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On the Monotonicity of Saturation Orders of Saturated Matrix Methods

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In the definition of saturation, some authors require the saturation orders to be monotonically decreasing, while others do not. In this paper, we do not require monotonicity, and give an example of a saturated method having no monotonic saturation order. Further, we present a class of methods having monotonic saturation orders provided they are saturated. We begin by quoting some related results partially known.

1. INTRODUCTION

We consider the space C of real, continuous, and 2π-periodic functions on the line equipped with the norm $\|f\| := \sup_{x \in \mathbb{R}} |f(x)|$. Let the Fourier series of $f \in C$ be given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) =: \sum_{k=0}^{\infty} A_k(x).$$

For an infinite matrix $B = (b_{nk}) = (b_{nk})_{n,k \in \mathbb{N}}$ in series-to-sequence form, satisfying

$$\sum_{k=0}^{\infty} |b_{nk}| < \infty \qquad (n \in \mathbb{N} := \{0, 1, 2, ...\}), \tag{1.1}$$

we define

$$\sigma_n(f, x) := \sum_{k=0}^{\infty} b_{nk} A_k(x) \qquad (n \in \mathbb{N}, f \in C, x \in \mathbb{R}). \tag{1.2}$$

Condition (1.1) guarantees that the series in (1.2) converge absolutely and uniformly [2, p. 45]. Hence $\sigma_n: C \to C$.

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We notice that the following continues to hold when $L^p_{2\pi}$ $(1 \leqslant p \leqslant \infty)$ is substituted for C. Let T_m $(m \in \mathbb{N})$ denote the set of real trigonometric polynomials of degree $\leqslant m$, c_0 the set of real-valued nullsequences, and define $c_0^+ := \{(s_n) \in c_0 \mid \forall n \in \mathbb{N}: s_n > 0\}$.

If $\varphi = (\varphi_n) \in c_0^+$, we denote

$$F := F^B(\varphi) := \{f \in C \mid \|f - \sigma_n\| = O(\varphi_n) \text{ as } n \to \infty\},$$

and consider the statements

$$\text{(i)} \quad \forall f \in C: \left(\lim_{n \to \infty} \frac{\|f - \sigma_n\|}{\varphi_n} = 0 \Rightarrow f \in T_m \right),$$

$$\text{(ii)} \quad \forall f \in C: \left(\liminf_{n \to \infty} \frac{\|f - \sigma_n\|}{\varphi_n} = 0 \Rightarrow f \in T_m \right),$$

$$\text{(iii)} \quad F \backslash T_m \neq \varnothing.$$

Definition. Let $B = (b_{nk})$ be a matrix satisfying (1.1).

(a) B is called saturated relative to T_m, if there is a $\varphi \in c_0^+$ satisfying (i) and (iii).

(b) B is called u-saturated relative to T_m, if there is a $\varphi \in c_0^+$ satisfying (ii) and (iii).

In both (a) and (b), φ is called a saturation order and F the saturation class of B relative to T_m.

Obviously, every u-saturated matrix is saturated. We use the term "u-saturated," because the saturation order of a u-saturated matrix B is unique in the sense that any two saturation orders $\varphi = (\varphi_n)$, $\psi = (\psi_n)$ of B satisfy the conditions $\varphi_n = O(\psi_n)$ and $\psi_n = O(\varphi_n)$ as $n \to \infty$ (see [4, pp. 50 ff.]) while these conditions are not satisfied in general, if B is merely saturated (see [4, p. 82 and pp. 100–102] and [6]). We notice that (b) is essentially the definition used by Tureckiĭ [8] and, in case $m = 0$, by Devore [4].

In the second section of this paper, we give characterizations of saturated matrix methods and u-saturated matrix methods, respectively, with which we show in the third section that certain saturated matrix methods B (in particular those with monotonically decreasing rows $(b_{nk})_{k \in \mathbb{N}}$ and $b_{n0} = 1$) always have monotonically decreasing saturation orders. Further, we show that this is not true in general (see Example 4) and that there are u-saturated matrices with monotonically decreasing rows having no monotonically decreasing saturation order (see Example 3).

2. Characterization of Saturated and u-Saturated Matrix Methods

Our first result can be found in similar versions by several authors (see for example [7, 6, 8, 4, 2]; in [2] see especially Problem 12.1.4, p. 439). The proof follows the same line used by these authors, so we omit it.

In the sequel, we employ the definition $a/0 := \infty$ for $a \geq 0$.

Theorem 1. *Let* $B = (b_{nk})$ *satisfy* (1.1).

(a) *Let* $\varphi = (\varphi_n) \in c_0^+$ *be given.*

(α) φ *satisfies* (i), *if and only if*

$$\forall k \geq m+1 : \liminf_{n \to \infty} \frac{\varphi_n}{|1 - b_{nk}|} < \infty. \tag{2.1}$$

(β) φ *satisfies* (ii), *if and only if*

$$\forall k \geq m+1 : \liminf_{n \to \infty} \frac{|1 - b_{nk}|}{\varphi_n} > 0. \tag{2.2}$$

(γ) φ *satisfies* (iii), *if and only if*

$$\exists k_0 \geq m+1 : \liminf_{n \to \infty} \frac{\varphi_n}{|1 - b_{nk_0}|} > 0. \tag{2.3}$$

(b) *B is saturated (respectively u-saturated) relative to T_m, if and only if there exists a* $\varphi \in c_0^+$ *satisfying* (2.1) (*respectively* (2.2)) *and* (2.3).

As an immediate consequence of Theorem 1, we obtain the following corollary (see [6] too).

Corollary 1. *Let* $B = (b_{nk})$ *satisfy* (1.1), *let all the entries* b_{nk} *be real, and suppose that there is an integer N such that $n \geq N$ implies*

$$\forall k \geq m+1 : b_{nk} \leq b_{n,m+1} \leq 1.$$

Further, let $\varphi = (\varphi_n) \in c_0^+$ *be given. Then* (2.1) *is equivalent to*

$$\liminf_{n \to \infty} \frac{\varphi_n}{1 - b_{n,m+1}} < \infty,$$

condition (2.2) *is equivalent to*

$$\liminf_{n \to \infty} \frac{1 - b_{n,m+1}}{\varphi_n} > 0,$$

and (2.3) *is equivalent to*

$$\liminf_{n\to\infty} \frac{\varphi_n}{1-b_{n,m+1}} > 0. \tag{2.4}$$

Our next result is a characterization of saturated and u-saturated methods using only properties of the matrix B (compare [3, Theorem 3.1; 4, Theorem 3.1]). To state the result, we denote

$$c_0^i := \{(s_n) \in c_0 \mid s_n \neq 0 \text{ for infinitely many } n \in \mathbb{N}\},$$

$$c_0^a := \{(s_n) \in c_0 \mid s_n \neq 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

THEOREM 2. *Let* $B=(b_{nk})$ *satisfy* (1.1).

(a) B *is saturated relative to* T_m, *if and only if there exists an integer* $k_0 \geqslant m+1$ *such that*

$$(1-b_{nk_0})_{n\in\mathbb{N}} \in c_0^i, \tag{2.5}$$

$$\forall k \geqslant m+1: \liminf_{n\to\infty} \frac{|1-b_{nk_0}|}{|1-b_{nk}|} < \infty. \tag{2.6}$$

In the case when (2.5) *and* (2.6) *hold, a saturation order* $\varphi=(\varphi_n)$ *is given by*

$$\varphi_n := \begin{cases} |1-b_{nk_0}| & (b_{nk_0} \neq 1) \\ \min(\{|1-b_{nj}| \mid m+1 \leqslant j \leqslant n \text{ and } b_{nj} \neq 1\} \cup \{a_n\}) & (\textit{otherwise}) \end{cases}$$

in which (a_n) *is any sequence of* c_0^+.

(b) B *is u-saturated relative to* T_m, *if and only if there exists an integer* $k_0 \geqslant m+1$ *such that*

$$(1-b_{nk_0})_{n\in\mathbb{N}} \in c_0^a, \tag{2.7}$$

$$\forall k \geqslant m+1: \liminf_{n\to\infty} \frac{|1-b_{nk}|}{|1-b_{nk_0}|} > 0. \tag{2.8}$$

In the case that (2.7) *and* (2.8) *hold, a saturation order* $\varphi=(\varphi_n)$ *is given by*

$$\varphi_n := \begin{cases} |1-b_{nk_0}| & (b_{nk_0} \neq 1) \\ 1 & (\textit{otherwise}). \end{cases}$$

The proof of Theorem 2 follows from Theorem 1.

We note that, because of our definition $a/0 := \infty$ $(a \geqslant 0)$, (2.6) implies $1-b_{nk_0} \neq 0$ for infinitely many n so that (2.5) could be weakened to $(1-b_{nk_0}) \in c_0$ (but compare Corollary 2(a) in this connection).

Further, we point out that the condition $(1-b_{nk_0})\in c_0$ cannot be omitted in Theorem 2 as can be shown by the example $B=(b_{nk})$, $b_{n0}:=1$, $b_{nk}:=0$ $(n\in\mathbb{N}, k\geqslant 1)$. B satisfies (2.6) and (2.8) with $k_0:=m+1$, but, by Corollary 1, B is neither saturated nor u-saturated relative to T_m, since (2.4) is not satisfied for any $\varphi\in c_0^+$. Therefore, [3, Theorem 3.1] and [4, Theorem 3.1 and some other results of Chap. 3] have to be modified in a corresponding manner. Moreover, the matrix $B=(b_{nk})$ of Example 1 below shows that the condition "$1-b_{nk_0}\neq 0$ for all but finitely many $n\in\mathbb{N}$" in Theorem 2(b) cannot be omitted either, since B given by (2.11) satisfies (2.8) with $k_0:=1$ (we consider the case $m=0$) and $(1-b_{n1})\in c_0\backslash c_0^a$, but is not u-saturated relative to T_0 (we remark that the kernel of the matrix given by (2.11) is not positive).

Finally, we call attention to the fact that, by Theorem 2(a) and [10, Theorem 2], for every matrix B saturated relative to T_m such that $(1-b_{nk})_{n\in\mathbb{N}}\in c_0$ for every $k\in\mathbb{N}$, there exists a matrix having the same summability domain, but not saturated relative to T_m. (The summability domain of a matrix $B=(b_{nk})$ is the set

$$c_B:=\left\{u=(u_n)\;\middle|\;\sum_{k=0}^{\infty} b_{nk}u_k \text{ exists for every } n\in\mathbb{N} \text{ and } \left(\sum_{k=0}^{\infty} b_{nk}u_k\right)_{n\in\mathbb{N}} \text{ is convergent}\right\}.)$$

The next result can easily be obtained as a consequence of Corollary 1.

COROLLARY 2. *Let $B=(b_{nk})$ satisfy the assumptions of Corollary 1.*

(a) *B is saturated relative to T_m, if and only if*

$$(1-b_{n,m+1})\in c_0^i. \tag{2.9}$$

In the case that (2.9) *holds, a saturation order $\varphi=(\varphi_n)$ is given by*

$$\varphi_n:=\begin{cases}1-b_{n,m+1} & (b_{n,m+1}\neq 1)\\ a_n & (\text{otherwise}),\end{cases}$$

where (a_n) is any sequence of c_0^+.

(b) *B is u-saturated relative to T_m, if and only if*

$$(1-b_{n,m+1})\in c_0^a. \tag{2.10}$$

In the case that (2.10) *holds, a saturation order $\varphi=(\varphi_n)$ is given by*

$$\varphi_n:=\begin{cases}1-b_{n,m+1} & (b_{n,m+1}\neq 1)\\ 1 & (\text{otherwise}).\end{cases}$$

Now it is easy to construct examples of matrices which are (u-)saturated or not.

EXAMPLE 1. By Corollary 2 we obtain that the matrix $B=(b_{nk})$ given by

$$b_{nk} := \begin{cases} 1-\dfrac{k}{n+1} & (n \text{ even and } k \leqslant n) \\ 1-\dfrac{\max\{0, k-1\}}{n} & (n \text{ odd and } k \leqslant n) \\ 0 & (\text{otherwise}) \end{cases} \tag{2.11}$$

is saturated relative to T_m for every $m \in \mathbb{N}$ and u-saturated relative to T_m for every $m \geqslant 1$ (in both cases $((n+1)^{-1})$ is a saturation order), but not u-saturated relative to T_0 (see [9] too). This example shows that Problem 12.1.1 in [2] is false (see [1, p. 87 ff.] too).

EXAMPLE 2. Let $g\colon [0, 1) \to [0, \infty)$ be a function. We consider order summability $[g]$ introduced by Jurkat–Peyerimhoff [5]. Define for $i, j, k \in \mathbb{N}$, $i \leqslant j$,

$$b_{ji}^{[g]}(k) := \begin{cases} 1-\dfrac{k}{j+1}\dfrac{g(i/(j+1))}{1+g(i/(j+1))} & (k<i) \\ 1-\dfrac{k}{j+1}\dfrac{g(i/(j+1))}{1+g(i/(j+1))}-\dfrac{k-i}{j+1-i}\dfrac{1}{1+g(i/(j+1))} & (i \leqslant k \leqslant j) \\ 0 & (\text{otherwise}), \end{cases}$$

and arrange the pairs (j, i) in lexicographic order so that (j, i) is the nth pair where $n = j(j+1)/2 + i$. If we denote $b_{nk}^{[g]} := b_{ji}^{[g]}(k)$, then $B_g := (b_{nk}^{[g]})$ is the series-to-sequence form of the matrix A^* given in the proof of Theorem 5.1 in [5] and equivalent to $[g]$.

By Corollary 2, it follows that B_g is saturated relative to T_m, since $1-b_{j0}^{[g]}(m+1) = (m+1)/(j+1) \neq 0$ for all $j \geqslant m+1$. Moreover, B_g is u-saturated relative to T_m, if and only if there exists an integer $j_0 \geqslant m+1$ such that $g(i/(j+1)) \neq 0$ for all $j \geqslant j_0$ and all i with $m+1 \leqslant i \leqslant j$.

3. MONOTONICITY OF SATURATION ORDERS

If B is a matrix u-saturated relative to T_0, then Vértesi [9, proof of Theorem 1.3] has shown that there exists a monotonically decreasing

$\psi \in c_0^+$ such that B is saturated relative to T_0 with saturation order ψ. If B is only saturated, this does not remain valid in general (see Example 4 below). But we can prove the following result.

THEOREM 3. *Let $B=(b_{nk})$ satisfy the assumptions of Corollary 1. Further, let B be saturated relative to T_m, and define $\psi=(\psi_n)$ by*

$$\psi_n := \sup\{1-b_{\nu,m+1} \mid \nu \geqslant n\}.$$

Then B is saturated relative to T_m, and ψ is a monotonically decreasing saturation order.

Since $(1-b_{n,m+1}) \in c_0^i$ by Corollary 2(a), we can argue as in the proof of Theorem 1.3 in [9]: There exists a strictly increasing sequence of positive integers (n_i) such that $\psi_{n_i}=1-b_{n_i,m+1}$ for all $i \in \mathbb{N}$. Because $\psi_n \geqslant 1-b_{n,m+1}$ for all $n \in \mathbb{N}$, the conclusion follows by Corollary 1 and Theorem 1(b).

The next example shows that an analogous result to Theorem 3 is not true for u-saturated matrices in general.

EXAMPLE 3. We consider the matrix $B=(b_{nk})$ defined by

$$b_{nk} := \begin{cases} 1-\dfrac{k}{n+1} & (n \text{ even and } k \leqslant n) \\ 1-\left(\dfrac{k}{n+1}\right)^2 & (n \text{ odd and } k \leqslant n) \\ 0 & (\text{otherwise}). \end{cases}$$

In virtue of Corollary 2(b), the matrix B is u-saturated relative to T_m, and $(1-b_{n,m+1})$ is a saturation order. Suppose that there exists a monotonically decreasing $\psi=(\psi_n) \in c_0^+$ such that B is u-saturated relative to T_m with saturation order ψ. By Theorem 1(b) and Corollary 1, we have

$$\liminf_{j \to \infty} \left(\frac{1-b_{2j-1,m+1}}{\psi_{2j-1}} \frac{\psi_{2j}}{1-b_{2j,m+1}} \right) > 0$$

which contradicts

$$\frac{1-b_{2j-1,m+1}}{\psi_{2j-1}} \frac{\psi_{2j}}{1-b_{2j,m+1}} \leqslant \frac{1-b_{2j-1,m+1}}{1-b_{2j,m+1}} = (m+1)\frac{2j+1}{(2j)^2} \to 0$$

as $j \to \infty$ $(2j-1 \geqslant m+1)$.

Finally, we show that there exists a matrix saturated relative to T_m, but having no monotonically decreasing saturation order.

EXAMPLE 4. Consider any sequence $(\phi_n) \in c_0^+$, and put

$$b_{nk} := \begin{cases} 1-\phi_n & (n \in \{j(j+1)/2+k \mid j \in \mathbb{N}, j \geqslant k\}) \\ 0 & (n<k) \\ 1 & (\text{otherwise}). \end{cases}$$

By Theorem 2, the matrix $B=(b_{nk})$ is saturated relative to T_m, and (ϕ_n) is a saturation order (consider any $k_0 \geqslant m+1$, and choose $a_n := \phi_n$), but B is not u-saturated relative to any T_m.

Now we consider a special sequence (ϕ_n). Since, for every $n \in \mathbb{N}$, there exists a uniquely determined $j=j(n) \in \mathbb{N}$ and a uniquely determined $k=k(n)$ such that $k \leqslant j$ and $n=j(j+1)/2+k$, we define

$$\phi_n := \left(\frac{1}{n+1}\right)^{k+1}.$$

Suppose that there exists a monotonically decreasing $\psi=(\psi_n) \in c_0^+$ such that B is saturated relative to T_m with saturation order ψ. By Theorem 1, the sequence ψ satisfies (2.1) and (2.3). Choose $k_0 \geqslant m+1$ according to (2.3). Then it follows from (2.1) that

$$\liminf_{n \to \infty} \frac{\psi_n}{|1-b_{n,k_0+1}|} < \infty.$$

Hence we can choose a strictly increasing sequence (n_i) of positive integers satisfying $n_0 \geqslant k_0+1$, $b_{n_i,k_0+1} \neq 1$ for all $i \in \mathbb{N}$ and

$$\liminf_{i \to \infty} \frac{|1-b_{n_i,k_0+1}|}{\psi_{n_i}} > 0. \tag{3.1}$$

Because of our definition of B, every n_i can be written as

$$n_i = \tfrac{1}{2} j_i(j_i+1)+k_0+1 \qquad (j_i \geqslant k_0+1).$$

Since ψ is monotonically decreasing and since

$$n_i + j_i = \tfrac{1}{2}(j_i+1)(j_i+2)+k_0,$$

we get

$$\begin{aligned} &\frac{|1-b_{n_i,k_0+1}|}{\psi_{n_i}} \frac{\psi_{n_i+j_i}}{|1-b_{n_i+j_i,k_0}|} \\ &\leqslant \frac{|1-b_{n_i,k_0+1}|}{|1-b_{n_i+j_i,k_0}|} = \frac{(n_i+j_i+1)^{k_0+1}}{(n_i+1)^{k_0+2}} \\ &\leqslant \left(\frac{2n_i+1}{n_i+1}\right)^{k_0+1} (n_i+1)^{-1} \to 0 \qquad \text{as} \quad i \to \infty, \end{aligned}$$

which contradicts (3.1) and (2.3). Hence there exists no monotonically decreasing saturation order of B.

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