ELSEVIER

Journal of Computational and Applied Mathematics 124 (2000) 245–261

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Piecewise linear methods for nonlinear equations and optimization

Eugene L. Allgower [1,*], Kurt Georg [1,2]

Department of Mathematics, Colorado State University, Fort Collins, CO 80523, USA

Received 7 April 1999; received in revised form 18 November 1999

Abstract

Piecewise linear methods had their beginning in the mid-1960s with Lemke's algorithm for calculating solutions to linear complementarity problems. In the 1970s and 1980s activity moved on to computing fixed points of rather general maps and economic equilibria. More recently, they have been used to approximate implicitly defined manifolds, with applications being made to computer graphics and approximations of integral over implicitly defined manifolds. In this paper we present the basic ideas of piecewise linear algorithms and a selection of applications. Further references to the literature on piecewise linear algorithms are indicated.

1. Introduction

Piecewise linear algorithms, also referred to in the literature as simplicial algorithms, can be used to generate piecewise linear manifolds which approximate the solutions of underdetermined systems of equations $H(x)=0$, where $H:\mathbb{R}^{N+K}\to\mathbb{R}^N$ may be a mapping having relaxed smoothness properties. Of particular interest and importance is the case $K=1$, in which case the algorithms produce an approximation of an implicitly defined curve. If the defining map H is itself piecewise linear, then $H^{-1}(0)$ is a polygonal path. More generally, H may also be piecewise smooth, or in some instances, an upper semi-continuous multi-valued map. Intuitively, the approximations which are produced result from traversing through cells of a tiling of $\mathbb{R}^{N+K}$ which intersect $H^{-1}(0)$. Often the tilings which are used are triangulations of $\mathbb{R}^{N+K}$ into simplices and hence the term simplicial algorithms occurs.

Piecewise linear algorithms have been used to find solutions to complementarity problems, fixed points of mappings, and economic equilibria [33]. Many classical theorems of analysis which can

* Corresponding author.

E-mail address: allgower@math.colostate.edu (E.L. Allgower).

[1] Partially supported by NSF via grant # DMS-9870274.

[2] http://www.math.colostate.edu/~georg/.

0377-0427/00/$ - see front matter
PII: S0377-0427(00)00427-1

be proven by means of homotopies or degree theory have been re-examined in terms of piecewise linear algorithms. Indeed, these algorithms can be viewed as a constructive approach to the Brouwer degree. Recent applications with $K > 1$ have been made to obtain computer graphical approximations of surfaces, surface and volume integrals, and solutions of differential–algebraic equations. In the first part of the paper we deal with the case $K = 1$, and in Section 9, we consider the more general case $K > 1$.

The first prominent example of a piecewise linear algorithm was designed in [27,26] to calculate a solution of the linear complementarity problem, see Section 7. This algorithm played a crucial role in the development of subsequent piecewise linear algorithms. Scarf [32] gave a numerically implementable proof of the Brouwer fixed point theorem, based upon Lemke's algorithm. Eaves [17] observed that a related class of algorithms can be obtained by considering piecewise linear approximations of homotopy maps. Concurrently, Merrill [28] gave a restart version for fixed points of upper semi-continuous maps. Thus the piecewise linear continuation methods began to emerge as a parallel to the classical embedding or predictor corrector numerical continuation methods (see [2,3]).

Piecewise linear methods require no smoothness of the underlying equations and hence have, at least in theory, a more general range of applicability than classical embedding methods. In fact, they can be used to calculate fixed points of set-valued maps. They are more combinatorial in nature and are closely related to the topological degree. Piecewise linear continuation methods are usually considered to be less efficient than the predictor corrector methods when the latter are applicable, especially for large N. The reasons for this lie in the fact that steplength adaptation and exploitation of special structure are more difficult to implement for piecewise linear methods. Some efforts to overcome this were given, e.g., in [31,35].

Many applications of piecewise linear algorithms for optimization and complementarity problems have recently been superceded by interior point methods which can handle the much larger (but more special) systems frequently occuring in practical problems (see, e.g., [25,29,37]).

We cast the notion of piecewise linear algorithms into the general setting of subdivided manifolds which we will call *piecewise linear manifolds.* Lack of space precludes an extensive bibliography. The older literature on the subject is well documented (see, e.g., [2,3]).

2. Basic facts

A piecewise linear algorithm consists of moving (pivoting) through cells which subdivide the domain of the map H. Let us formally introduce the basic notions.

Let $\mathbf{E}$ denote some ambient finite-dimensional Euclidean space which contains all points arising in the sequel. A *half-space* η and the corresponding *hyperplane* $\partial\eta$ are defined by $\eta = \{y \in \mathbf{E}: x^* y \leqslant \alpha\}$ and $\partial\eta = \{y \in \mathbf{E}: x^* y = \alpha\}$, respectively, for some $x \in \mathbf{E}$ with $x \neq 0$ and some $\alpha \in \mathbb{R}$. A finite intersection of half-spaces is called a *cell*. If σ is a cell and ξ a half-space such that $\sigma \subset \xi$ and $\tau := \sigma \cap \partial\xi \neq \emptyset$, then the cell τ is called a *face* of σ. For reasons of notation we consider σ also to be a face of itself, and all other faces are *proper* faces of σ. The *dimension* of a cell is the dimension of its affine hull. In particular, the dimension of a singleton is 0 and the dimension of the empty set is -1. If the singleton $\{v\}$ is a face of σ, then v is called a *vertex* of σ. If τ is a face of σ such that $\dim\tau = \dim\sigma - 1$, then τ is called a *facet* of σ. The *interior* of a cell σ consists of all points of σ which do not belong to a proper face of σ.

A *piecewise linear manifold* of dimension N is a system $\mathscr{M} \neq \emptyset$ of cells of dimension N such that the following conditions hold:

1. If $\sigma_1, \sigma_2 \in \mathscr{M}$, then $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2.
2. A cell τ of dimension $N-1$ can be a facet of at most two cells in $\mathscr{M}$.
3. The family $\mathscr{M}$ is *locally finite*, i.e., any relatively compact subset of

$$|\mathscr{M}| := \bigcup_{\sigma \in \mathscr{M}} \sigma \tag{1}$$

meets only finitely many cells $\sigma \in \mathscr{M}$.

A simple example of a piecewise linear manifold is $\mathbb{R}^N$ subdivided into unit cubes with integer vertices.

We introduce the *boundary* $\partial\mathscr{M}$ of $\mathscr{M}$ as the system of facets which are common to exactly one cell of $\mathscr{M}$. Generally, we cannot expect $\partial\mathscr{M}$ to again be a piecewise linear manifold. However, this is true for the case that $|\mathscr{M}|$ is convex. Two cells which have a common facet τ are called *adjacent*. Moving from one cell to another through a common facet is called *pivoting*.

It is typical of piecewise linear path following that at any particular step only one current cell is stored in the computer, along with some additional data, and the pivoting step is performed by calling a subroutine which makes use of the data to determine an adjacent cell which then becomes the new current cell.

A cell of particular interest is a *simplex* $\sigma = [v_1, v_2, \ldots, v_{N+1}]$ of dimension N which is defined as the convex hull of $N+1$ affinely independent points $v_1, v_2, \ldots, v_{N+1} \in \mathbf{E}$. These points are the vertices of σ. If a piecewise linear manifold $\mathscr{M}$ of dimension N consists only of simplices, then we call $\mathscr{M}$ a *pseudo-manifold* of dimension N. Such manifolds are of special importance, see, e.g., [34]. If a pseudo-manifold $\mathscr{T}$ subdivides a set $|\mathscr{T}|$, then we also say that $\mathscr{T}$ *triangulates* $|\mathscr{T}|$. We will use the notions pseudo-manifold and triangulation somewhat synonymously. Some triangulations of $\mathbb{R}^N$ of practical importance were already considered in [11,21]. Eaves [19] gave an overview of standard triangulations.

If σ is a simplex in a pseudo-manifold $\mathscr{T}$ and τ is a facet of σ which is not in the boundary of $\mathscr{T}$, then there is exactly one simplex $\tilde{\sigma}$ in $\mathscr{T}$ which is different from σ but contains the same facet τ, and there is exactly one vertex v of σ which is not a vertex of $\tilde{\sigma}$. We call v the vertex of σ *opposite* τ. There is also exactly one vertex $\tilde{v}$ of $\tilde{\sigma}$ opposite τ. We say that σ is pivoted across τ into $\tilde{\sigma}$, and that the vertex v of σ is pivoted into $\tilde{v}$.

A simple triangulation can be generated by the following pivoting rule (pivoting by reflection; see [12]) if

$$\sigma = [v_1, v_2, \ldots, v_i, \ldots, v_{N+1}]$$

is a simplex in $\mathbb{R}^N$, and τ is the facet opposite a vertex v_i, then σ is pivoted across τ into $\tilde{\sigma} = [v_1, v_2, \ldots, \tilde{v}_i, \ldots, v_{N+1}]$ by setting

$$\tilde{v}_i = \begin{cases} v_{i+1} + v_{i-1} - v_i & \text{for } 1 < i < N+1, \\ v_2 + v_{N+1} - v_1 & \text{for } i = 1, \\ v_N + v_1 - v_{N+1} & \text{for } i = N+1. \end{cases} \tag{2}$$

In fact, a minimal (nonempty) system of N-simplices in $\mathbb{R}^N$ which is closed under the above pivoting rule is a triangulation of $\mathbb{R}^N$. We note that the above-described triangulation maintains a consistent ordering of the vertices of the simplices.

3. Piecewise linear algorithms

Let $\mathscr{M}$ be a piecewise linear manifold of dimension $N+1$. We call $H:|\mathscr{M}| \to \mathbb{R}^N$ a *piecewise linear map* if the restriction $H_\sigma : \sigma \to \mathbb{R}^N$ of H to σ is an affine map for all $\sigma \in \mathscr{M}$. In this case, H_σ can be uniquely extended to an affine map on the affine space spanned by σ. The Jacobian H'_σ is piecewise constant and has the property $H'_\sigma(x-y) = H_\sigma(x) - H_\sigma(y)$ for x, y in this affine space. Note that under a choice of basis H'_σ corresponds to an $(N, N+1)$-matrix which has a one-dimensional kernel in case of nondegeneracy, i.e., if its rank is maximal.

If $\mathscr{M}$ is a pseudo-manifold triangulating a set $X = |\mathscr{M}|$, and if $\tilde{H}: X \to \mathbb{R}^k$ is a map, then the *piecewise linear approximation* of $\tilde{H}$ (with respect to $\mathscr{M}$) is defined as the unique piecewise linear map $H: X \to \mathbb{R}^k$ which coincides with $\tilde{H}$ on all vertices of $\mathscr{M}$, i.e., $\tilde{H}$ is affinely interpolated on the simplices of $\mathscr{M}$.

A piecewise linear algorithm is a method for following a polygonal path in $H^{-1}(0)$. To handle possible degeneracies, we introduce a concept of regularity. A point $x \in |\mathscr{M}|$ is called a *regular point* of H if x is not contained in any face of dimension $< N$, and if H'_τ has maximal rank N for all faces τ containing x. A value $y \in \mathbb{R}^N$ is a *regular value* of H if all points in $H^{-1}(y)$ are regular. By definition, y is vacuously a regular value if it is not contained in the range of H. If a point or value is not regular it is called *singular*.

The following analogue of Sard's theorem holds for piecewise linear maps (see, e.g., [18]). This enables us to confine ourselves to regular values. We note that degeneracies can also be handled via the closely related concept of lexicographical ordering (see [8,16,34]).

Theorem 3.1 (Perturbation Theorem). *Let $H: \mathscr{M} \to \mathbb{R}^N$ be a piecewise linear map where $\mathscr{M}$ is a piecewise linear manifold of dimension $N+1$. Then for any relatively compact subset $C \subset |\mathscr{M}|$ there are at most finitely many $\varepsilon > 0$ such that $C \cap H^{-1}(\vec{\varepsilon})$ contains a singular point of H. Consequently, $\vec{\varepsilon}$ is a regular value of H for almost all $\varepsilon > 0$. Here we use the notation*

$$\vec{\varepsilon} := \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^N \end{pmatrix}.$$

Let 0 be a regular value of H. This implies that $H^{-1}(0)$ consists of polygonal paths whose vertices are always in the interior of some facet. If σ is a cell, then $\sigma \cap H^{-1}(0)$ is a segment (two end points), a ray (one end point), a line (no end point) or empty. The latter two cases are not of interest for piecewise linear path following. A step of the method consists of following the ray or segment from one cell into a uniquely determined adjacent cell. The method is typically started at a point of the boundary or on a ray (coming from infinity), and it is typically terminated at a point

of the boundary or in a ray (going to infinity), The numerical linear algebra (*piecewise linear step*) required to perform one step of the method is typical for linear programming and usually involves $O(N^2)$ operations for dense matrices.

On the other hand, even if 0 is not a regular value of H, the above theorem helps us to do something similar. Namely, $\sigma \cap H^{-1}(\vec{\varepsilon})$ is a segment (two end points) for all sufficiently small $\varepsilon > 0$, a ray (one end point) for all sufficiently small $\varepsilon > 0$, a line (no end point) for all sufficiently small $\varepsilon > 0$ or empty for all sufficiently small $\varepsilon > 0$. This leads us to the following

Definition 3.1. We call an N-dimensional facet τ *completely labeled* with respect to an affine map $H : \tau \to \mathbb{R}^N$, if $\tau \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$ for all sufficiently small $\varepsilon > 0$. We call a cell σ of dimension $> N$ *transverse* with respect to an affine map $H : \sigma \to \mathbb{R}^N$, if $\sigma \cap H^{-1}(\vec{\varepsilon}) \neq \emptyset$ for all sufficiently small $\varepsilon > 0$.

Instead of following the paths $H^{-1}(0)$ for a regular value 0, we now follow more specifically the *regularized paths*

$$\bigcup \{H^{-1}(0) \cap \sigma : \sigma \text{ transverse}\}.$$

Of course, this set coincides with $H^{-1}(0)$ for the case that 0 is a regular value of H.

For $\varepsilon > 0$ sufficiently small and $\vec{\varepsilon}$ a regular value of H, a node of the polygonal paths $H^{-1}(\vec{\varepsilon})$ corresponds to a completely labeled facet (which is intersected), and hence the piecewise linear algorithm traces such completely labeled facets belonging to the same cell. The method is usually started either on the boundary, i.e., in a completely labeled facet $\tau \in \partial\mathscr{M}$, or on a ray, i.e., in a transverse cell $\sigma \in \mathscr{M}$ which has only one completely labeled facet.

Hence, a piecewise linear algorithm generates a succession of transverse cells σ_i and completely labeled facets τ_i such that σ_i, σ_{i+1} have the common facet τ_i. We are thus led to the following generic version:

Algorithm 3.1 (Piecewise Linear Algorithm)**.**

1. Start
   (a) start from the boundary
      (i) let $\tau_1 \in \partial\mathscr{M}$ be completely labeled
      (ii) find the unique $\sigma_1 \in \mathscr{M}$ such that $\tau_1 \subset \sigma_1$
   (b) start from a ray
      (i) let $\sigma_0 \in \mathscr{M}$ have precisely one completely labeled facet τ_1
      (ii) pivoting step:
         find $\sigma_1 \in \mathscr{M}, \sigma_1 \neq \sigma_0$ such that $\tau_1 \subset \sigma_1$
2. for $i = 1, 2, 3, \ldots$
   (a) if τ_i is the only completely labeled facet of σ_i
         then **stop** (ray termination)
   (b) else
         piecewise linear step:
         find the other completely labeled facet τ_{i+1} of σ_i

(c) if $\tau_{i+1} \in \partial\mathscr{M}$
    then **stop** (boundary termination)
(d) else
    pivoting step:
    find $\sigma_{i+1} \in \mathscr{M}$, $\sigma_{i+1} \neq \sigma_i$, such that $\tau_{i+1} \subset \sigma_{i+1}$

4. Numerical considerations

From a numerical point of view, two steps of a piecewise linear algorithm have to be efficiently implemented. Usually, a current cell σ and a completely labeled facet τ of σ are stored via some characteristic data.

A pivoting step consists of finding the adjacent cell $\tilde{\sigma}$ sharing the same facet τ. The implementation of this step is dependent on the special piecewise linear manifold under consideration. Typically, this step is performed by only a few operations. The pivoting rule (2) is a simple example.

A piecewise linear step consists of finding a second completely labeled facet $\tilde{\tau}$ of σ (if it exists, otherwise we have ray termination). This is usually computationally more expensive than the pivoting rule and typically involves some numerical linear algebra.

Let us consider an example. We assume that a cell of dimension $N+1$ is given by

$$\sigma := \{x \in \mathbb{R}^{N+1} : Lx \geqslant c\},$$

where $L : \mathbb{R}^{N+1} \to \mathbb{R}^m$ is a linear map and $c \in \mathbb{R}^m$ is a given value. Furthermore, let us assume that

$$\tau_i := \{x \in \mathbb{R}^{N+1} : Lx \geqslant c, e_i^* Lx = e_i^* c\},$$

for $i = 1, 2, \ldots, m$, is a numbering of all the facets of σ. Here and in the following e_i denotes the ith unit vector, i.e., the ith column of the identity matrix.

On the cell σ, the piecewise linear map $H : \mathscr{M} \to \mathbb{R}^N$ reduces to an affine map, and hence there is a linear map $A : \mathbb{R}^{N+1} \to \mathbb{R}^N$ and a vector $b \in \mathbb{R}^N$ such that the segment of the path in σ can be written as

$$\sigma \cap H^{-1}(0) = \{x \in \mathbb{R}^{N+1} : Ax = b, Lx \geqslant c\}. \tag{3}$$

Let τ_i be completely labeled. This implies that the rank of A is N. If we exclude degeneracies, then $\tau_i \cap H^{-1}(0) = \{x_0\}$ is a singleton, and there is a unique vector t in the one-dimensional kernel $A^{-1}(0)$ such that $e_i^* Lt = -1$. Since x_0 is in the interior of τ_i (by excluding degeneracies), we have

$$e_j^* Lx_0 > e_j^* c \quad \text{for } j = 1, \ldots, m,\ j \neq i$$

and hence $x_0 - \lambda t$ is in the interior of σ for small $\lambda > 0$.

If (3) is a ray, then $e_j^* L(x_0 - \lambda t) > e_j^* c$ for all $\lambda > 0$. Otherwise, we have $e_j^* Lt > 0$ for at least one index j, and since we are excluding degeneracies, the minimization

$$k := \arg\min \left\{ \frac{e_j^*(Lx_0 - c)}{e_j^* Lt} : j = 1, \ldots, m, e_j^* Lt > 0 \right\} \tag{4}$$

yields the unique completely labeled facet τ_k of σ with $k \neq i$. For the minimum

$$\lambda_0 := \frac{e_k^*(Lx_0 - c)}{e_k^* Lt} > 0,$$

we obtain: $\sigma \cap H^{-1}(0) = \{x_0 - \lambda t\colon\ 0 \leqslant \lambda \leqslant \lambda_0\}$.

Minimizations such as (4) are typical for linear programming, and the numerical linear algebra can be efficiently handled by standard routines. Successive linear programming steps can often make use of previous matrix factorizations via update methods. In the case of a pseudo-manifold $\mathscr{M}$ where the cell σ is a simplex, it is convenient to handle the numerical linear algebra with respect to the barycentric coordinates based on the vertices of σ. Then the equations become particularly simple, (see, e.g., [2, Sections 12.2–12.4] or [34] for details).

We now give some examples of how the piecewise linear path following methods are used.

5. Piecewise linear homotopy algorithms

Let us see how the above ideas can be used to approximate zero points of a map $G: \mathbb{R}^N \to \mathbb{R}^N$ by applying piecewise linear methods to an appropriate homotopy map. In order to also allow for applications to optimization problems or other nonlinear programming problems, we consider the case where G is not necessarily continuous, e.g., G might be a selection of a multi-valued map. For the case that $\bar{x}$ is a point of discontinuity of G, we have to generalize the notion of a zero point in an appropriate way, as described below.

Eaves [17] presented the first piecewise linear homotopy method for computing a fixed point. A restart method based on somewhat similar ideas was developed in [28]. Fixed point problems and zero point problems are obviously equivalent.

As an example of a piecewise linear homotopy algorithm, let us sketch the algorithm of Eaves and Saigal [20]. We consider a triangulation $\mathscr{T}$ of $\mathbb{R}^N \times (0,1]$ into $(N+1)$-simplices σ such that every simplex has vertices in adjacent levels $\mathbb{R}^N \times \{2^{-k}, 2^{-k-1}\}$ and a diameter $\leqslant C^k$ for some $k = 0, 1, \ldots$ and some $0 < C < 1$ which is not dependent on k. We call such a triangulation a refining triangulation (with refining factor C). Of course, the main point here is to obtain a triangulation which is easily implemented. The first such triangulation was proposed in [17]. Todd [34] gave a triangulation with refining factor $\frac{1}{2}$. Subsequently, many triangulations with arbitrary refining factors were developed (see the books [14,19]). To ensure success (i.e., convergence) of the algorithms, it is necessary to require a boundary condition.

Let us first introduce some notation. For $x \in \mathbb{R}^N$ we denote by $\mathscr{U}(x)$ the system of neighborhoods of x. By $\overline{\text{co}}(X)$ we denote the closed convex hull of a set $X \subset \mathbb{R}^N$. By $\mathbb{R}^N_\Sigma$ we denote the system of compact convex nonempty subsets of $\mathbb{R}^N$. We call the map $G: \mathbb{R}^N \to \mathbb{R}^N$ *asymptotically linear* if the following three conditions hold:

1. G is *locally bounded*, i.e., each point $x \in \mathbb{R}^N$ has a neighbourhood $U \in \mathscr{U}(x)$ such that $G(U)$ is a bounded set.
2. G is *differentiable at* ∞, i.e., there exists a linear map $G'_\infty: \mathbb{R}^N \to \mathbb{R}^N$ such that $||x||^{-1}||G(x) - G'_\infty x|| \to 0$ for $||x|| \to \infty$.
3. G'_∞ is nonsingular.

If a map $G:\mathbb{R}^N \to \mathbb{R}^N$ is locally bounded, then we define its *set-valued hull* $G_\Sigma:\mathbb{R}^N \to \mathbb{R}^N_\Sigma$ by setting

$$G_\Sigma(x) := \bigcap_{U\in\mathscr{U}(x)} \overline{\mathrm{co}}(G(U)).$$

It is not difficult to see that G_Σ is upper semi-continuous, and that G is continuous at x if and only if $G_\Sigma(x)$ is a singleton. By using a degree argument on the set-valued homotopy

$$H_\Sigma(x,\lambda) := (1-\lambda)G'_\infty x + \lambda G_\Sigma(x),$$

it can be seen that G_Σ has at least one zero point, i.e., a point $\bar{x}$ such that $0 \in G_\Sigma(\bar{x})$. Our aim here is to approximate such a solution numerically.

We now construct a piecewise linear homotopy for an asymptotically linear map $G:\mathbb{R}^N \to \mathbb{R}^N$. First we define $\tilde{H}:\mathbb{R}^N \times [0,\infty) \to \mathbb{R}^N$ by setting

$$\tilde{H}(x,\lambda) := \begin{cases} G'_\infty(x-x_1) & \text{for } \lambda = 1, \\ G(x) & \text{for } \lambda < 1. \end{cases}$$

Here x_1 is a chosen starting point of the method. Then we consider a refining triangulation $\mathscr{T}$ of $\mathbb{R}^N \times (0,1]$ as above, and we use the piecewise linear approximation H of $\tilde{H}$ (with respect to $\mathscr{T}$) to trace the polygonal path in $H^{-1}(0)$ which contains the starting point $(x_1,1)$.

The boundary $\partial\mathscr{T}$ is a pseudo-manifold which triangulates the sheet $\mathbb{R}^N \times \{1\}$. If we assume that the starting point $u_1 := (x_1,1)$ is in the interior of a facet $\tau_1 \in \partial\mathscr{T}$, then it is immediately clear that τ_1 is the only completely labeled facet of $\partial\mathscr{T}$. Hence, the piecewise linear algorithm started in τ_1 cannot terminate in the boundary, and since all cells of $\mathscr{T}$ are compact, it cannot terminate in a ray. Hence, it has no termination. Thus, the piecewise linear algorithm generates a sequence $\tau_1,\tau_2,\ldots$ of completely labeled facets of $\mathscr{T}$. Let us also consider the polygonal path generated by the piecewise linear algorithm. This path is characterized by the nodes $(x_1,\lambda_1),(x_2,\lambda_2),\ldots$ such that (x_i,λ_i) is the unique zero point of the piecewise linear homotopy H in τ_i for $i = 1,2,\ldots$. The resulting algorithm, i.e., applying Algorithm 3.1 to the above homotopy H, is due to Eaves [17] and Eaves and Saigal [20].

We call $\bar{x} \in \mathbb{R}^N$ an *accumulation point* of the algorithm if

$$\liminf_{i\to\infty} ||x_i - \bar{x}|| = 0.$$

The following convergence theorem holds.

Theorem 5.1. *The set A of accumulation points generated by the Eaves–Saigal algorithm is compact, connected and nonempty. Each point $\bar{x} \in A$ is a zero point of G_Σ, i.e., we have $0 \in G_\Sigma(\bar{x})$.*

A proof can be found in [3, p. 153].

As a consequence, if the set-valued hull G_Σ has only isolated zero points, then the sequence x_i generated by the Eaves–Saigal algorithm converges to a zero point of G_Σ.

As a simple example, we consider the situation of the celebrated Brouwer fixed-point theorem. Let $F:C \to C$ be a continuous map on a convex, compact, nonempty subset $C \subset \mathbb{R}^N$ with nonempty

interior. We define an asymptotically linear map $G:\mathbb{R}^N \to \mathbb{R}^N$ by setting

$$G(x) := \begin{cases} x - F(x) & \text{for } x \in C, \\ x - x_1 & \text{for } x \notin C. \end{cases}$$

Here, a point x_1 in the interior of C is used as a starting point. The above piecewise linear algorithm generates a point $\bar{x} \in \mathbb{R}^N$ such that $0 \in G_\Sigma(\bar{x})$. If $\bar{x} \notin C$, then $G_\Sigma(\bar{x}) = \{\bar{x} - x_1\}$, but $\bar{x} \neq x_1$ implies that this case is impossible. If $\bar{x}$ is an interior point of C, then $G_\Sigma(\bar{x}) = \{\bar{x} - F(\bar{x})\}$, and hence $\bar{x}$ is a fixed point of F. If $\bar{x}$ is in the boundary ∂C, then $G_\Sigma(\bar{x})$ is the convex hull of $\bar{x} - x_1$ and $\bar{x} - F(\bar{x})$, and hence $\bar{x} = (1-\lambda)x_1 + \lambda F(\bar{x})$ for some $0 \leqslant \lambda \leqslant 1$. But $\lambda < 1$ would imply that $\bar{x}$ is an interior point of C, and hence we have $\lambda = 1$, and again $\bar{x}$ is a fixed point of F. Hence, the above piecewise linear homotopy algorithm generates a fixed point of F in either case.

Many similar asymptotically linear maps can be constructed which correspond to important nonlinear problems (see e.g., [2, Chapter 13]). In particular, let us mention two examples that relate to nonlinear optimization (see [2, Examples 13.1.17 and 13.1.22]).

Consider the constrained minimization problem

$$\min_x \{\theta(x): \psi(x) \leqslant 0\}, \tag{5}$$

where $\theta, \psi : \mathbb{R}^N \to \mathbb{R}$ are convex. We assume the Slater condition

$$\{x: \psi(x) < 0,\ ||x - x_0|| < r\} \neq \emptyset$$

and the boundary condition that the problem

$$\min_x \{\theta(x): \psi(x) \leqslant 0,\ ||x - x_0|| \leqslant r\}$$

has no solution on the boundary $\{x: ||x - x_0|| = r\}$ for some suitable $x_0 \in \mathbb{R}^N$ and $r > 0$. This boundary condition is satisfied, for example, if

$$\{x: \psi(x) \leqslant 0\} \subset \{x: ||x - x_0|| < r\}$$

or more generally, if

$$\emptyset \neq \{x: \psi(x) \leqslant 0\} \cap \{x: \theta(x) \leqslant C\} \subset \{x: ||x - x_0|| < r\}$$

for some $C > 0$. Let us define the map $G:\mathbb{R}^N \to \mathbb{R}^N$ by

$$G(x) \in \begin{cases} \partial\theta(x) & \text{for } \psi(x) \leqslant 0 \text{ and } ||x - x_0|| < r, \\ \partial\psi(x) & \text{for } \psi(x) > 0 \text{ and } ||x - x_0|| < r, \\ \{x - x_0\} & \text{for } ||x - x_0|| \geqslant r, \end{cases}$$

where ∂ indicates the set of subdifferentials of a convex function. G is asymptotically linear with Jacobian $G'(\infty) = \mathrm{Id}$. Hence, by Theorem 5.1, we obtain a zero point $0 \in G_\Sigma(\bar{x})$. It can be shown that $\bar{x}$ solves the minimization problem (5).

As a second example let us consider the nonlinear complementarity problem: Find an $x \in \mathbb{R}^N$ such that

$$x \in \mathbb{R}^N_+, \quad g(x) \in \mathbb{R}^N_+, \quad x^* g(x) = 0, \tag{6}$$

where $g:\mathbb{R}^N \to \mathbb{R}^N$ is a continuous map.

Here $\mathbb{R}_+$ denotes the set of nonnegative real numbers, and in the sequel we also denote the set of positive real numbers by $\mathbb{R}_{++}$. For $x \in \mathbb{R}^N$ we also introduce the positive part $x_+ \in \mathbb{R}^N_+$ by setting $e_i^* x_+ := \max\{e_i^* x, 0\}$ for $i = 1, \ldots, N$ and the negative part $x_- \in \mathbb{R}^N_+$ by $x_- := (-x)_+$. The following formulae are then obvious: $x = x_+ - x_-$, $(x_+)^*(x_-) = 0$.

It can be seen that x is a solution of (6) if and only if $x = \bar{x}_+$ where $\bar{x}$ is a zero point of the map

$$x \mapsto g(x_+) - x_-. \tag{7}$$

We assume the following coercivity condition: There is a bounded open neighborhood $V \in \mathbb{R}^N$ such that

$$x^* g(x) > 0 \quad \text{for all } x \in \partial V \cap R^N_+.$$

We choose

$$\alpha > \max\{||g(x)||_\infty + ||x||_\infty : x \in \bar{V} \cap R^N_+\}$$

and define

$$\Omega := \{x \in \mathbb{R}^N : ||x_-||_\infty < \alpha,\ x_+ \in V\}.$$

Now define $G : \mathbb{R}^N \to \mathbb{R}^N$ by

$$G(x) = \begin{cases} x & \text{if } x \notin \Omega, \\ g(x_+) - x_- & \text{if } x \in \Omega. \end{cases}$$

Again, G is asymptotically linear and $G'(\infty) = \mathrm{Id}$. Hence, we have a zero point $0 \in G_\Sigma(\bar{x})$. It can be shown that $\bar{x}$ is a zero point of the map (7) and hence a solution of (6).

6. Index and orientation

Nearly all piecewise linear manifolds $\mathcal{M}$ which are of importance for practical implementations, are orientable. If $\mathcal{M}$ is orientable and of dimension $N+1$, and if $H : \mathcal{M} \to \mathbb{R}^N$ is a piecewise linear map, then it is possible to introduce an index for the piecewise linear solution manifold $H^{-1}(0)$ which has important invariance properties and also yields some useful information. It should be noted that this index is closely related to the topological index which is a standard tool in topology and nonlinear analysis. Occasionally, index arguments are used to guarantee a certain qualitative behavior of the solution path.

We begin with some basic definitions. Let $\mathbf{F}$ be a linear space of dimension k. An *orientation* of $\mathbf{F}$ is a function $\mathbf{or} : \boldsymbol{F}^k \to \{-1, 0, 1\}$ such that the following conditions hold:

1. $\mathbf{or}(b_1, \ldots, b_k) \neq 0$ if and only if $b_1, \ldots, b_k$ are linearly independent.
2. $\mathbf{or}(b_1, \ldots, b_k) = \mathbf{or}(c_1, \ldots, c_k) \neq 0$ if and only if the transformation matrix between $b_1, \ldots, b_k$ and $c_1, \ldots, c_k$ has positive determinant.

It is clear from the basic facts of linear algebra that any finite-dimensional linear space permits exactly two orientations.

Let σ be a cell of dimension k and $\operatorname{aff}\sigma$ its affine hull. We introduce the k-dimensional linear space $\operatorname{tng}\sigma := \{x - y : x, y \in \operatorname{aff}\sigma\}$ as the *tangent space* of σ. The cell σ is oriented by orienting

this tangent space. Such an orientation $\mathbf{or}_\sigma$ of σ *induces an orientation* $\mathbf{or}_{\tau,\sigma}$ on a facet τ of σ by the following convention:

$$\mathbf{or}_{\tau,\sigma}(b_1,\ldots,b_{k-1}) := \mathbf{or}_\sigma(b_1,\ldots,b_k)$$

whenever $b_1,\ldots,b_{k-1}$ is a basis for $\operatorname{tng}\tau$, and $b_1,\ldots,b_k$ is a basis for $\operatorname{tng}\sigma$ such that b_k points from τ into the interior of the cell σ. It is routine to check that the above definition of $\mathbf{or}_{\tau,\sigma}$ indeed satisfies the definition of an orientation.

If $\mathscr{M}$ is a piecewise linear manifold of dimension $N+1$, then an *orientation of* $\mathscr{M}$ is a choice of orientations $\{\mathbf{or}_\sigma\}_{\sigma\in\mathscr{M}}$ such that

$$\mathbf{or}_{\tau,\sigma_1} = -\mathbf{or}_{\tau,\sigma_2} \tag{8}$$

for each τ which is a facet of two different cells $\sigma_1,\sigma_2 \in \mathscr{M}$. By making use of the standard orientation

$$\mathbf{or}(b_1,\ldots,b_N) := \operatorname{sign}\det(b_1,\ldots,b_N)$$

of $\mathbb{R}^N$, it is clear that any piecewise linear manifold of dimension N which subdivides a subset of $\mathbb{R}^N$ is oriented in a natural way.

If $H:\mathscr{M}\to\mathbb{R}^N$ is a piecewise linear map on a piecewise linear manifold of dimension $N+1$ such that zero is a regular value of H, then it is clear that the system

$$\ker H := \{\sigma\cap H^{-1}(0)\}_{\sigma\in\mathscr{M}}$$

is a one-dimensional piecewise linear manifold which subdivides the solution set $H^{-1}(0)$. For the case that $\mathscr{M}$ is oriented, the orientation of $\mathscr{M}$ and the natural orientation of $\mathbb{R}^N$ induce an orientation of $\ker H$. Namely, for $\xi\in\ker H$, $v\in\operatorname{tng}(\xi)$ and $\sigma\in\mathscr{M}$ such that $\xi\subset\sigma$, the definition

$$\mathbf{or}_\xi(v) := \mathbf{or}_\sigma(b_1,\ldots,b_N,v)\ \operatorname{sign}\det(H'_\sigma b_1,\ldots,H'_\sigma b_N) \tag{9}$$

is independent of the special choice of $b_1,\ldots,b_N\in\operatorname{tng}(\sigma)$, provided the $b_1,\ldots,b_N$ are linearly independent. Clearly, an orientation of the one-dimensional manifold $\ker H$ is just a rule which indicates a direction for traversing each connected component of $\ker H$. Keeping this in mind, we now briefly indicate why the above definition indeed yields an orientation for $\ker H$.

Let τ be a facet of $\mathscr{M}$ which meets $H^{-1}(0)$ and does not belong to the boundary $\partial\mathscr{M}$, let $\sigma_1,\sigma_2\in\mathscr{M}$ be the two cells containing τ, and let $\xi_j := H^{-1}(0)\cap\sigma_j\in\ker H$ for $j=1,2$. If $b_1,\ldots,b_N$ is a basis of $\operatorname{tng}(\tau)$, and if $a_j\in\operatorname{tng}(\xi_j)$ points from τ into σ_j, then from condition (8) it follows that

$$\mathbf{or}_{\sigma_1}(b_1,\ldots,b_N,a_1) = -\mathbf{or}_{\sigma_2}(b_1,\ldots,b_N,a_2)$$

and hence (9) implies that

$$\mathbf{or}_{\xi_1}(a_1) = -\mathbf{or}_{\xi_2}(a_2),$$

which is exactly the right condition in the sense of (8) to ensure that the manifold $\ker H$ is oriented.

7. Lemke's algorithm

The first prominent example of a piecewise linear algorithm was designed in [26,27] to calculate a solution of the linear complementarity problem. Subsequently, several authors have studied complementarity problems from the standpoint of piecewise linear homotopy methods; see the references

in [3, Section 38]. Complementarity problems can also be handled via interior point methods (see [29,37]). Linear complementarity problems arise in quadratic programming, bimatrix games, variational inequalities and economic equilibria problems. Hence numerical methods for their solution have been of considerable interest. For further references, see [10].

We present the Lemke algorithm as an example of a piecewise linear algorithm since it played a crucial role in the development of subsequent piecewise linear algorithms. Let us consider the following *linear complementarity problem*: Given an affine map $g:\mathbb{R}^N \to \mathbb{R}^N$, find an $x \in \mathbb{R}^N$ such that

$$x \in \mathbb{R}^N_+, \quad g(x) \in \mathbb{R}^N_+, \quad x^* g(x) = 0. \tag{10}$$

If $g(0) \in \mathbb{R}^N_+$, then $x = 0$ is a trivial solution to the problem. Hence, this trivial case is always excluded and the additional assumption $g(0) \notin \mathbb{R}^N_+$ is made.

It is not difficult to show the following: Define $f:\mathbb{R}^N \to \mathbb{R}^N$ by $f(z) := g(z_+) - z_-$. If x is a solution of the linear complementarity problem, then $z := x - g(x)$ is a zero point of f. Conversely, if z is a zero point of f, then $x := z_+$ solves the linear complementarity problem.

The advantage which f provides is that it is obviously a piecewise linear map if we subdivide $\mathbb{R}^N$ into orthants. This is the basis for our description of Lemke's algorithm. For a fixed $d \in \mathbb{R}^N_{++}$ we define the homotopy $H:\mathbb{R}^N \times [0,\infty) \to \mathbb{R}^N$ by

$$H(x,\lambda) := f(x) + \lambda d. \tag{11}$$

For a given subset $I \subset \{1,2,\ldots,N\}$ an orthant can be written in the form

$$\sigma_I := \{(x,\lambda):\ \lambda \geqslant 0,\ e_i^* x \geqslant 0 \text{ for } i \in I,\ e_i^* x \leqslant 0 \text{ for } i \in I'\}, \tag{12}$$

where I' denotes the complement of I. The collection of all such orthants forms a piecewise linear manifold $\mathscr{M}$ (of dimension $N+1$) which subdivides $\mathbb{R}^N \times [0,\infty)$. Furthermore, it is clear that $H:\mathscr{M} \to \mathbb{R}^N$ is a piecewise linear map since $x \mapsto x_+$ switches its linearity character only at the co-ordinate hyperplanes.

Let us assume for simplicity that zero is a regular value of H. We note however, that the case of a singular value is treated in the same way by using perturbation techniques. Lemke's algorithm is started on a ray: if $\lambda > 0$ is sufficiently large, then

$$(-g(0) - \lambda d)_+ = 0 \quad \text{and} \quad (-g(0) - \lambda d)_- = g(0) + \lambda d \in \mathbb{R}^N_{++}$$

and consequently

$$H(-g(0) - \lambda d, \lambda) = 0.$$

Hence, the ray defined by

$$\lambda \in [\lambda_0, \infty) \mapsto -g(0) - \lambda d \in \sigma_\emptyset \tag{13}$$

$$\text{for} \quad \lambda_0 := \max_{i=1,\ldots,N} \frac{-e_i^* g(0)}{e_i^* d} \tag{14}$$

is used (for decreasing λ-values) to start the path following. Since the piecewise linear manifold $\mathscr{M}$ consists of the orthants of $\mathbb{R}^N \times [0,\infty)$, it is finite, and there are only two possibilities:

1. The algorithm terminates on the boundary $|\partial\mathscr{M}| = \mathbb{R}^N \times \{0\}$ at a point $(z,0)$. Then z is a zero point of f, and hence z_+ solves the linear complementarity problem.

2. The algorithm terminates on a secondary ray. Then it can be shown (see [9]), that the linear complementarity problem has no solution, at least if the Jacobian g' belongs to a certain class of matrices.

Let us illustrate the use of index and orientation by showing that the algorithm generates a solution in the sense that it terminates on the boundary under the assumption that all principal minors of the Jacobian g' are positive. Note that the Jacobian g' is a constant matrix since g is affine.

For $\sigma_I \in \mathscr{M}$, see (12), we immediately calculate the Jacobian

$$H'_{\sigma_I} = (f'_{\sigma_I}, d),$$

where

$$f'_{\sigma_I} e_i = \begin{cases} g' e_i & \text{for } i \in I, \\ e_i & \text{for } i \in I'. \end{cases} \tag{15}$$

If $\xi \in \ker H$ is a solution path in σ_I, then formula (9) yields

$$\mathbf{or}_\xi(v) = \operatorname{sign} \det f'_{\sigma_I} \ \mathbf{or}_{\sigma_I}(e_1, \ldots, e_N, v)$$

and since $\mathbf{or}_{\sigma_I}(e_1, \ldots, e_N, v) = \operatorname{sign}(v^* e_{N+1})$ by the standard orientation in $\mathbb{R}^{N+1}$, we have that $\det f'_{\sigma_I}$ is positive or negative if and only if the λ-direction is increasing or decreasing, respectively, while ξ is traversed according to its orientation. It is immediately seen from (15) that $\det f'_{\sigma_I}$ is obtained as a *principal minor* of g', i.e., by deleting all columns and rows of g' with index $i \in I'$ and taking the determinant of the resulting matrix (where the determinant of the "empty matrix" is assumed to be 1). Since we start in the negative orthant $\sigma_\emptyset$ where the principal minor is 1, we see that the algorithm traverses the primary ray against its orientation, because the λ-values are initially decreased. Hence, the algorithm continues to traverse $\ker H$ against its orientation. For the important case that all principal minors of g' are positive, the algorithm must continue to decrease the λ-values and thus it stops at the boundary $|\partial \mathscr{M}| = \mathbb{R}^N \times \{0\}$. Hence, in this case the algorithm finds a solution. Furthermore, it is clear that this solution is unique, since $\ker H$ can contain no other ray than the primary ray.

8. Further aspects of piecewise linear algorithms

Lack of space precludes the presentation of specific details of the extensive activity in piecewise linear methods which took place in the eighties and nineties. In particular, considerable activity took place on variable dimension algorithms, studies were made on the efficiency of triangulations, and on the complexity of piecewise linear methods. Literature of these developments until approximately 1994 can be found in [3]. The Netherlands school which works on piecewise linear methods continues to be active in this field, see, e.g., the recent publications and references cited therein: [15,23,24,36].

Many of the newer developments can be generally described in the following way: Very special piecewise linear manifolds are constructed for special classes of problems, e.g., special economic equilibrium problems or special complementarity problems. The aims are to fit the construction of the manifold to the problem in such a way that a convergence proof, leading to an existence theorem for

solutions, can be carried out, and/or the resulting piecewise linear algorithm is easily implemented and becomes very efficient.

9. Approximating manifolds

Let us now consider the case $K > 1$. The ideas of numerical continuation [3] and piecewise linear methods can be extended to the approximation of implicitly defined manifolds $\tilde{H}^{-1}(0)$ where $\tilde{H} : \mathbb{R}^{N+K} \to \mathbb{R}^N$.

For simplicity, we assume in this section that zero is a regular value of the smooth map $\tilde{H} : \mathbb{R}^{N+K} \to \mathbb{R}^N$. Hence $\tilde{\mathcal{M}} := \tilde{H}^{-1}(0)$ is a smooth K-dimensional manifold. Before we discuss the methods for obtaining piecewise linear approximations of $\tilde{\mathcal{M}}$, let us briefly indicate the well-known fact that the Gauss–Newton method can be used to obtain a nonlinear projector P from a neighborhood U of $\tilde{\mathcal{M}}$ onto $\tilde{\mathcal{M}}$.

More precisely, given a point $v_0 \in U$, the sequence

$$v_{i+1} = v_i - \tilde{H}'(v_i)^+ \tilde{H}(v_i), \quad i = 0, 1, \ldots \tag{16}$$

converges quadratically to a point $v_\infty := P(v_0) \in \tilde{\mathcal{M}}$. Here $\tilde{H}'(v_i)^+$ denotes the Moore–Penrose inverse.

Rheinboldt [30], has exploited this idea to project a standard triangulation of the tangent space at a point of $\tilde{\mathcal{M}}$ onto $\tilde{\mathcal{M}}$, which leads to a local triangulation of the manifold in a neighborhood of that point. This method will be discussed elsewhere in this volume. The method is well-suited for approximating smooth manifolds in which the dimension N is large, such as in multiple parameter nonlinear eigenvalue problems (see, e.g., [30]). It has been applied to the calculation of fold curves and to differential–algebraic equations (see [13]).

The approximation of implicit surfaces has also been an active area of research in computer graphics, where $H : \mathbb{R}^3 \to \mathbb{R}^1$ (see [7] for bibliography and references to software).

A global approximation of $\tilde{\mathcal{M}}$ can be obtained via piecewise linear algorithms. This has been developed in [5,6] (see also [2, Chapter 15; 3, Section 40.2]).

Piecewise linear algorithms have been applied to the visualization of body surfaces, and to the approximation of surface and body integrals [4] (see also [3, Section 41]). They can also be used as automatic mesh generators for boundary element methods [22]. For software for surface and volume approximation via piecewise linear methods; see the URL of the second author.

We begin with a description of the underlying ideas. Let us suppose that $\mathcal{T}$ triangulates the space $\mathbb{R}^{N+K}$. An important advantage of the usual standard triangulations is that any simplex can be very compactly stored and cheaply recovered by means of an $(N+K)$-tuple of integers m corresponding to its barycenter. It is also possible to perform the pivoting steps directly on the integer vector m and thereby to save some arithmetic operations.

As in Section 3, let H denote the piecewise linear approximation of $\tilde{H}$ with respect to $\mathcal{T}$. The definitions of regular points and regular values extend analogously to this context. We again obtain a perturbation theorem, i.e., the proof of Theorem 3.1 involving ε-perturbations, generalizes verbatim if 1 is replaced by K.

If zero is a regular value of H, the zero set $H^{-1}(0)$ carries the structure of a K-dimensional piecewise linear manifold. We formulate this last remark more precisely.

Theorem 9.1. *Let zero be a regular value of H. If $\sigma \in \mathscr{T}$ has a non-empty intersection with $H^{-1}(0)$, then $\mathscr{M}_\sigma := \sigma \cap H^{-1}(0)$ is a K-dimensional polytope, and the family*

$$\mathscr{M} := \{\mathscr{M}_\sigma : \sigma \in \mathscr{T},\ \sigma \cap H^{-1}(0) \neq \emptyset\}$$

is a K-dimensional piecewise linear manifold.

The following algorithm describes the fundamental steps for obtaining the piecewise linear manifold $\mathscr{M}$ approximating $\tilde{\mathscr{M}}$. We again make the assumptions that $\tilde{H} : \mathbb{R}^{N+K} \to \mathbb{R}^N$ is a smooth map, $\mathscr{T}$ is a triangulation of $\mathbb{R}^{N+K}$, and zero is a regular value of both $\tilde{H}$ and its piecewise linear approximation H. Analogously to the definitions preceding Algorithm 3.1, we call a simplex $\sigma \in \mathscr{T}$ *transverse* if it contains an N-face which is completely labeled with respect to H; see Definition 3.1. In the algorithm below, the input is one transverse simplex σ, and the output is the maximal set Σ of transverse simplices that meet a certain given compact domain D and are connected to σ. For any transverse simplex $\sigma \in \Sigma$, the dynamically varying set $V(\sigma)$ keeps track of all vertices which remain to be checked in order to find all possible new transverse simplices via pivoting.

Algorithm 9.1 (PL Approximation of a Manifold)**.**

1. input:
   (a) a transverse starting simplex $\sigma \in \mathscr{T}$
   (b) a compact subset $D \subset \mathbb{R}^{N+K}$ for bounding the search
2. initialization:
   $\Sigma := \{\sigma\}$ and $V(\sigma) :=$ set of vertices of σ
3. while $V(\sigma) \neq \emptyset$ for some $\sigma \in \Sigma$
   (a) get $\sigma \in \Sigma$ such that $V(\sigma) \neq \emptyset$, and get $v \in V(\sigma)$
   (b) pivot the vertex v into v' to get an adjacent simplex σ'
   (c) if $\sigma' \cap \sigma$ is not transverse or $\sigma' \cap D = \emptyset$
   delete v from $V(\sigma)$
   (d) else if σ' is not new, i.e., $\sigma' \in \Sigma$
   delete v from $V(\sigma)$ and v' from $V(\sigma')$
   (e) else σ' is added to the list Σ, i.e.,
   (i) $\Sigma := \Sigma \cup \{\sigma'\}$
   (ii) $V(\sigma') :=$ set of vertices of σ'
   (iii) delete v from $V(\sigma)$ and v' from $V(\sigma')$

For purposes of exposition, we have formulated the above algorithm in a very general way. A number of items remain to be discussed. We will show below how a starting simplex in 1a can be obtained in the neighborhood of a point $x \in \tilde{\mathscr{M}}$. The list Σ can be used to generate a K-dimensional connected piecewise linear manifold

$$\mathscr{M} := \{\mathscr{M}_\sigma\}_{\sigma \in \Sigma}$$

(see Theorem 9.1). This piecewise linear manifold approximates $\tilde{\mathscr{M}}$ quadratically in the mesh size of $\mathscr{T}$, as was shown in [1] (see also [3, Section 40.3]). If $\tilde{\mathscr{M}}$ is compact, the generated piecewise linear manifold will be compact without boundary, provided the mesh of the triangulation is sufficiently small and the bounding set D is sufficiently large. It is not really necessary to perform the pivot

in 3(b) if $\sigma' \cap \sigma$ is not transverse, since this will already be known from the current data. In the comparing process 3(d), it is crucial that compact exact storing is possible for standard triangulations. The list searching in 3(a) and 3(d) can be performed via efficient binary tree searching.

The piecewise linear manifold $\mathscr{M}$ furnishes an initial coarse piecewise linear approximation of $\tilde{\mathscr{M}}$. Several improvements are possible. The first is that a Gauss–Newton type method as in (16) can be used to project the nodes of $\mathscr{M}$ onto $\tilde{\mathscr{M}}$. Thus a new piecewise linear manifold $\mathscr{M}_1$ is generated which inherits the adjacency structure of the nodes from $\mathscr{M}$ and has nodes on $\tilde{\mathscr{M}}$.

In many applications (e.g., boundary element methods) it is desirable to uniformize the mesh $\mathscr{M}_1$. A very simple and successful means of doing this is "mesh smoothing". One such possible method consists of replacing each node of the mesh by the average of the nodes with which it shares an edge and by using the resulting point as a starting value for a Gauss–Newton type process to iterate back to $\tilde{\mathscr{M}}$. The edges or nodal adjacencies are maintained as before. Three or four sweeps of this smoothing process over all of the nodes of $\mathscr{M}_1$ generally yields a very uniform piecewise linear approximation of $\tilde{\mathscr{M}}$.

Another step which is useful for applications such as boundary element methods is to locally subdivide the cells of the piecewise linear manifolds $\mathscr{M}$ or $\mathscr{M}_1$ into simplices in such a way that the resulting manifold can be given the structure of the pseudo-manifold $\mathscr{M}_2$.

Once an approximating pseudo-manifold $\mathscr{M}_2$ has been generated, it is easy to refine it by, e.g., the well-known construction of halving all edges of each simplex $\tau \in \mathscr{M}_2$, triangulating it into 2^K subsimplices and projecting the new nodes back onto $\tilde{\mathscr{M}}$.

We have assumed that zero is a regular value of H. In fact, as in the Perturbation Theorem 3.1 and following remarks, $\vec{\varepsilon}$-perturbations and the corresponding general definition "completely labeled" automatically resolves singularities even if zero is not a regular value of H. The situation is similar to the case $K = 1$.

Let us next address the question of obtaining a transverse starting simplex. If we assume that a point x on $\tilde{\mathscr{M}}$ is given, then it can be shown that any $(N + K)$-simplex with barycenter x and sufficiently small diameter is transverse (see [3, Section 40.3]).

Algorithm 9.1 merely generates a list Σ of transverse simplices. For particular purposes such as boundary element methods, computer graphics, etc., a user will wish to have more information concerning the structure of the piecewise linear manifold $\mathscr{M}$, e.g., all nodes of the piecewise linear manifold $\mathscr{M}$ together with their adjacency structure. Hence, to meet such requirements, it is necessary to customize the above algorithm for the purpose at hand.

References

[1] E.L. Allgower, K. Georg, Estimates for piecewise linear approximations of implicitly defined manifolds, Appl. Math. Lett. 1 (5) (1989) 1–7.

[2] E.L. Allgower, K. Georg, Numerical Continuation Methods: An Introduction, Series in Computational Mathematics, Vol. 13, Springer, Berlin, 1990, p. 388.

[3] E.L. Allgower, K. Georg, Numerical path following, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, Vol. 5, North-Holland, Amsterdam, 1997, pp. 3–207.

[4] E.L. Allgower, K. Georg, R. Widmann, Volume integrals for boundary element methods, J. Comput. Appl. Math. 38 (1991) 17–29.

[5] E.L. Allgower, S. Gnutzmann, An algorithm for piecewise linear approximation of implicitly defined two-dimensional surfaces, SIAM J. Numer. Anal. 24 (1987) 452–469.

[6] E.L. Allgower, Ph.H. Schmidt, An algorithm for piecewise-linear approximation of an implicitly defined manifold, SIAM J. Numer. Anal. 22 (1985) 322–346.
[7] J. Bloomenthal, Introduction to Implicit Surfaces, Morgan Kaufmann, San Francisco, 1997.
[8] A. Charnes, Optimality and degeneracy in linear programming, Econometrica 20 (1952) 160–170.
[9] R.W. Cottle, Solution rays for a class of complementarity problems, Math. Programming Study 1 (1974) 58–70.
[10] R.W. Cottle, J.-S. Pang, R.E. Stone, The Linear Complementarity Problem, Academic Press, New York, 1992.
[11] H.S.M. Coxeter, Discrete groups generated by reflections, Ann. Math. 6 (1934) 13–29.
[12] H.S.M. Coxeter, Regular Polytopes, 3rd ed., MacMillan, New York, 1963.
[13] R.-X. Dai, W.C. Rheinboldt, On the computation of manifolds of fold points for parameter-dependent problems, SIAM J. Numer. Anal. 27 (1990) 437–466.
[14] C. Dang, Triangulations and Simplicial Methods, Lecture Notes in Economics and Mathematical Systems, Springer, Berlin, 1995.
[15] C. Dang, H. van Maaren, A simplicial approach to integer programming, Math. Oper. Res. 23 (1998) 403–415.
[16] G.B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, NJ, 1963.
[17] B.C. Eaves, Homotopies for the computation of fixed points, Math. Programming 3 (1972) 1–22.
[18] B.C. Eaves, A short course in solving equations with PL homotopies, R.W. Cottle, C.E. Lemke (Eds.), Nonlinear Programming, SIAM-AMS Proceedings, Vol. 9, American Mathematical Society, Providence, RI, 1976, pp. 73–143.
[19] B.C. Eaves, A Course in Triangulations for Solving Equations with Deformations, Lecture Notes in Economics and Mathematical Systems, Vol. 234, Springer, Berlin, 1984.
[20] B.C. Eaves, R. Saigal, Homotopies for computation of fixed points on unbounded regions, Math. Programming 3 (1972) 225–237.
[21] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, Ann. Math. 43 (1942) 580–582.
[22] K. Georg, Approximation of integrals for boundary element methods, SIAM J. Sci. Statist. Comput. 12 (1991) 443–453.
[23] P.J.J. Herings, A.J.J. Talman, Intersection theorems with a continuum of intersection points, J. Optim. Theory Appl. 96 (1998) 311–335.
[24] A. Kirman (Ed.), Elements of General Equilibrium Analysis, Blackwell, Oxford, 1998.
[25] M. Kojima, N. Megiddo, T. Noma, A. Yoshise, A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems, Lecture Notes in Computer Science, Vol. 538, Springer, Berlin, 1991.
[26] C.E. Lemke, Bimatrix equilibrium points and mathematical programming, Management Sci. 11 (1965) 681–689.
[27] C.E. Lemke, J.T. Howson, Equilibrium points of bimatrix games, SIAM J. Appl. Math. 12 (1964) 413–423.
[28] O. Merrill, Applications and extensions of an algorithm that computes fixed points of a certain upper semi-continuous point to set mapping, Ph.D. Thesis, University of Michigan, Ann Arbor, MI, 1972.
[29] Yu.E. Nesterov, A. Nemirovskii, Interior-Point Polynomial Algorithms in Convex Programming, SIAM, Philadelphia, PA, 1993.
[30] W.C. Rheinboldt, On the computation of multi-dimensional solution manifolds of parameterized equations, Numer. Math. 53 (1988) 165–182.
[31] R. Saigal, M.J. Todd, Efficient acceleration techniques for fixed point algorithms, SIAM J. Number. Anal. 15 (1978) 997–1007.
[32] H.E. Scarf, The approximation of fixed points of a continuous mapping, SIAM J. Appl. Math. 15 (1967) 1328–1343.
[33] H.E. Scarf, T. Hansen, The Computation on Economic Equilibria, Yale University Press, New Haven, CT, 1973.
[34] M.J. Todd, The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems, Vol. 124, Springer, Berlin, 1976.
[35] M.J. Todd, Piecewise-linear homotopy algorithms for sparse systems of nonlinear equations, SIAM J. Control Optim. 21 (1983) 204–214.
[36] G. van der Laan, A.J.J. Talman, Z. Yang, Existence and approximation of robust stationary points on polytopes, SIAM J. Control Optim., in preparation.
[37] S.J. Wright, Primal–Dual Interior-Point Methods, SIAM, Philadelphia, PA, 1997.