Nonlinear Elliptic Equations with Singular Boundary Conditions

Zhijun Zhang

Department of Mathematics, Northwest Normal University, Lanzhou, 730070, People’s Republic of China

Submitted by Howard A. Levine

Received March 29, 1996

We consider the model problem

$$\begin{align*}
\Delta u - k(u)(1 + |\nabla u|^q), & \quad u > 0 \text{ in } \Omega, \\
u(x) & \to +\infty \quad \text{as } d(x) = \text{dist}(x, \partial \Omega) \to 0,
\end{align*}$$

where $\Omega$ is a bounded region in $\mathbb{R}^N$ with smooth boundary, $q \in (0, 2)$, and $k(u) = u^p$ $(p > 1)$ or $e^u$. We show that there exists a classical solution.
through an analysis of the asymptotic behavior of solutions at the boundary.

In this note we present a new argument, which is more simple, for the existence of solutions of (1) with \( f(x, u, \nabla u) = e^u(1 + |\nabla u|^q) \) or \( f(x, u, \nabla u) = u^p(1 + |\nabla u|^q) \), where \( p > 1 \) and \( q \in [0, 2) \).

Our main results are as the following

**Theorem 1.** If \( f(x, u, \nabla u) = e^u(1 + |\nabla u|^q) \), where \( q \in [0, 2) \), then (1) has at least one solution \( u \in C^{2+\alpha}(\Omega) \) satisfying

\[
\limsup_{d(x) \to 0} \frac{u(x)}{|\ln d(x)|}
\]

is bounded.

**Theorem 2.** If \( f(x, u, \nabla u) = u^p(1 + |\nabla u|^q) \), \( p > 1 \), and \( q \in [0, 2) \), then (1) has at least one solution \( u \in C^{2+\alpha}(\Omega) \).

2. **Proofs**

Let \( w \) be the unique solution of

\[
\begin{align*}
-\Delta u &= 1 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

As is well known, \( 0 < w \) in \( \Omega \), \( w \in C^{2+\alpha}(\Omega) \), and \( \partial w(x)/\partial n < 0 \) for all \( x \in \partial \Omega \), where \( \vec{n} \) denotes the outward normal to \( \partial \Omega \) at \( x \).

We can easily obtain that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1d(x) \leq w(x) \leq C_2d(x) \quad \text{on } \Omega.
\]

Put \( |u|_\infty = \max\{||u(x)|| \mid x \in \Omega, u \in C(\overline{\Omega})\} \).

**Proof of Theorem 1.** The change of variable \( v = e^{-u} \) transforms the problem (1) into the equivalent problem

\[
\begin{align*}
-\Delta v + |\nabla v|^2 = 1 + |\nabla u|^q, & \quad v > 0 \text{ in } \Omega, \\
v|_{\partial \Omega} = 0.
\end{align*}
\]

Let \( \bar{u} = M_0w^{\alpha_1} \), where \( M_0 \) is a large positive constant to be chosen, and \( \alpha_1 \) is chosen to satisfy the following conditions:

For \( q \in [0, 1] \), \( \alpha_1 \in (0, 1) \), is an arbitrary constant;

For \( q \in (1, 2) \), \( \alpha_1 = 2 - q \).
Then
\[-\Delta \varphi \geq 1 + \frac{|
abla \varphi|^q}{\varphi^q} \quad \text{in } \Omega, \quad (5)\]
provided
\[M_0 \geq |w|^2 - \alpha_1^q |w|^q + \alpha_1^q |\nabla w|^q \mid \min_{x \in \Omega} \left[ \alpha_1 (1 - \alpha_2) \left| \nabla w(x) \right|^2 + \alpha_2 w(x) \right].\]
i.e., \( \overline{\varphi} \) is a supersolution of (4). To construct a subsolution \( \underline{\varphi} \) of (4), a good candidate is \( \underline{\varphi} = m_0 w^2 \), where \( m_0 \) is a small positive constant to be chosen. Indeed one has
\[-\Delta \underline{\varphi} + \frac{|\nabla \underline{\varphi}|^2}{\underline{\varphi}} = 2m_0 \left( w + |\nabla w|^2 \right) \leq 1 \quad \text{in } \Omega, \quad (6)\]
provided \( m_0 \leq \left[2 |w + |\nabla w|^2| \right]^{-1}. \)
By the maximum principle, we can easily obtain
\[\underline{\varphi} \leq \overline{\varphi} \quad \text{on } \overline{\Omega}. \]

It follows by the proofs in the following Appendix that (4) has at least one solution \( \varphi \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega}) \) and
\[\underline{\varphi} \leq \varphi \leq \overline{\varphi} \quad \text{on } \overline{\Omega}, \]
i.e., (1) has at least one solution \( u = \ln(1/\varphi) \in C^{2,\alpha}(\Omega) \) satisfying
\[\ln m_0 + 2 \ln w \leq -u \leq \ln M_0 + \alpha_1 \ln w \quad \text{in } \Omega. \]
Consequently
\[\lim_{d(x) \to 0} \sup_{d(x) \to 0} \frac{u(x)}{\ln w(x)} \text{ is bounded.} \]
By (3), we know Theorem 1 is true.

**Proof of Theorem 2.** We follow the same arguments as in the proof of Theorem 1. The change of variable \( \varphi = u^{-p} \) transforms the problem (1) into the equivalent problem
\[
\begin{cases}
-\Delta v + \frac{(p + 1) |\nabla v|^2}{pv} = pv^{1/p} + \frac{p^{1-q} |\nabla v|^q}{v^{(1+1/p)q}}, & \text{in } \Omega, \\
v|_{\partial \Omega} = 0.
\end{cases} \quad (7)
\]
Then \( \bar{\varphi} = M_0 w^{\alpha_2} \) satisfies

\[
-\Delta \bar{\varphi} \geq p^{1/p} + \frac{p^{1-q} |\nabla \bar{\varphi}|^q}{p(1+1/p)^q} \quad \text{in } \Omega, \tag{8}
\]

where

\[
M_0 \geq \max\left\{1, \left[ w_\infty^{(2-q)-((p+q)/(p+1))\alpha_2} |w_\infty^{((1+q)/(p+1))\alpha_2+q} + p^{1+q(\alpha_2+q)} |\nabla w|^q \right] \times \min_{x \in \Omega} \left[ (1 - \alpha_2) |\nabla w(x)|^2 + \alpha_2 w(x) \right] \right\},
\]

and \( \alpha_2 \) is chosen to satisfy the condition

\[
0 < \alpha_2 < \min\left\{1, \frac{p(2-q)}{p+q} \right\}.
\]

Thus \( \bar{\varphi} \) is a supersolution of (7). To construct a subsolution of (7), a good candidate is \( \underline{\varphi} = m_\alpha w^{\beta_1} \), where \( \beta_1 = 2p/(p-1) \), and \( m_\alpha \) is a small positive constant. Then \( \underline{\varphi} \) satisfies

\[
-\Delta \underline{\varphi} + \frac{(p + 1) |\nabla \underline{\varphi}|^2}{p \underline{\varphi}} = m_\alpha \beta_1 w^{\beta_1-2} \left[ w + \frac{p + 1}{p - 1} |\nabla w|^2 \right] \leq p m_\alpha^{1/p} \beta_1^{1/p} - p \underline{\varphi}^{1/p} \quad \text{in } \Omega, \tag{9}
\]

provided \( m_\alpha \leq \left[ p/C_0 \beta_1/p \right]^{(p-1)/(p+1)} \), where \( C_0 = \max_{x \in \Omega} \left[ w_\infty + \left( (p+1)/(p-1) \right) |\nabla w(x)|^2 \right] \). Thus \( \underline{\varphi} \) is a subsolution of (7).

The conclusion follows as in the proof of Theorem 1 and is omitted.

Remark. He does not know, if \( q \geq 2 \), whether (1) has a solution or not for \( f(x, u, \nabla u) \) in Theorem 1 or Theorem 2.

**APPENDIX**

In this section we only need to complete the proof of Theorem 1. The proof is similar to that in [13]. Let

\[
f(s, \eta) = 1 + |\eta|^q/s^q - |\eta|^2/s, \quad \text{where } 0 < s, \quad \text{and } \eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^N.
\]
For any $\varepsilon \in (0,1)$, we can easily prove that there exists a function $f_\varepsilon(s, \eta) \in C^1(\mathbb{R} \times \mathbb{R}^N)$ such that

\begin{align}
(i) & \quad f(s, \eta) = f_\varepsilon(s, \eta), \quad \forall (s, \eta) \in [\varepsilon, +\infty) \times \mathbb{R}^N, \quad (10) \\
(ii) & \quad |f_\varepsilon(s, \eta)| \leq \left(1 + \frac{2}{\varepsilon^{1+q}}\right)(1 + |\eta|^2), \quad \forall (s, \eta) \in \mathbb{R} \times \mathbb{R}^N. \quad (11)
\end{align}

Consider the perturbed problems

\[
\begin{aligned}
-\Delta v &= 1 + \frac{|\nabla v|^q}{v^q} - \frac{|\nabla v|^2}{v}, \quad v > \varepsilon, \text{ in } \Omega, \\
v|_{\partial \Omega} &= \varepsilon,
\end{aligned}
\]

and

\[
\begin{aligned}
-\Delta v &= f_\varepsilon(v, \nabla v), \quad v > \varepsilon, \text{ in } \Omega, \\
v|_{\partial \Omega} &= \varepsilon.
\end{aligned}
\]

Let $\overline{v}_\varepsilon = M_1(w + \varepsilon)^{\alpha_1}$ and $\underline{v}_\varepsilon = m_0 w^2 + \varepsilon = v + \varepsilon$, where

\[
M_1 \geq 1 + |w + 1|_{\infty}^{2-q-\alpha_1} \left( |w + 1|^q + \alpha_2 |\nabla w|^q \right)_{\infty} \\
\times \min_{x \in \Omega} \left\{ \alpha_3 (1 - \alpha_1) |\nabla w(x)|^2 + \alpha_3 (w(x) + 1) \right\}.
\]

Then $\overline{v}_\varepsilon \in C^{2+\alpha}(\overline{\Omega})$, $\underline{v}_\varepsilon \in C^{2+\alpha}(\overline{\Omega})$, and

\[
\begin{aligned}
-\Delta \overline{v}_\varepsilon &\geq 1 + \frac{|\nabla \overline{v}_\varepsilon|^q}{\overline{v}_\varepsilon^q} \geq 1 + \frac{|\nabla \overline{v}_\varepsilon|^2}{\overline{v}_\varepsilon} = f_\varepsilon(\overline{v}_\varepsilon, \nabla \overline{v}_\varepsilon), \quad \text{in } \Omega, \\
\overline{v}_\varepsilon|_{\partial \Omega} &= \varepsilon,
\end{aligned}
\]

\[
\begin{aligned}
-\Delta \underline{v}_\varepsilon &\leq 1 - \frac{|\nabla \underline{v}_\varepsilon|^2}{\underline{v}_\varepsilon} = f_\varepsilon(\underline{v}_\varepsilon, \nabla \underline{v}_\varepsilon), \quad \text{in } \Omega, \\
\underline{v}_\varepsilon|_{\partial \Omega} &= \varepsilon,
\end{aligned}
\]

i.e., $\overline{v}_\varepsilon$ is a (usual) supersolution of (12) and (13), $\underline{v}_\varepsilon$ is a (usual) subsolution of (12) and (13). By the maximum principle (or let $M_1$ be sufficiently large), we can easily prove

\[
\underline{v}_\varepsilon \leq \overline{v}_\varepsilon \quad \text{on } \overline{\Omega}. \quad (16)
\]
It follows by the first theorem of Amann [15] that (13) has a maximal solution \( v_\varepsilon \in C^{2+\alpha}(\Omega) \) in order interval \([v_\varepsilon, \bar{v}_\varepsilon]\).

Since
\[
\varepsilon \leq 2m_0w^2 + \varepsilon \leq M_1(w + \varepsilon)^{\alpha_1}
\]
on \( \Omega \),

we see
\[
f_\varepsilon(v_\varepsilon(x), \nabla v_\varepsilon(x)) = f(v_\varepsilon(x), \Delta v_\varepsilon(x)), \quad \forall x \in \Omega,
\]
i.e., \( v_\varepsilon \) is a solution of (12).

Now we need to estimate \( v_\varepsilon \).

For any \( C^{2+\alpha} \)-smooth domain \( \Omega' \Subset \Omega \), take \( \Omega_i, i = 1, 2, 3 \), with \( C^{2+\alpha} \)-smooth boundaries such that
\[
\Omega' \Subset \Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \Omega.
\]

Let \( m = m_0 \min_{x \in \Omega_3} w^2 \) with \( m_0 |w|_2^2 < 1 \), and
\[
f_\varepsilon(x) = f_m(v_\varepsilon(x), \nabla v_\varepsilon(x)), \quad x \in \Omega_3,
\]
where \( f_m \) is a function as in (10) and (11) with \( \varepsilon = m \). By \( w > 0 \) in \( \Omega \), we see that \( 0 < m < 1 \).

Since \(-\Delta v_\varepsilon = \hat{f}_\varepsilon(x)\) on \( \Omega_3 \), by the interior estimate theorem of Ladyzhenskaja and Ural'tseva (see [16, Theorem 3.1, p. 266]), we get a positive constant \( C_1 \) independent of \( \varepsilon \) such that
\[
\max_{x \in \Pi_2} |\nabla v_\varepsilon(x)| \leq C_1 \max_{x \in \Pi_3} v_\varepsilon(x).
\]

From (17) and \( 0 < \varepsilon < 1 \) we see that \( |\nabla v_\varepsilon(x)| \) is uniformly bounded on \( \Omega_2 \). It follows that \( |\hat{f}_\varepsilon(x)| \) is uniformly bounded on \( \Omega_2 \), and hence \( \hat{f}_\varepsilon \in L^p(\Omega_2) \) for any \( p > 1 \).

Since
\[
-\Delta v_\varepsilon = \hat{f}_\varepsilon(x), \quad x \in \Omega_2,
\]
it follows that by Theorem 9.11 of [17] that there exists a positive constant \( C_2 \) independent of \( \varepsilon \) such that
\[
\|v_\varepsilon\|_{W^{2,\alpha}(\Omega_2)} \leq C_2 \left( \|\hat{f}_\varepsilon\|_{L^p(\Omega_2)} + \|v_\varepsilon\|_{L^p(\Omega_2)} \right),
\]
i.e., \( \|v_\varepsilon\|_{W^{2,\alpha}(\Omega_2)} \) is uniformly bounded. Taking \( p > N \) such that \( \alpha < 1 - N/p \) and applying Sobolev's embedding inequality, we see that \( \|v_\varepsilon\|_{C^{1+\alpha}(\Pi_3)} \) is uniformly bounded. Therefore \( \hat{f}_\varepsilon \in C^\alpha(\Omega_1) \) and \( \|\hat{f}_\varepsilon\|_{C^\alpha(\Pi_1)} \) is uniformly bounded. By Schauder's interior estimate theorem (see [17, Chap. 1,
Introduction, p. 2), we see that there exists a positive constant $C_3$ independent of $\varepsilon$ such that
\[
\|v_{\varepsilon}\|_{C^{2+\alpha}(\Omega')} \leq C_3 \left( \|v_{\varepsilon}\|_{C(\Omega')} + \|f_{\varepsilon}\|_{C^{\alpha}(\Omega')} \right).
\]
It follows that $\|v_{\varepsilon}\|_{C^{2+\alpha}(\Omega')}$ is uniformly bounded.

From the above proof, we see that $\{v_{\varepsilon}\}$ is uniformly bounded for arbitrary $\Omega' \Subset \Omega$. Using the Ascoli–Arzela theorem and the diagonal sequential process, we see that $\{v_{\varepsilon}\}$ has a subsequence which converges uniformly in the $C^2(\Omega')$ norm to a function $v \in C^2(\Omega)$ and $v$ satisfies Eq. (4).

From (17) we get
\[
m_0 w^2(x) \leq v(x) \leq M_0 w^{\alpha}(x), \quad \forall x \in \Omega,
\]
which implies that $\lim_{x \to \partial \Omega} v(x) = 0$. Let $v|_{\partial \Omega} = 0$. We, thus, get a solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$. Applying Schauder’s interior regularity theorem we see $u \in C^{2+\alpha} (\overline{\Omega})$ and, thus, Theorem 1 is proved.

ACKNOWLEDGMENTS

The author is greatly indebted to the referees and to Professors Wenyuan Chen and Lishang Jiang for their suggestions and encouragement.

REFERENCES