a matrix in which at least one of the last four columns has four ones and this case is already done. □

Corollary 2.2. *A regular matroid has a special* 6*-set if and only if it has a minor isomorphic to* $M(G_{10})$ *or* $M(K_4)\oplus_2 M(K_4)$.

Proof. The proof follows from the previous proposition and the fact that M_1 and M_2 are not regular. □

The next theorem is the main result of this section.

Theorem 2.3. *Let M be a binary matroid with a special k-set X, for some $k\geqslant 6$. Then M has a special* 6*-set.*

Proof. As a consequence of Proposition 1.5, it is sufficient to prove that, if M has a special 8-set then M also has a special 6-set. The theorem follows by induction. Proposition 1.4 implies that M has a minor N in which X is a circuit and a cocircuit and $r(N)=r^*(N)=7$. Therefore, N is a 14-element, rank-7 binary matroid. Consider a standard representation for N of the form $[I_7|D]$. We may assume that $X=\{1,2,\dots,8\}$, so 8 is a column of ones. Since X is cospanning, the set of columns $Y=\{9,10,\dots,14\}$ is independent. Moreover, as in the proof of Proposition 2.1, each column in Y may have 2, 4, or 6 ones. We shall show that, in all cases, N has a special 6-set. Let $i\in Y$. Suppose i has 6 ones, say, $\bar{i}=(1111110)^{\mathrm{T}}$. Then $\{1,2,3,4,5,6,i\}$ is a circuit whose intersection with the cocircuit X has 6 elements. Next, suppose i has 2 ones, say, $\bar{i}=(1100000)^{\mathrm{T}}$. Then $\{1,2,i\}$ is a 3-circuit in N. Since N is binary, Proposition 1.6 implies that $X\triangle C$ is a disjoint union of circuits. However, since $X\triangle C=(X-C)\cup i$, it is a circuit. The intersection of $X\triangle C$ with X has 6 elements. We may now assume that all the columns in Y have exactly four ones. Without loss of generality, assume that $\bar{i}=(1111000)^{\mathrm{T}}$. A pair of columns may have 1, 2, or 3 ones in common. If all the columns in $Y-i$ meet i in exactly 2 ones, then in the dual $\{1,2,3,4\}$ would be a circuit. This is a contradiction since $\{1,2,3,4\}$ is contained in X, which is a cocircuit. So there is a column j that has 1 or 3 ones in common with i. First suppose that $\bar{j}=(1000111)^{\mathrm{T}}$. Then $\{1,8,i,j\}$ is a circuit. Since N is binary, $X\triangle\{1,8,i,j\}$, which is $\{2,3,4,5,6,7,i,j\}$, is a disjoint union of circuits. Suppose there is a circuit C properly contained in $\{2,3,4,5,6,7,i,j\}$. Then i or $j\in C$, say $i\in C$. However, the only circuits in $X\cup i$ containing i are $\{1,2,3,4,i\}$ and $\{5,6,7,8,i\}$. So $i\notin C$ and similarly $j\notin C$. Therefore, C is properly contained in X, which is a contradiction. Hence $\{2,3,4,5,6,7,i,j\}$ is a circuit and its intersection with X has 6 elements. Finally, if j has 3 ones in common with i, we may assume that $\bar{j}=(1110100)^{\mathrm{T}}$. Then $\{4,5,i,j\}$ is a circuit and by an argument similar to the previous one, we will find a special 6-set. □

Finally, to see that there is a 3-connected binary (in fact, graphic) matroid that has a special 6-set, yet not every pair of elements is in a special 6-set, consider the 3-connected matroid $M(G_{10})$ shown in Fig. 1. It has only one special 6-set, namely $\{1,2,3,4,5,6\}$.

3. The regular matroids without special 6-sets

In this section the regular matroids without special 6-sets are determined.

Let $\mathscr{M} = \{M(W_r)$ for $r \geqslant 3$, $M(K_5)$, $M(K_5 \backslash e)$, $M(K_{3,p}), M(K'_{3,p})$, $M(K''_{3,p})$, $M(K'''_{3,p})$ for $p \geqslant 3$, $R_{10}\}$. Let $\mathscr{M}^*$ be the set containing the duals of the above matroids.

Theorem 3.1. *M is a connected regular matroid with no special 6-set if and only if M is a series–parallel network or a series–parallel extension of a matroid in $\mathscr{M}$ or $\mathscr{M}^*$.*

Proof. It is easy to see that a series–parallel network has no special 6-set. Next, observe that the matroids in $\mathscr{M}$ are 3-connected. Suppose M is a series–parallel extension of a matroid in $\mathscr{M}$. It is sufficient by Corollary 2.2, to show that M has no minor isomorphic to $M(G_{10})$ or $M(K_4) \oplus_2 M(K_4)$. Since $M(G_{10})$ has a minor isomorphic to $M^*(K_5 \backslash e)$-minor, it follows from Theorem 1.6 that M has no minor isomorphic to $M(G_{10})$. It follows from Theorem 1.1 that M has no minor isomorphic to $M(K_4) \oplus_2 M(K_4)$. Finally, since the class of regular matroids without special 6-sets is closed under duality, series–parallel extensions of the matroids in $\mathscr{M}^*$ have no special 6-sets.

Next, suppose M is a regular matroid with no special 6-set. We will first show that M is either a series–parallel network or a series–parallel extension of a 3-connected matroid. Suppose M is not a series–parallel network. If M itself is 3-connected, then there is nothing to show. Otherwise Theorem 1.1 implies that $M \cong M_1 \oplus_2 M_2$ where M_1 and M_2 are isomorphic to proper minors of M, and $E(M_1) \cap E(M_2) = \{p\}$. Suppose both M_1 and M_2 have 3-connected minors with at least four elements. Then by a result of Seymour [9] each of M_1 and M_2 has an $M(K_4)$-minor containing p. Therefore, M has a minor isomorphic to $M(K_4) \oplus_2 M(K_4)$. This is a contradiction since $M(K_4) \oplus_2 M(K_4)$ has a special 6-set. Therefore, one of M_1 or M_2 is a series–parallel network, say M_2, and M is a series–parallel extension of M_1. If M_1 is not 3-connected, we can repeat the above argument until we find a 3-connected minor N of M such that M is a series–parallel extension of N. Finally, it remains to show that N is in $\mathscr{M}$ or $\mathscr{M}^*$. Since N is regular, Theorem 1.2 implies that N is graphic, or cographic, or has a minor isomorphic to R_{10} or R_{12}. Since R_{12} has $M(G_{10})$ as a minor and the latter has a special 6-set, N cannot have an R_{12}-minor. Since R_{10} is a splitter for the class of regular matroids, if N has an R_{10}-minor, then $N \cong R_{10}$ and so N is in $\mathscr{M}$. Therefore, we may assume that N is graphic or cographic. We will first assume that N is a 3-connected graphic matroid without special 6-sets. Suppose, if possible, N has an $M^*(K_5 \backslash e)$-minor. The Splitter Theorem [7] implies that N has as a minor, a 3-connected single-element extension or coextension of $M^*(K_5 \backslash e)$. However, $M^*(K_5 \backslash e)$ has no 3-connected graphic single-element coextension, and $M(G_{10})$ is the only 3-connected graphic single-element extension of $M^*(K_5 \backslash e)$. So N must have an $M(G_{10})$-minor. This is a contradiction and therefore N has no $M^*(K_5 \backslash e)$-minor. Theorem 1.7 implies that N is a graphic matroid in $\mathscr{M}$. By duality, if N is a 3-connected cographic matroid without special 6-sets, then M is a cographic matroid in $\mathscr{M}^*$. $\square$

We know that series–parallel networks are the only matroids whose circuit and cocircuit intersections are of size at most two. Similarly, Theorems 2.3 and 3.1 imply that series–parallel networks and series–parallel extensions of the matroids in $\mathscr{M}$ and $\mathscr{M}^*$ are the only regular matroids whose circuit and cocircuit intersections are of size at most four.

4. Intersections of circuits and cocircuits in graphs

In this section we will prove that in the case of graphic matroids, if for some $k \geqslant 4$, a circuit and a cocircuit intersect in k elements, then there must be a circuit and a cocircuit that intersect in $(k-2)$-elements. It is useful to note that a circuit in a graph is a cycle and a cocircuit is a minimal edge cut. If X is a subset of edges of the graph G, denote by $G-X$ the graph G with the edges in X deleted.

Lemma 4.1. *Let $M(G)$ be a graphic matroid with a special k-set X, for some $k \geqslant 4$. Then $M(G)$ has a connected minor $M(H)$ such that:*

(i) *X is both a circuit and a cocircuit in $M(H), r(M(H)) = r^*(M(H)) = k-1$, and $|E(M(H))| = 2(k-1)$.*

(ii) *$H-X$ has two connected components T_1 and T_2, each of which is a tree with $k/2$ vertices and $(k-2)/2$ edges, such that, every edge in X has one end-vertex in T_1 and the other in T_2.*

Proof. Part (i) follows from Proposition 1.4 and the fact that $|E(M(H))| = r(M(H)) + r^*(M(H))$. Next, since X is a spanning set, $M(H)$ is connected. We may assume that H has no isolated vertices. Since X is a cospanning set, $H-X$ is independent. Therefore, $H-X$ has no circuits, that is, it is a forest. Again, since X is a spanning circuit, H has exactly k vertices. The graph $H-X$ has k vertices and $(k-2)$ edges and therefore has 2 components, say T_1 and T_2. Since X is a cocircuit in $M(H)$, it is a minimal edge cut in H. Therefore, each edge in X has one end-vertex in T_1 and the other in T_2 and T_1 and T_2 are trees, each with $k/2$ vertices and $(k-2)/2$ edges. □

Theorem 4.2. *Let $M(G)$ be a graphic matroid with a special k-set X, for some $k \geqslant 4$. Then $M(G)$ has at least four special $(k-2)$-sets contained in X.*

Proof. The result holds for $k=4$, since every pair of elements in a connected graph is a special 2-set. Therefore, assume $k \geqslant 6$. Lemma 4.1 implies that $M(G)$ has a minor $M(H)$ with rank and corank equal to $k-1$, such that $H-X$ has two connected components T_1 and T_2, each of which is a tree. Each of T_1 and T_2 has at least two leaves, that is, vertices of degree 1. We will show that each leaf in T_1 and T_2 gives rise to a special $(k-2)$-set contained in X, and that different leaves yield different special $(k-2)$-sets. Let v be a leaf in T_1 or T_2, say T_1. The vertex v is incident with exactly two edges of X. Therefore, the degree of v in H is 3. Let C^* be the set of edges incident on v. Since $M(H)$ is connected, C^* is a cocircuit of size three. Let e be the edge of T_1 incident on v. Proposition 1.6 implies that the set $X \triangle C^*$ is a disjoint union of cocircuits. Since $X \triangle C^* = (X - C^*) \cup e$, it is a cocircuit. Observe that $X - C^*$ is the intersection of the cocircuit $(X - C^*) \cup e$ and the circuit X, and $|X - C^*| = k - 2$. Therefore, $X - C^*$ is a special $(k-2)$-set in $M(H)$, and hence in $M(G)$. It remains to show that different leaves yield different special $(k-2)$-sets. Let v_1 and v_2 be any two leaves of T_1. Let C_1^* and C_2^* be the sets of edges incident on v_1 and v_2, respectively. Then $C_1^* \cap C_2^*$ is nonempty only if v_1 and v_2 are adjacent in T_1. However, since $k \geqslant 6$, T_1 has at least three vertices, so the leaves v_1 and v_2 are nonadjacent. Therefore $C_1^* \cap C_2^*$ is empty and $X - C_1^* \neq X - C_2^*$. Finally, let v_1 and v_2 be leaves of T_1 and T_2, respectively. Then $|C_1^* \cap C_2^* \cap X| \leqslant 1$. Therefore, $C_1^* \cap X$ and $C_2^* \cap X$ are distinct, and again, $X - C_1^* \neq X - C_2^*$. $\square$

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