



## Combining linear-time temporal logic with constructiveness and paraconsistency

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### ABSTRACT

It is known that linear-time temporal logic (LTL), which is an extension of classical logic, is useful for expressing temporal reasoning as investigated in computer science. In this paper, two constructive and bounded versions of LTL, which are extensions of intuitionistic logic or Nelson's paraconsistent logic, are introduced as Gentzen-type sequent calculi. These logics, IB[I] and PB[I], are intended to provide a useful theoretical basis for representing not only temporal (linear-time), but also constructive, and paraconsistent (inconsistency-tolerant) reasoning. The time domain of the proposed logics is bounded by a fixed positive integer. Despite the restriction on the time domain, the logics can derive almost all the typical temporal axioms of LTL. As a merit of bounding time, faithful embeddings into intuitionistic logic and Nelson's paraconsistent logic are shown for IB[I] and PB[I], respectively. Completeness (with respect to Kripke semantics), cut-elimination, normalization (with respect to natural deduction), and decidability theorems for the newly defined logics are proved as the main results of this paper. Moreover, we present sound and complete display calculi for IB[I] and PB[I].

In [P. Maier, Intuitionistic LTL and a new characterization of safety and liveness, in: Proceedings of Computer Science Logic 2004, in: Lecture Notes in Computer Science, vol. 3210, Springer-Verlag, Berlin, 2004, pp. 295–309] it has been emphasized that intuitionistic linear-time logic (ILTL) admits an elegant characterization of safety and liveness properties. The system ILTL, however, has been presented only in an algebraic setting. The present paper is the first semantical and proof-theoretical study of bounded constructive linear-time temporal logics containing either intuitionistic or strong negation.

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## 1. Introduction

### 1.1. Constructive linear-time temporal logics

It is known that *linear-time temporal logic* (LTL) is very useful for verifying and specifying concurrent systems [8]. Gentzen-type sequent calculi for LTL and its neighbors have been introduced by many researchers. For example, a sequent calculus  $LT_\omega$  for LTL, which is precisely a system for Kröger's *infinitary temporal logic* [18], was introduced by Kawai [17], who proved cut-elimination and completeness theorems for this calculus. An alternative proof of the cut-elimination theorem for  $LT_\omega$  was given by introducing an embedding of  $LT_\omega$  into a sequent calculus for *infinitary logic*, see [15].

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In the present paper, two constructive (or intuitionistic) and bounded versions of  $LT_\omega$ , which have embeddings into intuitionistic logic and Nelson's paraconsistent logic rather than infinitary logic, are studied. The first one, which is an extension of intuitionistic logic, is called *intuitionistic bounded linear-time temporal logic* (denoted as  $IB[l]$ ), and the second one, which is an extension of Nelson's paraconsistent logic N4 [2,21,33], is called *paraconsistent bounded linear-time temporal logic* (denoted as  $PB[l]$ ). Completeness (w.r.t. Kripke semantics), embedding, cut-elimination, normalization (w.r.t. natural deduction) and decidability theorems for  $IB[l]$  and  $PB[l]$  are proved as the main results of this paper. The logics  $IB[l]$  and  $PB[l]$  are intended to give a useful theoretical basis for adequately representing not only temporal (linear-time), but also constructive and paraconsistent (inconsistency-tolerant) reasoning.

Whereas the Hilbert-style axiom scheme for the temporal operators  $G$  (globally) and  $X$  (next):  $G\alpha \leftrightarrow (\alpha \wedge X\alpha \wedge X^2\alpha \wedge \dots \wedge X^i\alpha)$ , where  $X^i\alpha$  means  $\overbrace{XX \dots X}^i \alpha$ , is characteristic of  $LT_\omega$ , the axiom scheme:  $G\alpha \leftrightarrow (\alpha \wedge X\alpha \wedge X^2\alpha \wedge \dots \wedge X^l\alpha)$ , which may be regarded as a finite approximation of the original scheme, is characteristic of the logics  $IB[l]$  and  $PB[l]$ . Then the following very informal correspondences are useful to understand these logics:  $G\alpha$  in  $LT_\omega$  corresponds to the infinite conjunction  $\bigwedge_{j=0}^{\infty} X^j\alpha$  in infinitary logic (extended by  $X^i$ ), and  $G\alpha$  in  $IB[l]$  and  $PB[l]$  corresponds to the finite conjunction  $\bigwedge_{j=0}^l X^j\alpha$  in intuitionistic or Nelson's paraconsistent logic (extended by  $X^i$ ).

### 1.2. Why do we bound the time domain?

Although the standard LTL has an infinite (unbounded) time domain, namely the set  $\omega$  of all natural numbers, the logics  $IB[l]$  and  $PB[l]$  have a *bounded time domain* which is restricted by a fixed positive integer  $l$ , i.e., the set  $\omega_l := \{x \in \omega \mid x \leq l\}$ . Despite the restriction on the time domain,  $IB[l]$  and  $PB[l]$  can derive almost all the typical temporal axioms of LTL, such as a *time induction axiom*. As mentioned before,  $IB[l]$  and  $PB[l]$  allow us to obtain simple embeddings into intuitionistic logic and Nelson's paraconsistent logic, respectively. Using the embedding results, cut-elimination and decidability theorems for these logics can be derived. Moreover, a completeness theorem (w.r.t. Kripke semantics) and a normalization theorem (w.r.t. natural deduction) can be obtained. Such a theoretical merit may not be obtained for an unbounded and intuitionistic version of LTL, because the unbounded time domain requires some infinite inference rules. Such infinite inference rules are neither familiar to nor welcomed by researchers who study automated reasoning, since these rules cannot be implemented efficiently. Indeed, the replacement of such infinite rules of certain proof systems by finitary rules is known as an important issue.

To restrict the time domain in LTL is not a new idea. Such an idea was discussed, for instance, in [5,7,12]. By using and introducing a bounded time domain and the notion of bounded validity, *bounded tableaux calculi* (with temporal constraints) for propositional and first-order LTLs were studied by Cerrito, Mayer and Prand [7]. It is also known that to restrict the time domain is a technique that may be applied to obtain a decidable or efficient fragment of LTL [12]. Restricting the time domain implies not only some purely theoretical merits as mentioned above, but also some practical merits for describing temporal databases [7] and for implementing an efficient model checking algorithm, called *bounded model checking* [5]. Such practical merits are important due to the fact that there are problems in computer science and artificial intelligence where only a finite fragment of the time sequence is of interest [7]. We hope that  $IB[l]$  and  $PB[l]$  provide a good proof-theoretical basis for such practical applications as well as a good tool for automated reasoning with (bounded) linear-time formalisms.

### 1.3. Why do we use constructive and paraconsistent logics?

In (extensions of) standard classical propositional logic, the law of excluded middle  $\alpha \vee \neg\alpha$  is valid. This means that the information represented by classical logic is *complete information*: every formula  $\alpha$  is either true or not true in a model. Representing only complete information is plausible in classical mathematics, which is a discipline handling eternal truth and falsehood. The statements of classical mathematics do not change their truth value in the course of time, and the classical mathematician may assume every situation to support either the truth or the falsity of such a statement. The assumption of complete information is, however, inadequate when it comes to representing the information available to real world agents. We wish to explore the consequences of *incomplete* information about computer and information systems, and then it is desirable to avail of a logic which is *paracomplete* in the sense of not validating the law of excluded middle [22,37]. For representing the development of incomplete information over time, it turned out that constructive logics are useful as base logics for temporal reasoning. Indeed, *constructive (intuitionistic) modal and temporal logics* have been studied by several researchers, the *constructive concurrent dynamic logic* of Wijesekera and Nerode [37] being just one example of such logics. Particularly relevant for the present concerns is the intuitionistic linear-time temporal logic (ILTL) introduced in [19], which is a system that can be used to express properties relating finite and infinite behaviors. In [19], a logical characterization of safety and liveness properties is given: For every formula  $\alpha$ ,  $\alpha$  is (expresses) an intuitionistic safety (or liveness) property iff  $(F\perp \rightarrow \alpha) \rightarrow \alpha$  (or  $F\perp \rightarrow \alpha$ , resp.) is valid in ILTL. Moreover, the following decomposition theorem holds: For every formula  $\alpha$ ,  $\alpha \leftrightarrow ((F\perp \rightarrow \alpha) \wedge (F\perp \vee \alpha))$  is valid in ILTL. The system ILTL, however, is presented only in an algebraic setting. The

present paper is the first proof-theoretical and model-theoretical study of bounded constructive linear-time temporal logics containing either intuitionistic or strong negation.<sup>1</sup>

We wish to handle *inconsistent* as well as incomplete information, since some real systems such as software systems need to ensure inconsistency-tolerance. *Paraconsistent model checking* based on *many-valued temporal logics*, for instance, which was suggested by Easterbrook and Chechik [10], is intended to represent inconsistent information for requirements elicitation in software engineering. Whereas incomplete information calls for *paracomplete* logics, handling inconsistent information within a logic requires *paraconsistent logics* such as Nelson's N4, Dunn's and Belnap's four-valued logic, da Costa's C systems, or annotated logics. The present paper's approach is based on N4, since N4 is known as a very useful paraconsistent logic in philosophical logic, computer science, and AI (see, e.g., [22–24,32–34]) and because N4 is based on positive intuitionistic logic. A systematic and historical survey of paraconsistent logic can be found in [26,27].

The idea of combining time with paraconsistency is not a new idea. In order to express inconsistent states in temporal reasoning, *annotated temporal logics*  $\Delta^*\tau$ , which are combinations of annotated logics and LTL, were proposed by Abe and Akama [1]. The motivation for using PB[l] in the present paper is basically the same as the motivation given in [1]. Whereas Abe and Akama's approach is only semantical, the present approach is both semantical and proof-theoretical. A general theory of combining logics has been developed, for example, in [6].

## 2. Intuitionistic bounded linear-time temporal logic

### 2.1. Sequent calculus

*Formulas* of IB[l] are constructed from (countably many) propositional variables,  $\perp$  (the falsity constant),  $\rightarrow$  (implication),  $\wedge$  (conjunction),  $\vee$  (disjunction), G (globally), F (eventually) and X (next). Lower-case letters  $p, q, \dots$  are used to denote propositional variables, Greek lower-case letters  $\alpha, \beta, \dots$  are used to denote formulas, and Greek capital letters  $\Gamma, \Delta, \dots$  are used to represent finite (possibly empty) sequences of formulas. For any  $\sharp \in \{G, F, X\}$ , the expression  $\sharp\Gamma$  is used to denote the sequence  $\langle \sharp\gamma \mid \gamma \in \Gamma \rangle$ . The symbol  $\equiv$  is used to denote the equality of sequences of symbols. The symbol  $\omega$  or  $N$  is used to represent the set of natural numbers. Let  $l$  be a fixed positive integer. The symbol  $\omega_l$  or  $N_l$  is used to represent the set  $\{i \in \omega \mid i \leq l\}$ . The expression  $X^i\alpha$  for any  $i \in \omega$  is inductively defined by  $(X^0\alpha \equiv \alpha)$  and  $(X^{n+1}\alpha \equiv X^nX\alpha)$ . Lower-case letters  $i, j$  and  $k$  are used to denote any natural numbers. An expression of the form  $\Gamma \Rightarrow \Delta$  where  $\Delta$  is empty or a single formula is called a *sequent* (for IB[l]). An expression  $L \vdash S$  is used to denote the fact that a sequent  $S$  is provable in a sequent calculus  $L$ .

**Definition 1** (IB[l]). Let  $l$  be a fixed positive integer. In the following definition,  $\Delta$  represents the empty sequence or a single formula.

The initial sequents of IB[l] are of the following form, where  $p$  is any propositional variable:

$$X^i p \Rightarrow X^i p \quad X^i \perp \Rightarrow$$

The structural rules of IB[l] are of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \alpha} \text{ (we-right)}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (co)} \quad \frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Delta} \text{ (ex)}$$

The logical inference rules of IB[l] are of the following form, for any  $k \in \omega_l$  and any positive integer  $m$ :

$$\frac{\Gamma \Rightarrow X^i \alpha \quad X^i \beta, \Sigma \Rightarrow \Delta}{X^i(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta} \text{ (}\rightarrow\text{left)} \quad \frac{X^i \alpha, \Gamma \Rightarrow X^i \beta}{\Gamma \Rightarrow X^i(\alpha \rightarrow \beta)} \text{ (}\rightarrow\text{right)}$$

$$\frac{X^i \alpha, \Gamma \Rightarrow \Delta}{X^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left1)} \quad \frac{X^i \beta, \Gamma \Rightarrow \Delta}{X^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left2)}$$

$$\frac{\Gamma \Rightarrow X^i \alpha \quad \Gamma \Rightarrow X^i \beta}{\Gamma \Rightarrow X^i(\alpha \wedge \beta)} \text{ (}\wedge\text{right)}$$

$$\frac{X^i \alpha, \Gamma \Rightarrow \Delta \quad X^i \beta, \Gamma \Rightarrow \Delta}{X^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}\vee\text{left)}$$

$$\frac{\Gamma \Rightarrow X^i \alpha}{\Gamma \Rightarrow X^i(\alpha \vee \beta)} \text{ (}\vee\text{right1)} \quad \frac{\Gamma \Rightarrow X^i \beta}{\Gamma \Rightarrow X^i(\alpha \vee \beta)} \text{ (}\vee\text{right2)}$$

<sup>1</sup> In [9], a Curry–Howard isomorphism for intuitionistic linear-time temporal logic in the language based on X and intuitionistic implication is established. Note, however, that the author extends this positive constructive logic by *classical* negation and that he uses a natural deduction system with time-annotated derivability relations inspired by [20]. Natural deduction proof systems and typed  $\lambda$ -calculi for bounded intuitionistic linear-time temporal logics are surveyed in [16]. See also [3].

$$\begin{array}{c}
\frac{X^l\alpha, \Gamma \Rightarrow \Delta}{X^{l+m}\alpha, \Gamma \Rightarrow \Delta} \text{ (Xleft)} \\
\frac{X^{i+k}\alpha, \Gamma \Rightarrow \Delta}{X^iG\alpha, \Gamma \Rightarrow \Delta} \text{ (Gleft)} \\
\frac{\{X^{i+j}\alpha, \Gamma \Rightarrow \Delta\}_{j \in \omega_l}}{X^iF\alpha, \Gamma \Rightarrow \Delta} \text{ (Fleft)} \\
\frac{\Gamma \Rightarrow X^l\alpha}{\Gamma \Rightarrow X^{l+m}\alpha} \text{ (Xright)} \\
\frac{\{\Gamma \Rightarrow X^{i+j}\alpha\}_{j \in \omega_l}}{\Gamma \Rightarrow X^iG\alpha} \text{ (Gright)} \\
\frac{\Gamma \Rightarrow X^{i+k}\alpha}{\Gamma \Rightarrow X^iF\alpha} \text{ (Fright)}
\end{array}$$

Note that for any formula  $\alpha$ , the sequent  $X^i\alpha \Rightarrow X^i\alpha$  is provable in  $IB[l]$ . This can be shown by induction on  $\alpha$ . Thus, the sequents of the form  $X^i\alpha \Rightarrow X^i\alpha$  can also be regarded as initial sequents.

It is remarked that  $IB[l]$  is just a logic parameterized by a fixed concrete positive integer  $l$ . Thus, before any detailed discussion, we have to fix  $IB[l]$  as a concrete logic such as  $IB[5]$ . Indeed, for example,  $IB[2]$  is different from  $IB[1]$ :  $p \wedge Xp \Rightarrow Gp$  is provable in  $IB[1]$ , but it is not provable in  $IB[2]$ . A proof of  $p \wedge Xp \Rightarrow Gp$  in  $IB[1]$  is presented below:

$$\begin{array}{c}
\frac{p \Rightarrow p}{Xp, p \Rightarrow p} \text{ (we-left)} \\
\frac{Xp \Rightarrow Xp}{Xp, p \Rightarrow Xp} \text{ (we-right)} \\
\frac{Xp, p \Rightarrow Xp}{Xp, p \Rightarrow Xp} \text{ (ex)} \\
\frac{Xp, p \Rightarrow Gp}{p \wedge Xp, p \Rightarrow Gp} \text{ (\wedge left2)} \\
\frac{p \wedge Xp, p \Rightarrow Gp}{p, p \wedge Xp \Rightarrow Gp} \text{ (ex)} \\
\frac{p, p \wedge Xp \Rightarrow Gp}{p \wedge Xp, p \wedge Xp \Rightarrow Gp} \text{ (\wedge left1)} \\
\frac{p \wedge Xp, p \wedge Xp \Rightarrow Gp}{p \wedge Xp \Rightarrow Gp} \text{ (co)}
\end{array}$$

It is noted that (Gright) and (Fleft) have  $l+1$  (i.e., a finite number of) premises. In (Gleft) and (Fright), the number  $k$  is bounded by  $l$ . Then  $IB[l]$  has the Hilbert-style axiom schemes  $G\alpha \leftrightarrow (\alpha \wedge X\alpha \wedge X^2\alpha \wedge \dots \wedge X^l\alpha)$  and  $F\alpha \leftrightarrow (\alpha \vee X\alpha \vee X^2\alpha \vee \dots \vee X^l\alpha)$ . By (Xleft) and (Xright), the nest of the outermost occurrences of  $X$  in a formula can be bounded by  $l$ . Indeed, (Xleft) and (Xright) correspond to the Hilbert-style axiom scheme  $X^{l+m}\alpha \leftrightarrow X^l\alpha$ .

We may regard  $IB[l]$  as an intuitionistic and bounded version of Kawai's sequent calculus  $LT_\omega$  for LTL [17].  $LT_\omega$  has no  $l$ -bounded rules  $\{(Xleft) (Xright)\}$ , and uses  $\omega$  instead of  $\omega_l$ .

**Proposition 2.** Let  $\Delta$  be the empty sequence or a single formula. The rule of the form:

$$\frac{\Gamma \Rightarrow \Delta}{X\Gamma \Rightarrow X\Delta} \text{ (Xregu)}$$

is admissible in cut-free  $IB[l]$ .

**Proof.** By induction on proofs  $P$  of  $\Gamma \Rightarrow \Delta$  in cut-free  $IB[l]$ . We distinguish the cases according to the last inference of  $P$ . We show some cases.

Case  $(X^i\perp \Rightarrow)$ . The last inference of  $P$  is of the form:  $X^i\perp \Rightarrow$ . In this case, we have  $IB[l] \vdash XX^i\perp \Rightarrow$ .

Case (Gleft). The last inference of  $P$  is of the form:

$$\frac{X^{i+k}\alpha, \Sigma \Rightarrow \Delta}{X^iG\alpha, \Sigma \Rightarrow \Delta} \text{ (Gleft)}$$

By induction hypothesis, we obtain:

$$\begin{array}{c}
\vdots \\
\vdots \\
\frac{XX^{i+k}\alpha, X\Sigma \Rightarrow X\Delta}{XX^iG\alpha, X\Sigma \Rightarrow X\Delta} \text{ (Gleft)}
\end{array}$$

Case  $(\rightarrow\text{left})$ . The last inference of  $P$  is of the form:

$$\frac{\Pi \Rightarrow X^i\alpha \quad X^i\beta, \Sigma \Rightarrow \Delta}{X^i(\alpha \rightarrow \beta), \Pi, \Sigma \Rightarrow \Delta} \text{ (\rightarrow left)}$$

By induction hypothesis, we obtain:

$$\begin{array}{c}
\vdots \\
\vdots \\
\frac{X\Pi \Rightarrow XX^i\alpha \quad XX^i\beta, X\Sigma \Rightarrow X\Delta}{XX^i(\alpha \rightarrow \beta), X\Pi, X\Sigma \Rightarrow X\Delta} \text{ (\rightarrow left)}
\end{array}$$

□

Note that the rule (Xregu) is more expressive than the following standard inference rules for the normal modal logic K and KD, respectively:

$$\frac{\Gamma \Rightarrow \alpha}{X\Gamma \Rightarrow X\alpha} \quad \frac{\Gamma \Rightarrow \gamma}{X\Gamma \Rightarrow X\gamma}$$

where  $\gamma$  can be empty.

**Proposition 3.** An expression  $\alpha \Leftrightarrow \beta$  means the sequents  $\alpha \Rightarrow \beta$  and  $\beta \Rightarrow \alpha$ . The following sequents are provable in IB[I], for any formulas  $\alpha, \beta$  and any  $i \in \omega$ :

1.  $X^i \perp \Leftrightarrow \perp$ ,
2.  $X^i(\alpha \circ \beta) \Leftrightarrow X^i \alpha \circ X^i \beta$  where  $\circ \in \{\rightarrow, \wedge, \vee\}$ ,
3.  $X^i G\alpha \Leftrightarrow GX^i \alpha$ ,
4.  $G\alpha \Rightarrow X\alpha$ ,
5.  $G\alpha \Rightarrow XG\alpha$ ,
6.  $G\alpha \Rightarrow GG\alpha$ ,
7.  $\alpha, G(\alpha \rightarrow X\alpha) \Rightarrow G\alpha$  (time induction),
8.  $G\alpha \Leftrightarrow \alpha \wedge X\alpha \wedge X^2\alpha \wedge \dots \wedge X^l \alpha$ ,
9.  $F\alpha \Leftrightarrow \alpha \vee X\alpha \vee X^2\alpha \vee \dots \vee X^l \alpha$ ,
10.  $X^{l+i} \alpha \Leftrightarrow X^{l+i} G\alpha$ ,
11.  $X^{l+i} \alpha \Leftrightarrow X^{l+i} F\alpha$ ,
12.  $X^{l+i} G\alpha \Leftrightarrow X^{l+i} F\alpha$ .

**Proof.** We show some cases.

$$(5) \quad \frac{\begin{array}{c} \vdots \\ X\alpha \Rightarrow X\alpha \\ G\alpha \Rightarrow X\alpha \end{array} \text{ (Gleft)} \quad \begin{array}{c} \vdots \\ X^2\alpha \Rightarrow X^2\alpha \\ G\alpha \Rightarrow X^2\alpha \end{array} \text{ (Gleft)} \quad \dots \quad \begin{array}{c} \vdots \\ X^n\alpha \Rightarrow X^n\alpha \\ G\alpha \Rightarrow X^n\alpha \end{array} \text{ (Gleft)} \quad \dots}{G\alpha \Rightarrow XG\alpha} \text{ (Gright)}$$

$$(6) \quad \frac{G\alpha \Rightarrow G\alpha \quad \begin{array}{c} \vdots \\ G\alpha \Rightarrow XG\alpha \end{array} \quad \begin{array}{c} \vdots \\ G\alpha \Rightarrow X^2G\alpha \end{array} \quad \dots \quad \begin{array}{c} \vdots \\ G\alpha \Rightarrow X^nG\alpha \end{array} \quad \dots}{G\alpha \Rightarrow GG\alpha} \text{ (Gright)}$$

where  $\vdash G\alpha \Rightarrow X^j G\alpha$  for any  $j \in \omega_l$  can be shown in a similar way as in (5).

(7) In the following proofs, the applications of (ex) are omitted.

$$\frac{\begin{array}{c} \vdots \\ \{\alpha, G(\alpha \rightarrow X\alpha) \Rightarrow X^k \alpha\}_{k \in \omega_l} \end{array}}{\alpha, G(\alpha \rightarrow X\alpha) \Rightarrow G\alpha} \text{ (Gright)}$$

where  $\vdash \alpha, G(\alpha \rightarrow X\alpha) \Rightarrow X^k \alpha$  for any  $k \in \omega_l$  is shown by mathematical induction on  $k$  as follows: the base step is obvious, and the induction step can be shown by

$$\begin{array}{c} \vdots \text{ ind.hyp.} \\ \frac{\alpha, G(\alpha \rightarrow X\alpha) \Rightarrow X^k \alpha \quad X^{k+1} \alpha \Rightarrow X^{k+1} \alpha}{\alpha, G(\alpha \rightarrow X\alpha), X^k(\alpha \rightarrow X\alpha) \Rightarrow X^{k+1} \alpha} (\rightarrow \text{left}) \\ \frac{\alpha, G(\alpha \rightarrow X\alpha), X^k(\alpha \rightarrow X\alpha) \Rightarrow X^{k+1} \alpha}{\alpha, G(\alpha \rightarrow X\alpha), G(\alpha \rightarrow X\alpha) \Rightarrow X^{k+1} \alpha} \text{ (Gleft)} \\ \frac{\alpha, G(\alpha \rightarrow X\alpha), G(\alpha \rightarrow X\alpha) \Rightarrow X^{k+1} \alpha}{\alpha, G(\alpha \rightarrow X\alpha) \Rightarrow X^{k+1} \alpha} \text{ (co)} \end{array}$$

$$(10) \quad \frac{\begin{array}{c} \vdots \\ X^l \alpha \Rightarrow X^l \alpha \end{array} \text{ (Xright)}}{\{X^l \alpha \Rightarrow X^{l+j} \alpha\}_{j \in \omega_l} \text{ (Gright)}} \quad \frac{\begin{array}{c} \vdots \\ X^l \alpha \Rightarrow X^l G\alpha \\ X^l \alpha \Rightarrow X^{l+i} G\alpha \end{array} \text{ (Xright)}}{\frac{X^l \alpha \Rightarrow X^{l+i} G\alpha}{X^{l+i} \alpha \Rightarrow X^{l+i} G\alpha} \text{ (Xleft)}} \quad \frac{\begin{array}{c} \vdots \\ X^{l+i} \alpha \Rightarrow X^{l+i} \alpha \end{array}}{X^{l+i} G\alpha \Rightarrow X^{l+i} \alpha} \text{ (Gleft)} \quad \square$$

**Definition 4 (LJ).** A sequent calculus LJ for propositional intuitionistic logic is obtained from IB[l] by deleting (Xleft), (Xright), (Gleft), (Gright), (Fleft), (Fright), and replacing  $X^i$  by  $X^0$ . The modified inference rules for LJ by replacing  $i$  by 0 are denoted by using “LJ” as a superscript, e.g.,  $(\rightarrow\text{left}^{\text{LJ}})$ .

Expressions like  $\bigwedge\{\alpha_i \mid i \in \omega_l\}$  and  $\bigvee\{\alpha_i \mid i \in \omega_l\}$  where  $\{\alpha_i \mid i \in \omega_l\}$  is a multiset mean  $\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_l$  and  $\alpha_0 \vee \alpha_1 \vee \dots \vee \alpha_l$ , respectively. For example,  $\bigwedge\{\alpha, \alpha, \beta\}$  means  $\alpha \wedge \alpha \wedge \beta$ . The following definition of the embedding function  $f$  is regarded as a finite analogue of the definition of the embedding function of  $\text{LT}_\omega$  into infinitary logic [15].

**Definition 5.** We fix a countable nonempty set  $\Phi$  of propositional variables and define the sets  $\Phi_i := \{p_i \mid p \in \Phi\}$  ( $1 \leq i \in \omega$ ) and  $\Phi_0 := \Phi$  of propositional variables. The language  $\mathcal{L}_{\text{IB}[l]}$  of IB[l] is defined by using  $\Phi$ ,  $\perp$ ,  $\rightarrow$ ,  $\wedge$ ,  $\vee$ , X, G and F. The language  $\mathcal{L}_{\text{LJ}}$  of LJ is defined by using  $\bigcup_{i \in \omega} \Phi_i$ ,  $\perp$ ,  $\rightarrow$ ,  $\wedge$  and  $\vee$ .

A mapping  $f$  from  $\mathcal{L}_{\text{IB}[l]}$  to  $\mathcal{L}_{\text{LJ}}$  is defined by the following clause, for any  $i \in \omega$  and any positive integer  $m$ :

1.  $f(X^i \perp) := \perp$ ,
2.  $f(X^i p) := p_i \in \Phi_i$  for any  $p \in \Phi$  (especially,  $f(p) := p \in \Phi_0$ ),
3.  $f(X^i(\alpha \circ \beta)) := f(X^i \alpha) \circ f(X^i \beta)$  where  $\circ \in \{\rightarrow, \wedge, \vee\}$ ,
4.  $f(X^{l+m} \alpha) := f(X^l \alpha)$ ,
5.  $f(X^i G \alpha) := \bigwedge\{f(X^{i+j} \alpha) \mid j \in \omega_l\}$ ,
6.  $f(X^i F \alpha) := \bigvee\{f(X^{i+j} \alpha) \mid j \in \omega_l\}$ .

The expression  $f(\Gamma)$  denotes the result of replacing every occurrence of a formula  $\alpha$  in  $\Gamma$  by an occurrence of  $f(\alpha)$ .

Strictly speaking, the embedding function  $f$  strongly depends on the time bound  $l$ , i.e.,  $f$  should be denoted as  $f_l$ . Indeed,  $f_3(Gp)$  and  $f_5(Gp)$  are different. But, for the sake of brevity, we will just use  $f$  in the following.

**Theorem 6 (Embedding).** Let  $\Gamma$  be a sequence of formulas in  $\mathcal{L}_{\text{IB}[l]}$ ,  $\Delta$  be the empty sequence or a formula in  $\mathcal{L}_{\text{IB}[l]}$ , and  $f$  be the mapping defined in Definition 5.

1.  $\text{IB}[l] \vdash \Gamma \Rightarrow \Delta$  iff  $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$ .
2.  $\text{IB}[l]\text{-cut} \vdash \Gamma \Rightarrow \Delta$  iff  $\text{LJ}\text{-cut} \vdash f(\Gamma) \Rightarrow f(\Delta)$ .

**Proof.** Since (2) follows from (1), we show only (1).

( $\Rightarrow$ ) By induction on proofs  $P$  of  $\Gamma \Rightarrow \Delta$  in IB[l]. We distinguish the cases according to the last inference of  $P$  and show some cases.

Case  $(X^i p \Rightarrow X^i p)$ . The last inference of  $P$  is of the form:  $X^i p \Rightarrow X^i p$ . In this case, we obtain  $f(X^i p) \Rightarrow f(X^i p)$ , i.e.,  $p_i \Rightarrow p_i$  ( $p_i \in \Phi_i$ ). This is an initial sequent of LJ.

Case  $(\rightarrow\text{left})$ . The last inference of  $P$  is of the form:

$$\frac{\Gamma \Rightarrow X^i \alpha \quad X^i \beta, \Sigma \Rightarrow \Delta}{X^i(\alpha \rightarrow \beta), \Gamma, \Sigma \Rightarrow \Delta} (\rightarrow\text{left})$$

By induction hypothesis, we have  $\text{LJ} \vdash f(\Gamma) \Rightarrow f(X^i \alpha)$  and  $\text{LJ} \vdash f(X^i \beta), f(\Sigma) \Rightarrow f(\Delta)$ . Then we obtain

$$\frac{\begin{array}{c} \vdots \\ f(\Gamma) \Rightarrow f(X^i \alpha) \end{array} \quad \begin{array}{c} \vdots \\ f(X^i \beta), f(\Sigma) \Rightarrow f(\Delta) \end{array}}{f(X^i \alpha) \rightarrow f(X^i \beta), f(\Gamma), f(\Sigma) \Rightarrow f(\Delta)} (\rightarrow\text{left}^{\text{LJ}})$$

where  $f(X^i \alpha) \rightarrow f(X^i \beta) = f(X^i(\alpha \rightarrow \beta))$  by the definition of  $f$ .

Case (Xleft). The last inference of  $P$  is of the form:

$$\frac{X^l \alpha, \Gamma \Rightarrow \Delta}{X^{l+m} \alpha, \Gamma \Rightarrow \Delta} (\text{Xleft})$$

By induction hypothesis, we have  $\text{LJ} \vdash f(X^l \alpha), f(\Gamma) \Rightarrow f(\Delta)$ , and  $f(X^l \alpha) = f(X^{l+m} \alpha)$  by the definition of  $f$ . Thus, we obtain  $\text{LJ} \vdash f(X^{l+m} \alpha), f(\Gamma) \Rightarrow f(\Delta)$ .

Case (Gleft). The last inference of  $P$  is of the form:

$$\frac{X^{i+k} \alpha, \Gamma \Rightarrow \Delta}{X^i G \alpha, \Gamma \Rightarrow \Delta} (\text{Gleft})$$

By induction hypothesis, we have  $\text{LJ} \vdash f(X^{i+k}\alpha), f(\Gamma) \Rightarrow f(\Delta)$ , and hence we obtain

$$\begin{array}{c} \vdots \\ f(X^{i+k}\alpha), f(\Gamma) \Rightarrow f(\Delta) \\ \vdots (\wedge\text{left}^{\text{LJ}}) \\ \bigwedge \{f(X^{i+j}\alpha) \mid j \in \omega_l\}, f(\Gamma) \Rightarrow f(\Delta) \end{array}$$

where  $\bigwedge \{f(X^{i+j}\alpha) \mid j \in \omega_l\} = f(X^i G\alpha)$  by the definition of  $f$ , and  $f(X^{i+k}\alpha)$  is in the multiset  $\{f(X^{i+j}\alpha) \mid j \in \omega_l\}$ . It is remarked that the case  $i > l$  is also included in this proof. In such a case,  $f(X^{i+k}\alpha)$  and  $\bigwedge \{f(X^{i+j}\alpha) \mid j \in \omega_l\}$  mean  $f(X^l\alpha)$

and  $f(X^l\alpha) \wedge f(X^l\alpha) \wedge \dots \wedge f(X^l\alpha)$ , respectively.

Case (Gright). The last inference of  $P$  is of the form:

$$\frac{\{ \Gamma \Rightarrow X^{i+j}\alpha \}_{j \in \omega_l}}{\Gamma \Rightarrow X^i G\alpha} \text{ (Gright)}$$

By induction hypothesis, we have  $\text{LJ} \vdash f(\Gamma) \Rightarrow f(X^{i+j}\alpha)$  for all  $j \in \omega_l$ . Let  $\Phi$  be the multiset  $\{f(X^{i+j}\alpha) \mid j \in \omega_l\}$ . We obtain

$$\begin{array}{c} \vdots \\ \{f(\Gamma) \Rightarrow f(X^{i+j}\alpha)\}_{f(X^{i+j}\alpha) \in \Phi} \\ \vdots (\wedge\text{right}^{\text{LJ}}) \\ f(\Gamma) \Rightarrow \bigwedge \Phi \end{array}$$

where  $\bigwedge \Phi = f(X^i G\alpha)$  by the definition of  $f$ .

( $\Leftarrow$ ) By induction on proofs  $Q$  of  $f(\Gamma) \Rightarrow f(\Delta)$  in LJ. We distinguish the cases according to the last inference of  $Q$ , and show only the following case.

Case ( $\wedge\text{right}^{\text{LJ}}$ ). The last inference of  $Q$  is of the form:

$$\frac{f(\Gamma) \Rightarrow f(X^i\alpha) \quad f(\Gamma) \Rightarrow f(X^i\beta)}{f(\Gamma) \Rightarrow f(X^i(\alpha \wedge \beta))} (\wedge\text{right}^{\text{LJ}})$$

where  $f(X^i(\alpha \wedge \beta)) = f(X^i\alpha) \wedge f(X^i\beta)$  by the definition of  $f$ . By induction hypothesis, we have  $\text{IB}[l] \vdash \Gamma \Rightarrow X^i\alpha$  and  $\text{IB}[l] \vdash \Gamma \Rightarrow X^i\beta$ . Then we obtain

$$\frac{\begin{array}{c} \vdots \\ \Gamma \Rightarrow X^i\alpha \end{array} \quad \begin{array}{c} \vdots \\ \Gamma \Rightarrow X^i\beta \end{array}}{\Gamma \Rightarrow X^i(\alpha \wedge \beta)} (\wedge\text{right}) \quad \square$$

Using this theorem, we can prove the following.

**Theorem 7 (Cut-elimination).** *The rule (cut) is admissible in cut-free  $\text{IB}[l]$ .*

**Proof.** Suppose  $\text{IB}[l] \vdash \Gamma \Rightarrow \Delta$ . Then we have  $\text{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta)$  by Theorem 6(1), and hence  $\text{LJ}-(\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$  by the well-known cut-elimination theorem for LJ. By Theorem 6(2), we obtain  $\text{IB}[l]-(\text{cut}) \vdash \Gamma \Rightarrow \Delta$ .  $\square$

Although in this paper the cut-elimination theorem for  $\text{IB}[l]$  is proved via an embedding theorem, a direct syntactical cut-elimination proof for  $\text{IB}[l]$  may be obtained using the standard way of Gentzen.

**Theorem 8 (Decidability).**  *$\text{IB}[l]$  is decidable.*

**Proof.** By Theorem 6, provability in  $\text{IB}[l]$  can be reduced to provability in LJ. Since LJ is decidable,  $\text{IB}[l]$  is also decidable.  $\square$

## 2.2. Kripke semantics

The symbols  $\geq$  and  $\leq$  are used to represent the linear order on  $\omega$ .

**Definition 9.** Let  $l$  be a fixed positive integer. A *Kripke frame* is a structure  $\langle M, N, N_l, R \rangle$  satisfying the following conditions.

1.  $M$  is a nonempty set.

2.  $N$  is the set of natural numbers and  $N_l := \{i \in N \mid i \leq l\}$ .
3.  $R$  is a reflexive and transitive binary relation on  $M$ .

The set  $M$  can be understood as a set of information states, and the set  $N$  can be understood as a set of time points.

**Definition 10.** A valuation  $\models$  on a Kripke frame  $\langle M, N, N_l, R \rangle$  is a mapping from the set  $\Psi$  of all propositional variables to the power set  $2^{M \times N}$  of the direct product  $M \times N$  such that for any  $p \in \Psi$ , any  $i \in N$ , and any  $x, y \in M$ , if  $(x, i) \in \models (p)$  and  $xRy$ , then  $(y, i) \in \models (p)$ . We will write  $(x, i) \models p$  for  $(x, i) \in \models (p)$ . Each valuation  $\models$  is extended to a mapping from the set  $\Phi$  of all formulas to  $2^{M \times N}$  by the following clauses:

1.  $(x, i) \models \perp$  does not hold,
2.  $(x, i) \models \alpha \rightarrow \beta$  iff  $\forall y \in M [xRy \text{ and } (y, i) \models \alpha \text{ imply } (y, i) \models \beta]$ ,
3.  $(x, i) \models \alpha \wedge \beta$  iff  $(x, i) \models \alpha$  and  $(x, i) \models \beta$ ,
4.  $(x, i) \models \alpha \vee \beta$  iff  $(x, i) \models \alpha$  or  $(x, i) \models \beta$ ,
5.  $(x, i) \models X\alpha$  iff  $(x, i+1) \models \alpha$ ,
6.  $(x, i) \models X^l\alpha$  iff  $(x, l) \models \alpha$ ,
7.  $(x, i) \models G\alpha$  iff  $\forall j \in N_l [i \leq j \text{ implies } (x, j) \models \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models \alpha$ ,
8.  $(x, i) \models F\alpha$  iff  $\exists j \in N_l [i \leq j \text{ and } (x, j) \models \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models \alpha$ .

Conditions 5 and 6 in Definition 10 are intended to express that for any positive integer  $m$ ,  $(x, l+m) \models \alpha$  iff  $(x, 0) \models X^{l+m}\alpha$  iff  $(x, 0) \models X^l\alpha$  iff  $(x, l) \models \alpha$ . The statement  $(x, i) \models \alpha$  can be read as “ $\alpha$  is true at the information state  $x$  and the time  $i$ .”

**Proposition 11.** Let  $\models$  be a valuation on a Kripke frame  $\langle M, N, N_l, R \rangle$ . For any formula  $\alpha$ , any  $i \in N$ , and any  $x, y \in M$ , if  $(x, i) \models \alpha$  and  $xRy$ , then  $(y, i) \models \alpha$ .

**Proof.** By induction on the complexity of  $\alpha$ .  $\square$

In the following discussion, Proposition 11 will often be used implicitly.

Note that the time-hereditary condition:  $\forall i, j \in N \forall x \in M [(x, i) \models \alpha \text{ and } i \leq j \text{ imply } (x, j) \models \alpha]$  is not assumed in this semantics.

An expression  $\Gamma^\wedge$  means  $\gamma_1 \wedge \gamma_2 \wedge \dots \wedge \gamma_n$  if  $\Gamma \equiv \langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle$  ( $0 \leq n$ ). Let  $\Delta$  be the empty sequence or a sequence consisting of a single formula. An expression  $\Delta^*$  means  $\alpha$  or  $\perp$  if  $\Delta \equiv \langle \alpha \rangle$  or  $\emptyset$ , respectively. An expression  $(\Gamma \Rightarrow \Delta)^*$  means  $\Gamma^\wedge \rightarrow \Delta^*$  if  $\Gamma$  is not empty, and means  $\Delta^*$  otherwise.

**Definition 12.** A Kripke model is a structure  $\langle M, N, N_l, R, \models \rangle$  such that (1)  $\langle M, N, N_l, R \rangle$  is a Kripke frame, and (2)  $\models$  is a valuation on  $\langle M, N, N_l, R \rangle$ .

A formula  $\alpha$  is true in a Kripke model  $\langle M, N, N_l, R, \models \rangle$  if  $(x, 0) \models \alpha$  for any  $x \in M$ , and valid in a Kripke frame  $\langle M, N, N_l, R \rangle$  if it is true for any valuation  $\models$  on the Kripke frame.

A sequent  $\Gamma \Rightarrow \Delta$  is true in a Kripke model  $\langle M, N, N_l, R, \models \rangle$  if the formula  $(\Gamma \Rightarrow \Delta)^*$  is true in the Kripke model, and valid in a Kripke frame  $\langle M, N, N_l, R \rangle$  if it is true for any valuation  $\models$  on the Kripke frame.

**Theorem 13 (Soundness).** Let  $C$  be the class of all Kripke frames,  $L := \{\Gamma \Rightarrow \Delta \mid \text{IB}[I] \vdash \Gamma \Rightarrow \Delta\}$  and  $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \text{ is valid in all frames of } C\}$ . Then  $L \subseteq L(C)$ .

**Proof.** It is sufficient to show that for any sequent  $\Gamma \Rightarrow \Delta$ , if  $\text{IB}[I] \vdash \Gamma \Rightarrow \Delta$ , then  $\Gamma \Rightarrow \Delta$  is valid in  $\langle M, N, N_l, R \rangle \in C$ , i.e., for any valuation  $\models$  on  $\langle M, N, N_l, R \rangle$  and any  $x \in M$ ,  $(x, 0) \models \Gamma \Rightarrow \Delta$ . This is proved by induction on proofs  $P$  of  $\Gamma \Rightarrow \Delta$  in  $\text{IB}[I]$ . To show this, we distinguish the cases according to the last inference of  $P$ . Since the proof is straightforward, we show only the following cases.

Case (Gright): The last inference of  $P$  is of the form:

$$\frac{\{\Gamma \Rightarrow X^{i+j}\alpha\}_{j \in \omega_l}}{\Gamma \Rightarrow X^i G\alpha} \text{ (Gright)}$$

In the following, we consider only the case  $\Gamma \neq \emptyset$ . Let  $\models$  be a valuation on  $\langle M, N, N_l, R \rangle$ . By the induction hypothesis, we have  $\forall j \in N_l \forall x \in M [(x, 0) \models \Gamma \Rightarrow X^{i+j}\alpha]$ . First, we show the case for  $i < l$  as follows.  $\forall j \in N_l \forall x \in M [(x, 0) \models \Gamma \Rightarrow X^{i+j}\alpha]$  iff  $\forall j \in N_l \forall x \in M [(x, 0) \models \Gamma^\wedge \rightarrow X^{i+j}\alpha]$  iff  $\forall j \in N_l \forall x \in M \forall y \in M [xRy \text{ and } (y, 0) \models \Gamma^\wedge \text{ imply } (y, i+j) \models \alpha]$  iff  $\forall x, y \in M [xRy \text{ and } (y, 0) \models \Gamma^\wedge \text{ imply } \forall j \in N_l [(y, i+j) \models \alpha]]$  iff  $\forall x, y \in M [xRy \text{ and } (y, 0) \models \Gamma^\wedge \text{ imply } (y, i) \models G\alpha]$  iff  $\forall x \in M [(x, 0) \models \Gamma \Rightarrow X^i G\alpha]$ . Second, we consider the case for  $i \geq l$  as follows. This case can be shown similarly. The difference is



that the part “ $\forall j \in N_l [(y, i + j) \models \alpha]$ ” can be replaced by the following:  $\forall j \in N_l [(y, i + j) \models \alpha]$  iff  $(y, l) \models \alpha$  iff  $(y, l) \models G\alpha$  iff  $(y, i) \models G\alpha$ . Thus, we obtain the required fact.

Case (Xright): The last inference of  $P$  is of the form:

$$\frac{\Gamma \Rightarrow X^l \alpha}{\Gamma \Rightarrow X^{l+m} \alpha} \text{ (Xright)}$$

In the following, we consider only the case  $\Gamma \neq \emptyset$ . Let  $\models$  be a valuation on  $\langle M, N, N_l, R \rangle$ . By the induction hypothesis, we have  $\forall x \in M [(x, 0) \models X^l \alpha]$  where  $(x, 0) \models X^l \alpha$  iff  $(x, l) \models \alpha$  iff  $(x, m) \models X^l \alpha$  iff  $(x, 0) \models X^{l+m} \alpha$ . We thus obtain the required fact that  $\forall x \in M [(x, 0) \models X^{l+m} \alpha]$ .  $\square$

Prior to the detailed presentation of the completeness proof, we prove a Lindenbaum lemma.

**Definition 14.** Let  $x$  and  $y$  be sets of formulas. The pair  $(x, y)$  is *consistent* iff for any  $\alpha_1, \dots, \alpha_m \in x$  and any  $\beta_1, \dots, \beta_n \in y$  with  $(m, n \geq 0)$ , the sequent  $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1 \vee \dots \vee \beta_n$  is not provable in  $\text{IB}[I]$ . The pair  $(x, y)$  is *maximal consistent* iff it is consistent and for every formula  $\alpha$ ,  $\alpha \in x$  or  $\alpha \in y$ .<sup>2</sup>

The following lemma can be proved using (cut).

**Lemma 15.** Let  $x$  and  $y$  be sets of formulas. If the pair  $(x, y)$  is consistent, then there is a maximal consistent pair  $(x', y')$  such that  $x \subseteq x'$  and  $y \subseteq y'$ .

**Proof.** Let  $\gamma_1, \gamma_2, \dots$  be an enumeration of all formulas of  $\text{IB}[I]$ . Define a sequence of pairs  $(x_n, y_n)$  ( $n = 0, 1, \dots$ ) inductively by  $(x_0, y_0) := (x, y)$ , and  $(x_{m+1}, y_{m+1}) := (x_m, y_m \cup \{\gamma_{m+1}\})$  if  $(x_m, y_m \cup \{\gamma_{m+1}\})$  is consistent, and  $(x_{m+1}, y_{m+1}) := (x_m \cup \{\gamma_{m+1}\}, y_m)$  otherwise. We can obtain the fact that if  $(x_m, y_m)$  is consistent, then so is  $(x_{m+1}, y_{m+1})$ . To verify this, suppose  $(x_{m+1}, y_{m+1})$  is not consistent. Then there are formulas  $\alpha_1, \dots, \alpha_i, \alpha'_1, \dots, \alpha'_j \in x_m$  and  $\beta_1, \dots, \beta_k, \beta'_1, \dots, \beta'_l \in y_m$  such that  $\text{IB}[I] \vdash \alpha_1, \dots, \alpha_i \Rightarrow \beta_1 \vee \dots \vee \beta_k \vee \gamma_{m+1}$  and  $\text{IB}[I] \vdash \alpha'_1, \dots, \alpha'_j, \gamma_{m+1} \Rightarrow \beta'_1 \vee \dots \vee \beta'_l$ . By using (cut) and some other rules, we can obtain  $\text{IB}[I] \vdash \alpha_1, \dots, \alpha_i, \alpha'_1, \dots, \alpha'_j \Rightarrow \beta_1 \vee \dots \vee \beta_k \vee \beta'_1 \vee \dots \vee \beta'_l$ . This contradicts the consistency of  $(x_m, y_m)$ . Hence, a pair  $(x_k, y_k)$  produced by the construction is consistent for any  $k$ . We thus obtain a maximal consistent pair  $(\bigcup_{n=0}^{\infty} x_n, \bigcup_{n=0}^{\infty} y_n)$ .  $\square$

We now start to prove the completeness theorem for  $\text{IB}[I]$ .

The expression  $\{\Gamma\}$  means the set of all formulas occurring in  $\Gamma$ . Suppose that  $\Gamma \Rightarrow \Delta$  is not provable in  $\text{IB}[I]$ . Then the pair  $(\{\Gamma\}, \{\Delta\})$  is consistent. By Lemma 15, there is a maximal consistent pair  $(u, v)$  such that  $\{\Gamma\} \subseteq u$  and  $\{\Delta\} \subseteq v$ . Note that if  $\Delta \equiv \langle \alpha \rangle$ , then  $\alpha \notin u$  by the consistency of  $(u, v)$ .

**Definition 16.** Let  $M_L$  be the set of all maximal consistent pairs. The binary relation  $R_L$  on  $M_L$  is defined by  $(x, w)R_L(y, z)$  iff  $x \subseteq y$ . The valuation  $\models_L(p)$  for any propositional variable  $p$  is defined by  $\{(x, w), i\} \in M_L \times N \mid X^i p \in x$ .

**Lemma 17.** The structure  $\langle M_L, N, N_l, R_L, \models_L \rangle$  is a Kripke model such that for any formula  $\alpha$ , any  $i \in N$ , and any  $(x, w) \in M_L$ ,  $X^i \alpha \in x$  iff  $((x, w), i) \models_L \alpha$ .

**Proof.** It can be shown that (1)  $M_L$  is a nonempty set, because  $(u, v) \in M_L$  by the discussion above, (2)  $R_L$  is a reflexive and transitive relation on  $M_L$ , and (3) for any propositional variable  $p$  and any  $(x, w), (y, z) \in M_L$ , if  $(x, w)R_L(y, z)$  and  $((x, w), i) \models_L(p)$ , then  $((y, z), i) \models_L(p)$ . Thus, the structure  $\langle M_L, N, N_l, R_L, \models_L \rangle$  is a Kripke model.

It remains to be shown that in this model, for any formula  $\alpha$ , any  $i \in N$ , and any  $(x, w) \in M_L$ ,  $X^i \alpha \in x$  iff  $((x, w), i) \models_L \alpha$ . This is shown by induction on the complexity of  $\alpha$ .

*Base step.* By Definition 16.

*Induction step.*

- Case  $\alpha \equiv \perp$ . By the consistency of  $(x, w)$ ,  $X^i \perp \in x$  does not hold.
- Case  $\alpha \equiv \gamma \rightarrow \delta$ . Suppose  $X^i(\gamma \rightarrow \delta) \in x$ . We will show  $((x, w), i) \models_L \gamma \rightarrow \delta$ , i.e.,  $\forall (y, z) \in M_L [(x, w)R_L(y, z) \text{ and } ((y, z), i) \models_L \gamma \text{ imply } ((y, z), i) \models_L \delta]$ . Suppose  $(x, w)R_L(y, z)$  and  $((y, z), i) \models_L \gamma$ . Then we have (\*):  $X^i(\gamma \rightarrow \delta) \in y$  by the definition of  $R_L$ , and obtain (\*\*):  $X^i \gamma \in y$  by the induction hypothesis. Since (\*), (\*\*), and  $\text{IB}[I] \vdash X^i(\gamma \rightarrow \delta), X^i \gamma \Rightarrow X^i \delta$ , the fact that  $X^i \delta \in z$  contradicts the consistency of  $(y, z)$ , and hence  $X^i \delta \notin z$ . By the maximality of  $(y, z)$ , we obtain  $X^i \delta \in y$ . By the induction hypothesis, we obtain the required fact  $((y, z), i) \models_L \delta$ . Conversely, suppose  $X^i(\gamma \rightarrow \delta) \notin x$ . Then  $X^i(\gamma \rightarrow \delta) \in w$  by the maximality of  $(x, w)$ . Then the pair  $(x \cup \{X^i \gamma\}, \{X^i \delta\})$  is consistent for the following reason. If it is

<sup>2</sup> For example, the pair  $(\{p\}, \{q\})$  where  $p$  and  $q$  are distinct propositional variables is consistent, and the pair  $(\{p\}, \{p\})$  is inconsistent.

not consistent, then  $IB[l] \vdash \Gamma, X^i \gamma \Rightarrow X^i \delta$  for some  $\Gamma$  consisting of formulas in  $x$ , and hence  $IB[l] \vdash \Gamma \Rightarrow X^i(\gamma \rightarrow \delta)$ . This fact contradicts the consistency of  $(x, w)$ . By Lemma 15, there is a maximal consistent pair  $(y, z)$  such that  $x \cup \{X^i \gamma\} \subseteq y$  and  $\{X^i \delta\} \subseteq z$  (thus, we have  $X^i \delta \notin y$  by the consistency of  $(y, z)$ ). As a consequence, we have  $(x, w)R_L(y, z)$ ,  $((y, z), i) \models_L \gamma$  and not  $[(y, z), i] \models_L \delta$  by the induction hypothesis. Therefore  $((x, w), i) \models_L \gamma \rightarrow \delta$  does not hold.

- Case  $\alpha \equiv \gamma \wedge \delta$ . Suppose  $X^i(\gamma \wedge \delta) \in x$ . Since  $IB[l] \vdash X^i(\gamma \wedge \delta) \Rightarrow X^i \gamma$ , the fact that  $X^i \gamma \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i \gamma \in x$ . Similarly, we obtain  $X^i \delta \in x$ . By the induction hypothesis, we obtain  $((x, w), i) \models_L \gamma$  and  $((x, w), i) \models_L \delta$ , and hence  $((x, w), i) \models_L \gamma \wedge \delta$ . Conversely, suppose  $((x, w), i) \models_L \gamma \wedge \delta$ , i.e.,  $((x, w), i) \models_L \gamma$  and  $((x, w), i) \models_L \delta$ . Then we obtain  $X^i \gamma \in x$  and  $X^i \delta \in x$  by the induction hypothesis. Since  $IB[l] \vdash X^i \gamma, X^i \delta \Rightarrow X^i(\gamma \wedge \delta)$ , the fact that  $X^i(\gamma \wedge \delta) \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i(\gamma \wedge \delta) \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i(\gamma \wedge \delta) \in x$ .
- Case  $\alpha \equiv \gamma \vee \delta$ . Suppose  $X^i(\gamma \vee \delta) \in x$ . Since  $IB[l] \vdash X^i(\gamma \vee \delta) \Rightarrow X^i \gamma \vee X^i \delta$ , the fact that  $X^i \gamma, X^i \delta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i \gamma \notin w$  or  $X^i \delta \notin w$ . Thus, we obtain  $X^i \gamma \in x$  or  $X^i \delta \in x$  by the maximality of  $(x, w)$ . By the induction hypothesis, we obtain  $((x, w), i) \models_L \gamma$  or  $((x, w), i) \models_L \delta$ , and hence  $((x, w), i) \models_L \gamma \vee \delta$ . Conversely, suppose  $((x, w), i) \models_L \gamma \vee \delta$ , i.e.,  $((x, w), i) \models_L \gamma$  or  $((x, w), i) \models_L \delta$ . By the induction hypothesis, we obtain  $X^i \gamma \in x$  or  $X^i \delta \in x$ . Since  $IB[l] \vdash X^i \gamma \Rightarrow X^i(\gamma \vee \delta)$  and  $IB[l] \vdash X^i \delta \Rightarrow X^i(\gamma \vee \delta)$ , the fact that  $X^i(\gamma \vee \delta) \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i(\gamma \vee \delta) \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i(\gamma \vee \delta) \in x$ .
- Case  $\alpha \equiv X\beta$ .  $X^i X\beta \in x$  iff  $X^{i+1}\beta \in x$  iff  $((x, w), i+1) \models_L \beta$  (by the induction hypothesis) iff  $((x, w), i) \models_L X\beta$ .
- Case  $\alpha \equiv X^l\beta$ .  $X^i X^l\beta \in x$  iff  $X^{l+i}\beta \in x$  iff  $((x, w), l+i) \models_L \beta$  (by the induction hypothesis) iff  $((x, w), i) \models_L X^l\beta$ .
- Case  $\alpha \equiv G\beta$ .

(Subcase  $i < l$ ): Suppose  $X^i G\beta \in x$  with  $i < l$ . Since  $IB[l] \vdash X^i G\beta \Rightarrow X^{i+k}\beta$  for any  $k \in N_l$ , the fact that  $X^{i+k}\beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^{i+k}\beta \notin w$ . Thus, by the maximality of  $(x, w)$ , we obtain  $X^{i+k}\beta \in x$  for any  $k \in N_l$ , i.e.,  $\forall j \in N_l [i \leq j \text{ implies } X^j \beta \in x]$ . By the induction hypothesis, we obtain  $\forall j \in N_l [i \leq j \text{ implies } ((x, w), j) \models_L \beta]$ , i.e.,  $((x, w), i) \models_L G\beta$ . Conversely, suppose  $((x, w), i) \models_L G\beta$ , i.e.,  $\forall j \in N_l [i \leq j \text{ implies } ((w, w), j) \models_L \beta]$ . By the induction hypothesis, we obtain  $\forall j \in N_l [i \leq j \text{ implies } X^j \beta \in x]$ , i.e.,  $\forall k \in N_l [X^{i+k}\beta \in x]$ . Since  $IB[l] \vdash X^i \beta, X^{i+1}\beta, \dots, X^{i+l}\beta \Rightarrow X^i G\beta$ , the fact that  $X^i G\beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i G\beta \notin w$ . Thus, we obtain  $X^i G\beta \in x$  by the maximality of  $(x, w)$ .

(Subcase  $i \geq l$ ): Suppose  $X^i G\beta \in x$  with  $i \geq l$ . Since  $IB[l] \vdash X^i G\beta \Rightarrow X^i \beta$  ( $i \geq l$ ), the fact that  $X^i \beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i \beta \notin w$ . Thus, by the maximality of  $(x, w)$ , we obtain  $X^i \beta \in x$ . By the induction hypothesis, we obtain  $((x, w), i) \models_L \beta$ , and then  $((x, w), i) \models_L \beta$  iff  $((x, w), i) \models_L G\beta$ . Conversely, suppose  $((x, w), i) \models_L G\beta$ , i.e.,  $((w, w), i) \models_L \beta$ . By the induction hypothesis, we obtain  $X^i \beta \in x$ . Since  $IB[l] \vdash X^i \beta \Rightarrow X^i G\beta$  ( $i \geq l$ ), the fact that  $X^i G\beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i G\beta \notin w$ . Thus, we obtain  $X^i G\beta \in x$  by the maximality of  $(x, w)$ .

- Case  $\alpha \equiv F\beta$ .

(Subcase  $i < l$ ): Suppose  $X^i F\beta \in x$  with  $i < l$ . Since  $IB[l] \vdash X^i F\beta \Rightarrow X^i \beta \vee X^{i+1}\beta \vee \dots \vee X^{i+l}\beta$ , the fact that  $\forall i \in N_l [X^{i+l}\beta \in w]$  contradicts the consistency of  $(x, w)$ , and hence  $\exists k \in N_l [X^{i+k}\beta \notin w]$ . By the maximality of  $(x, w)$ , we obtain  $\exists k \in N_l [X^{i+k}\beta \in x]$ , i.e.,  $\exists j \in N_l [i \leq j \text{ and } X^j \beta \in x]$ . By the induction hypothesis, we obtain  $\exists j \in N_l [i \leq j \text{ and } ((x, w), j) \models_L \beta]$ , i.e.,  $((x, w), i) \models_L F\beta$ . Conversely, suppose  $((x, w), i) \models_L F\beta$ , i.e.,  $\exists j \in N_l [i \leq j \text{ and } ((x, w), j) \models_L \beta]$ . By the induction hypothesis, we obtain  $\exists j \in N_l [i \leq j \text{ and } X^j \beta \in x]$ , i.e.,  $\exists k \in N_l [X^{i+k}\beta \in x]$ . Since  $IB[l] \vdash X^{i+k}\beta \Rightarrow X^i F\beta$  (for any  $k \in N_l$ ), the fact that  $X^i F\beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i F\beta \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i F\beta \in x$ .

(Subcase  $i \geq l$ ): Suppose  $X^i F\beta \in x$  with  $i \geq l$ . Since  $IB[l] \vdash X^i F\beta \Rightarrow X^i \beta$  ( $i \geq l$ ), the fact that  $X^i \beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i \beta \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i \beta \in x$ . By the induction hypothesis, we obtain  $((x, w), i) \models_L \beta$ , and then  $((x, w), i) \models_L \beta$  iff  $((x, w), i) \models_L F\beta$ . Conversely, suppose  $((x, w), i) \models_L F\beta$ , i.e.,  $((x, w), i) \models_L \beta$ . By the induction hypothesis, we obtain  $X^i \beta \in x$ . Since  $IB[l] \vdash X^i \beta \Rightarrow X^i F\beta$  ( $i \geq l$ ), the fact that  $X^i F\beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i F\beta \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i F\beta \in x$ .  $\square$

**Theorem 18 (Completeness).** Let  $C$  be the class of all Kripke frames,  $L := \{\Gamma \Rightarrow \Delta \mid IB[l] \vdash \Gamma \Rightarrow \Delta\}$  and  $L(C) := \{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \text{ is valid in all frames of } C\}$ . Then  $L = L(C)$ .

**Proof.** In order to prove this theorem, by Theorem 13, it is sufficient to show that  $L(C) \subseteq L$ , i.e., for any sequent  $\Gamma \Rightarrow \Delta$ , if  $\Gamma \Rightarrow \Delta$  is valid in an arbitrary frame in  $C$ , then it is provable in  $IB[l]$ . To show this, we show that if  $\Gamma \Rightarrow \Delta$  is not provable in  $IB[l]$ , then there is a frame  $F = \langle M_L, N, N_l, R_L \rangle \in C$  such that  $\Gamma \Rightarrow \Delta$  is not valid in  $F$ , i.e., there is a Kripke model  $\langle M_L, N, N_l, R_L, \models_L \rangle$  such that  $\Gamma \Rightarrow \Delta$  is not true in it.

Then, our goal is to show that  $((u, v), 0) \models_L \Gamma \Rightarrow \Delta$  does not hold in the constructed model. Here we consider only the case  $\Gamma \neq \emptyset$ . We show that  $((u, v), 0) \models_L \Gamma \wedge \rightarrow \Delta^*$  does not hold, i.e.,  $\exists(x, z) \in M_L [((u, v)R_L(x, z) \text{ and } ((x, z), 0) \models_L \Gamma \wedge) \text{ and } [((x, z), 0) \models_L \Delta^* \text{ does not hold}]]$ . Taking  $(u, v)$  for  $(x, z)$  and 0 for  $i$ , we can verify that there is  $(u, v) \in M_L$  such that  $[(u, v)R_L(u, v) \text{ and } ((u, v), 0) \models_L \Gamma \wedge] \text{ and } [((u, v), 0) \models_L \Delta^* \text{ does not hold}]$ . The first argument is obvious because of the reflexivity of  $R_L$  and the fact that  $\{\Gamma\} \subseteq u$ . The second argument is shown below. The case  $\Delta \equiv \emptyset$  is obvious because  $((u, v), 0) \models_L \perp$  does not hold. The case  $\Delta \equiv \langle \alpha \rangle$  can be proved by using Lemma 17 and the fact that  $\alpha \notin u$ , because we have the fact that  $\alpha \notin u$  iff  $[(u, v), 0) \models_L \alpha \text{ does not hold}]$  by Lemma 17.  $\square$

### 3. Paraconsistent bounded linear-time temporal logic

#### 3.1. Sequent calculus

The language of PB[l] is obtained from that of IB[l] by adding a strong negation connective  $\sim$ . The notations and conventions used for PB[l] are almost the same as those for IB[l]. An expression of the form  $\Gamma \Rightarrow \gamma$  where  $\gamma$  is a single formula is called a *sequent* (for PB[l]).

**Definition 19.** PB[l] is obtained from IB[l] by restricting each succedent of sequents to a single formula (i.e.,  $\Delta$  used in IB[l] is just a single formula  $\gamma$ ), deleting the initial sequents of the form  $X^i \perp \Rightarrow$  and the structural rule (we-right), adding initial sequents  $X^i \sim p \Rightarrow X^i \sim p$ , and adding (for any  $k \in \omega_l$ ) the logical inference rules of the form:

$$\begin{array}{c} \frac{X^i \alpha, \Gamma \Rightarrow \gamma}{X^i \sim \sim \alpha, \Gamma \Rightarrow \gamma} (\sim\sim\text{left}) \quad \frac{\Gamma \Rightarrow X^i \alpha}{\Gamma \Rightarrow X^i \sim \sim \alpha} (\sim\sim\text{right}) \\ \frac{X^i \alpha, X^i \sim \beta, \Gamma \Rightarrow \gamma}{X^i \sim (\alpha \rightarrow \beta), \Gamma \Rightarrow \gamma} (\sim\rightarrow\text{left}) \quad \frac{\Gamma \Rightarrow X^i \alpha \quad \Gamma \Rightarrow X^i \sim \beta}{\Gamma \Rightarrow X^i \sim (\alpha \rightarrow \beta)} (\sim\rightarrow\text{right}) \\ \frac{X^i \sim \alpha, \Gamma \Rightarrow \gamma \quad X^i \sim \beta, \Gamma \Rightarrow \gamma}{X^i \sim (\alpha \wedge \beta), \Gamma \Rightarrow \gamma} (\sim\wedge\text{left}) \\ \frac{\Gamma \Rightarrow X^i \sim \alpha}{\Gamma \Rightarrow X^i \sim (\alpha \wedge \beta)} (\sim\wedge\text{right1}) \quad \frac{\Gamma \Rightarrow X^i \sim \beta}{\Gamma \Rightarrow X^i \sim (\alpha \wedge \beta)} (\sim\wedge\text{right2}) \\ \frac{X^i \sim \alpha, \Gamma \Rightarrow \gamma}{X^i \sim (\alpha \vee \beta), \Gamma \Rightarrow \gamma} (\sim\vee\text{left1}) \quad \frac{X^i \sim \beta, \Gamma \Rightarrow \gamma}{X^i \sim (\alpha \vee \beta), \Gamma \Rightarrow \gamma} (\sim\vee\text{left2}) \\ \frac{\Gamma \Rightarrow X^i \sim \alpha \quad \Gamma \Rightarrow X^i \sim \beta}{\Gamma \Rightarrow X^i \sim (\alpha \vee \beta)} (\sim\vee\text{right}) \\ \frac{\{X^{i+j} \sim \alpha, \Gamma \Rightarrow \gamma\}_{j \in \omega_l}}{X^i \sim G\alpha, \Gamma \Rightarrow \gamma} (\sim\text{Gleft}) \quad \frac{\Gamma \Rightarrow X^{i+k} \sim \alpha}{\Gamma \Rightarrow X^i \sim G\alpha} (\sim\text{Gright}) \\ \frac{X^{i+k} \sim \alpha, \Gamma \Rightarrow \gamma}{X^i \sim F\alpha, \Gamma \Rightarrow \gamma} (\sim\text{Fleft}) \quad \frac{\{\Gamma \Rightarrow X^{i+j} \sim \alpha\}_{j \in \omega_l}}{\Gamma \Rightarrow X^i \sim F\alpha} (\sim\text{Fright}) \\ \frac{X^i \sim \alpha, \Gamma \Rightarrow \gamma}{\sim X^i \alpha, \Gamma \Rightarrow \gamma} (\sim\text{Xleft}) \quad \frac{\Gamma \Rightarrow X^i \sim \alpha}{\Gamma \Rightarrow \sim X^i \alpha} (\sim\text{Xright}) \end{array}$$

We use the same names for the modified single-succedent inference rules.

Note that the rules ( $\sim\text{Xleft}$ ) and ( $\sim\text{Xright}$ ) imply  $\text{PB}[l] \vdash \sim X^i \alpha \Leftrightarrow X^i \sim \alpha$  for any formula  $\alpha$ . Also remark that the following sequents are provable in cut-free PB[l]: for any formulas  $\alpha$  and  $\beta$ ,

1.  $\sim \sim \alpha \Leftrightarrow \alpha$ ,
2.  $\sim (\alpha \wedge \beta) \Leftrightarrow \sim \alpha \vee \sim \beta$ ,
3.  $\sim (\alpha \vee \beta) \Leftrightarrow \sim \alpha \wedge \sim \beta$ ,
4.  $\sim (\alpha \rightarrow \beta) \Leftrightarrow \alpha \wedge \sim \beta$ ,
5.  $\sim G\alpha \Leftrightarrow F\sim \alpha$ ,
6.  $\sim F\alpha \Leftrightarrow G\sim \alpha$ ,
7.  $\sim X\alpha \Leftrightarrow X\sim \alpha$ .

**Definition 20** (LN4). A sequent calculus LN4 for Nelson's paraconsistent logic N4 is obtained from PB[l] by deleting the inference rules (Xleft), (Xright), ( $\sim\text{Xleft}$ ), ( $\sim\text{Xright}$ ), (Gleft), (Gright), (Fleft), (Fright), ( $\sim\text{Gleft}$ ), ( $\sim\text{Gright}$ ), ( $\sim\text{Fleft}$ ), ( $\sim\text{Fright}$ ), and replacing  $X^i$  by  $X^0$ .

For more information on sequent calculi for N4, see, for instance, [24,33].

**Definition 21.** We fix a countable nonempty set  $\Phi$  of propositional variables and define the sets  $\Phi_i := \{p_i \mid p \in \Phi\}$  ( $1 \leq i \in \omega$ ) and  $\Phi_0 := \Phi$  of propositional variables. The language  $\mathcal{L}_{\text{PB}[l]}$  of PB[l] is defined by using  $\Phi$ ,  $\rightarrow$ ,  $\wedge$ ,  $\vee$ ,  $\sim$ , X, G and F. The language  $\mathcal{L}_{\text{LN4}}$  of LN4 is defined by using  $\bigcup_{i \in \omega} \Phi_i$ ,  $\rightarrow$ ,  $\wedge$ ,  $\vee$  and  $\sim$ .

A mapping  $f$  from  $\mathcal{L}_{\text{PB}[I]}$  to  $\mathcal{L}_{\text{LN4}}$  is obtained from Definition 5 by replacing Condition 1 by the following condition, for any  $i \in \omega$ :

$$1. f(X^i \sim \alpha) = f(\sim X^i \alpha) = \sim f(X^i \alpha).$$

**Theorem 22 (Embedding).** Let  $\Gamma$  be a sequence of formulas in  $\mathcal{L}_{\text{PB}[I]}$ ,  $\gamma$  be a formula in  $\mathcal{L}_{\text{PB}[I]}$ , and  $f$  be the mapping defined in Definition 21.

1.  $\text{PB}[I] \vdash \Gamma \Rightarrow \gamma$  iff  $\text{LN4} \vdash f(\Gamma) \Rightarrow f(\gamma)$ .
2.  $\text{PB}[I]\text{-}(\text{cut}) \vdash \Gamma \Rightarrow \gamma$  iff  $\text{LN4}\text{-}(\text{cut}) \vdash f(\Gamma) \Rightarrow f(\gamma)$ .

**Proof.** Similar to the proof of Theorem 6. We show only the direction ( $\Rightarrow$ ) of (1) by induction on proofs  $P$  of  $\Gamma \Rightarrow \Delta$  in  $\text{PB}[I]$ . We distinguish the cases according to the last inference of  $P$  and show some cases.

Case ( $\sim$ Xright). The last inference of  $P$  is of the form:

$$\frac{\Gamma \Rightarrow X^i \sim \alpha}{\Gamma \Rightarrow \sim X^i \alpha} (\sim\text{Xright})$$

By induction hypothesis, we have  $\text{LN4} \vdash f(\Gamma) \Rightarrow f(X^i \sim \alpha)$ , and hence obtain the required fact  $\text{LN4} \vdash f(\Gamma) \Rightarrow f(\sim X^i \alpha)$ , since  $f(X^i \sim \alpha) = f(\sim X^i \alpha)$  by the definition of  $f$ .

Case ( $\sim \wedge$ right1). The last inference of  $P$  is of the form:

$$\frac{\Gamma \Rightarrow X^i \sim \alpha}{\Gamma \Rightarrow X^i \sim (\alpha \wedge \beta)} (\sim \wedge \text{right1})$$

By induction hypothesis, we have  $\text{LN4} \vdash f(\Gamma) \Rightarrow f(X^i \sim \alpha)$  where  $f(X^i \sim \alpha) = \sim f(X^i \alpha)$  by the definition of  $f$ . Then we obtain  $\text{LN4} \vdash f(\Gamma) \Rightarrow f(X^i \sim (\alpha \wedge \beta))$  by:

$$\frac{\begin{array}{c} \vdots \\ f(\Gamma) \Rightarrow \sim f(X^i \alpha) \end{array}}{f(\Gamma) \Rightarrow \sim (f(X^i \alpha) \wedge f(X^i \beta))} (\sim \wedge \text{right1}^{\text{LN4}})$$

where  $\sim (f(X^i \alpha) \wedge f(X^i \beta)) = \sim (f(X^i (\alpha \wedge \beta))) = f(X^i \sim (\alpha \wedge \beta))$  by the definition of  $f$ .  $\square$

**Theorem 23 (Cut-elimination).** The rule (cut) is admissible in cut-free  $\text{PB}[I]$ .

**Proof.** Similar to the proof of Theorem 7.  $\square$

**Theorem 24 (Decidability).**  $\text{PB}[I]$  is decidable.

**Proof.** Similar to the proof of Theorem 8.  $\square$

**Definition 25.** Let  $\sharp$  be a negation connective. A sequent calculus  $L$  is called *explosive* with respect to  $\sharp$  iff for any formulas  $\alpha$  and  $\beta$ , the sequent  $\alpha, \sharp \alpha \Rightarrow \beta$  is provable in  $L$ . It is called *paraconsistent* with respect to  $\sharp$  iff it is not explosive with respect to  $\sharp$ .

**Proposition 26 (Paraconsistency).**  $\text{PB}[I]$  is paraconsistent with respect to  $\sim$ .

**Proof.** Consider a sequent  $p, \sim p \Rightarrow q$  where  $p$  and  $q$  are distinct propositional variables. Then the unprovability of this sequent is guaranteed by using Theorem 23, i.e., we cannot construct a cut-free proof of  $p, \sim p \Rightarrow q$ .  $\square$

### 3.2. Kripke semantics

The same kind of Kripke frames which is used for  $\text{IB}[I]$  is also used for  $\text{PB}[I]$ . It is known that the Kripke semantics for logics with strong negation uses two kinds of valuations  $\models^+$  (representing *verification*) and  $\models^-$  (representing *refutation*). For information on this type of semantics see, for example, [29,30,33]. Kripke models for  $\text{PB}[I]$  also use such valuations.

**Definition 27.** Valuations  $\models^+$  and  $\models^-$  on a Kripke frame  $\langle M, N, N_i, R \rangle$  are mappings from the set  $\Psi$  of all propositional variables to the power set  $2^{M \times N}$  such that for any  $p \in \Psi$ , any  $i \in N$ , and any  $x, y \in M$ ,

1. if  $(x, i) \in \models^+(p)$  and  $xRy$ , then  $(y, i) \in \models^+(p)$ ,
2. if  $(x, i) \in \models^-(p)$  and  $xRy$ , then  $(y, i) \in \models^-(p)$ .

We will write  $(x, i) \models^* p$  for  $(x, i) \in \models^* (p)$  where  $* \in \{+, -\}$ . The valuations  $\models^+$  and  $\models^-$  are extended to mappings from the set  $\Phi$  of all formulas to  $2^{M \times N}$  by the following clauses:

1.  $(x, i) \models^+ \alpha \rightarrow \beta$  iff  $\forall y \in M [xRy \text{ and } (y, i) \models^+ \alpha \text{ imply } (y, i) \models^+ \beta]$ ,
2.  $(x, i) \models^+ \alpha \wedge \beta$  iff  $(x, i) \models^+ \alpha$  and  $(x, i) \models^+ \beta$ ,
3.  $(x, i) \models^+ \alpha \vee \beta$  iff  $(x, i) \models^+ \alpha$  or  $(x, i) \models^+ \beta$ ,
4.  $(x, i) \models^+ X\alpha$  iff  $(x, i+1) \models^+ \alpha$ ,
5.  $(x, i) \models^+ X^l\alpha$  iff  $(x, l) \models^+ \alpha$ ,
6.  $(x, i) \models^+ G\alpha$  iff  $\forall j \in N_l [i \leq j \text{ implies } (x, j) \models^+ \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models^+ \alpha$ ,
7.  $(x, i) \models^+ F\alpha$  iff  $\exists j \in N_l [i \leq j \text{ and } (x, j) \models^+ \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models^+ \alpha$ ,
8.  $(x, i) \models^+ \sim\alpha$  iff  $(x, i) \not\models^+ \alpha$ ,
9.  $(x, i) \models^- \alpha \rightarrow \beta$  iff  $(x, i) \models^+ \alpha$  and  $(x, i) \models^- \beta$ ,
10.  $(x, i) \models^- \alpha \wedge \beta$  iff  $(x, i) \models^- \alpha$  or  $(x, i) \models^- \beta$ ,
11.  $(x, i) \models^- \alpha \vee \beta$  iff  $(x, i) \models^- \alpha$  and  $(x, i) \models^- \beta$ ,
12.  $(x, i) \models^- X\alpha$  iff  $(x, i+1) \models^- \alpha$ ,
13.  $(x, i) \models^- X^l\alpha$  iff  $(x, l) \models^- \alpha$ ,
14.  $(x, i) \models^- G\alpha$  iff  $\exists j \in N_l [i \leq j \text{ and } (x, j) \models^- \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models^- \alpha$ ,
15.  $(x, i) \models^- F\alpha$  iff  $\forall j \in N_l [i \leq j \text{ implies } (x, j) \models^- \alpha]$  if  $i < l$ , and otherwise  $(x, l) \models^- \alpha$ ,
16.  $(x, i) \models^- \sim\alpha$  iff  $(x, i) \not\models^- \alpha$ .

**Proposition 28.** Let  $\models^+$  and  $\models^-$  be valuations on a Kripke frame  $\langle M, N, N_l, R \rangle$ . For any formula  $\alpha$ , any  $i \in N$ , and any  $x, y \in M$ , (1) if  $(x, i) \models^+ \alpha$  and  $xRy$ , then  $(y, i) \models^+ \alpha$ , and (2) if  $(x, i) \models^- \alpha$  and  $xRy$ , then  $(y, i) \models^- \alpha$ .

**Definition 29.** A paraconsistent Kripke model is a structure  $\langle M, N, N_l, R, \models^+, \models^- \rangle$  such that (1)  $\langle M, N, N_l, R \rangle$  is a Kripke frame, and (2)  $\models^+$  and  $\models^-$  are valuations on  $\langle M, N, N_l, R \rangle$ .

A formula  $\alpha$  is true in a paraconsistent Kripke model  $\langle M, N, N_l, R, \models^+, \models^- \rangle$  if  $(x, 0) \models^+ \alpha$  for any  $x \in M$ , and  $p$ -valid in a Kripke frame  $\langle M, N, N_l, R \rangle$  if it is true for any valuations  $\models^+$  and  $\models^-$  on the Kripke frame.

A sequent  $\Gamma \Rightarrow \gamma$  is true in a paraconsistent Kripke model  $\langle M, N, N_l, R, \models^+, \models^- \rangle$  if the formula  $(\Gamma \Rightarrow \gamma)^*$  is true in the paraconsistent Kripke model, and  $p$ -valid in a Kripke frame  $\langle M, N, N_l, R \rangle$  if it is true for any valuations  $\models^+$  and  $\models^-$  on the Kripke frame.

We sketch the proof of the following completeness theorem for PB[I].

**Theorem 30 (Completeness).** Let  $C$  be the class of all Kripke frames,  $L := \{\Gamma \Rightarrow \gamma \mid \text{PB[I]} \vdash \Gamma \Rightarrow \gamma\}$  and  $L(C) := \{\Gamma \Rightarrow \gamma \mid \Gamma \Rightarrow \gamma \text{ is } p\text{-valid in all frames of } C\}$ . Then  $L = L(C)$ .

In order to prove  $L(C) \subseteq L$ , almost the same arguments as those for IB[I] will be employed, i.e., the notions of consistent and maximal consistent pairs, and modifications of Lemmas 15 and 17 are used. In the following, only some particularly different points will be explained.

The canonical model  $\langle M_L, N, N_l, R_L, \models_L^+, \models_L^- \rangle$  for PB[I] is defined as follows.

**Definition 31.** Let  $M_L$  be the set of all maximal consistent pairs. The binary relation  $R_L$  on  $M_L$  is defined in the same manner as in Definition 16. Valuations  $\models_L^+(p)$  and  $\models_L^-(p)$  for any propositional variable  $p$  are defined by  $\{(x, w), i \in M_L \times N \mid X^i p \in x\}$  and  $\{(x, w), i \in M_L \times N \mid X^i \sim p \in x\}$ , respectively.

Using this definition, the PB[I] version of Lemma 17 can be formalized and proved as follows.

**Lemma 32.** The structure  $\langle M_L, N, N_l, R_L, \models_L^+, \models_L^- \rangle$  is a paraconsistent Kripke model such that for any formula  $\alpha$ , any  $i \in N$ , and any  $(x, w) \in M_L$ , (1)  $X^i \alpha \in x$  iff  $((x, w), i) \models_L^+ \alpha$ , and (2)  $X^i \sim \alpha \in x$  iff  $((x, w), i) \models_L^- \alpha$ .

**Proof.** Since the structure  $\langle M_L, N, N_l, R_L, \models_L^+, \models_L^- \rangle$  is a paraconsistent Kripke model, it is shown that in this model, for any formula  $\alpha$ , any  $i \in N$ , and any  $(x, w) \in M_L$ , (1)  $X^i \alpha \in x$  iff  $((x, w), i) \models_L^+ \alpha$ , and (2)  $X^i \sim \alpha \in x$  iff  $((x, w), i) \models_L^- \alpha$ . This is shown by (simultaneous) induction on the complexity of  $\alpha$ . We show only the following critical cases.

- Case  $\alpha \equiv \sim\beta$ . First we show (1).  $X^i \sim\beta \in x$  iff  $((x, w), i) \models_L^- \beta$  (by the induction hypothesis for (2)) iff  $((x, w), i) \models_L^+ \sim\beta$ . Next, we show (2). Suppose  $X^i \sim\sim\beta \in x$ . Since  $\text{PB[I]} \vdash X^i \sim\sim\beta \Rightarrow X^i \beta$ , the fact that  $X^i \beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^i \beta \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^i \beta \in x$ . By the induction hypothesis for (1), we obtain  $((x, w), i) \models_L^+ \beta$ , and hence  $((x, w), i) \models_L^- \sim\beta$ .

- Case  $\alpha \equiv X\beta$ . First, we show (1).  $X^i(X\beta) \in x$  iff  $X^{i+1}\beta \in x$  iff  $((x, w), i+1) \models_L^+ \beta$  (by the induction hypothesis for (1)) iff  $((x, w), i) \models_L^+ X\beta$ . Second, we show (2). Suppose  $X^i \sim(X\beta) \in x$ . Since  $PB[I] \vdash X^i \sim(X\beta) \Rightarrow X^{i+1} \sim \beta$  by using  $(\sim Xleft)$  and  $(\sim Xright)$ , the fact that  $X^{i+1} \sim \beta \in w$  contradicts the consistency of  $(x, w)$ , and hence  $X^{i+1} \sim \beta \notin w$ . By the maximality of  $(x, w)$ , we obtain  $X^{i+1} \sim \beta \in x$ . By the induction hypothesis for (2), we obtain  $((x, w), i+1) \models_L^- \beta$ , and hence  $((x, w), i) \models_L^- X\beta$ .  $\square$

#### 4. Natural deduction

This section assumes basic knowledge of Gentzen-type natural deduction systems (for detailed information, see e.g., [25,31]). First, we introduce a natural deduction system  $NPB[I]$  for  $PB[I]$  and show the normalization theorem for  $NPB[I]$ . Second, we discuss a natural deduction system  $NIB[I]$  for  $IB[I]$  and the normalization theorem for  $NIB[I]$ . The systems  $NIB[I]$  and  $NPB[I]$  are defined as (modified) extensions of the natural deduction system  $NJ$  for intuitionistic logic and a natural deduction system for Nelson's  $N4$ , respectively. A survey of natural deduction systems for  $N4$  is presented in [14]. The treatment of linear time in  $NIB[I]$  and  $NPB[I]$  is adopted from [16]. In [16], the strong normalization theorem for a typed  $\lambda$ -calculus for the  $\{\rightarrow, \wedge, X, G\}$ -fragment of  $IB[I]$  is shown, but the (strong) normalization theorem for the full system is not discussed. There are a lot of natural deduction systems and typed- $\lambda$ -calculi for LTL and its neighbors, and a survey of such systems is also given in [3,9,16]. Note that our systems somewhat resemble Baratella's and Masini's system  $PNJ$  for an intuitionistic LTL which is called a *logic of positions* [3].

**Definition 33** ( $NPB[I]$ ). The inference rules of  $NPB[I]$  are of the following form, for any  $k \in \omega_l := \{i \in \omega \mid i \leq l\}$  and any positive integer  $m$ :

$$\begin{array}{c}
 [X^i\alpha] \\
 \vdots \\
 \frac{X^i\beta}{X^i(\alpha \rightarrow \beta)} \ (\rightarrow I) \quad \frac{X^i(\alpha \rightarrow \beta) \quad X^i\alpha}{X^i\beta} \ (\rightarrow E) \\
 \\
 \frac{X^i\alpha_1 \quad X^i\alpha_2}{X^i(\alpha_1 \wedge \alpha_2)} \ (\wedge I) \quad \frac{X^i(\alpha_1 \wedge \alpha_2)}{X^i\alpha_1} \ (\wedge E1) \quad \frac{X^i(\alpha_1 \wedge \alpha_2)}{X^i\alpha_2} \ (\wedge E2) \\
 \\
 \frac{X^i\alpha_1}{X^i(\alpha_1 \vee \alpha_2)} \ (\vee I1) \quad \frac{X^i\alpha_2}{X^i(\alpha_1 \vee \alpha_2)} \ (\vee I2) \quad \frac{X^i(\alpha_1 \vee \alpha_2) \quad \begin{array}{c} [X^i\alpha_1] \\ \vdots \\ \beta \end{array} \quad \begin{array}{c} [X^i\alpha_2] \\ \vdots \\ \beta \end{array}}{\beta} \ (\vee E) \\
 \\
 \frac{X^i\alpha}{X^{l+m}\alpha} \ (XI) \quad \frac{X^{l+m}\alpha}{X^i\alpha} \ (XE) \\
 \\
 \frac{\{X^{i+j}\alpha\}_{j \in \omega_l}}{X^iG\alpha} \ (GI) \quad \frac{X^iG\alpha}{X^{i+k}\alpha} \ (GE) \\
 \\
 \frac{X^{i+k}\alpha}{X^iF\alpha} \ (FI) \quad \frac{X^iF\alpha \quad \begin{array}{c} [X^i\alpha] \\ \vdots \\ \beta \end{array} \quad \begin{array}{c} [X^{i+1}\alpha] \\ \vdots \\ \beta \end{array} \quad \cdots \quad \begin{array}{c} [X^{i+l}\alpha] \\ \vdots \\ \beta \end{array}}{\beta} \ (FE) \\
 \\
 \frac{X^i\alpha}{X^i\sim\sim\alpha} \ (\sim I) \quad \frac{X^i\sim\sim\alpha}{X^i\alpha} \ (\sim E) \\
 \\
 \frac{X^i(\alpha \wedge \sim\beta)}{X^i\sim(\alpha \rightarrow \beta)} \ (\sim\rightarrow I) \quad \frac{X^i\sim(\alpha \rightarrow \beta)}{X^i(\alpha \wedge \sim\beta)} \ (\sim\rightarrow E) \\
 \\
 \frac{X^i(\sim\alpha \vee \sim\beta)}{X^i\sim(\alpha \wedge \beta)} \ (\sim\wedge I) \quad \frac{X^i\sim(\alpha \wedge \beta)}{X^i(\sim\alpha \vee \sim\beta)} \ (\sim\wedge E) \\
 \\
 \frac{X^i(\sim\alpha \wedge \sim\beta)}{X^i\sim(\alpha \vee \beta)} \ (\sim\vee I) \quad \frac{X^i\sim(\alpha \vee \beta)}{X^i(\sim\alpha \wedge \sim\beta)} \ (\sim\vee E) \\
 \\
 \frac{X^iF\sim\alpha}{X^i\sim G\alpha} \ (\sim GI) \quad \frac{X^i\sim G\alpha}{X^iF\sim\alpha} \ (\sim GE) \\
 \\
 \frac{X^iG\sim\alpha}{X^i\sim F\alpha} \ (\sim FI) \quad \frac{X^i\sim F\alpha}{X^iG\sim\alpha} \ (\sim FE)
 \end{array}$$

$$\frac{X^i \sim \alpha}{\sim X^i \alpha} (\sim XI) \quad \frac{\sim X^i \alpha}{X^i \sim \alpha} (\sim XE)$$

For the sake of simplicity, the  $(l + 1)$ -premises rule (FE) is sometimes denoted as:

$$\frac{X^i F\alpha \quad \begin{array}{c} \{[X^{i+j}\alpha]\}_{j \in \omega_l} \\ \vdots \\ \beta \end{array}}{\beta} (FE)$$

The inference rules  $(\rightarrow I)$ ,  $(\wedge I)$ ,  $(\vee I1)$ ,  $(\vee I2)$ ,  $(XI)$ ,  $(GI)$ ,  $(FI)$ ,  $(\sim I)$ ,  $(\sim \rightarrow I)$ ,  $(\sim \wedge I)$ ,  $(\sim \vee I)$ ,  $(\sim GI)$ ,  $(\sim FI)$ , and  $(\sim XI)$  are called *introduction rules*, and the inference rules  $(\rightarrow E)$ ,  $(\wedge E1)$ ,  $(\wedge E2)$ ,  $(\vee E)$ ,  $(XE)$ ,  $(GE)$ ,  $(FE)$ ,  $(\sim E)$ ,  $(\sim \rightarrow E)$ ,  $(\sim \wedge E)$ ,  $(\sim \vee E)$ ,  $(\sim GE)$ ,  $(\sim FE)$ , and  $(\sim XE)$  are called *elimination rules*. The usual terminology of *major and minor premises* of inference rules is used. The notions of *proof* (of NPB[l]), (*open and discharged*) *assumptions* of a proof, and *end-formula* of a proof are defined as usual. A formula  $\alpha$  is said to be *provable* in NPB[l] iff there exists a proof of NPB[l] with no open assumption whose end-formula is  $\alpha$ . This terminology and these standard notions are from the well-known text books [25,31]. For example, the major and minor premises of (FE) are  $X^i F\alpha$  and  $\beta$ , respectively, and the discharged assumptions of (FE) are the square bracketed assumptions  $[X^i \alpha]$ ,  $[X^{i+1} \alpha]$ ,  $\dots$ ,  $[X^{i+l} \alpha]$ .

Note that NPB[l] includes the Gentzen-type natural deduction system GN for positive intuitionistic logic. Taking 0 for  $i$  in  $X^i$ , the rules of NPB[l] comprise the usual inference rules for GN. As a result, all the provable formulas (without temporal operators) in GN can be proved in NPB[l]. Thus, NPB[l] is an extension and generalization of GN.

We give an example proof in NPB[l] below, where we prove the temporal induction axiom for the case  $l = 1$ .

$$\frac{\frac{\frac{[\alpha \wedge G(\alpha \rightarrow X\alpha)]}{\alpha} (\wedge E1) \quad \frac{[\alpha \wedge G(\alpha \rightarrow X\alpha)]}{\alpha} (\wedge E1) \quad \frac{[\alpha \wedge G(\alpha \rightarrow X\alpha)]}{G(\alpha \rightarrow X\alpha)} (GE)}{\alpha \rightarrow X\alpha} (\rightarrow I)}{X\alpha} (GI) \quad \frac{[\alpha \wedge G(\alpha \rightarrow X\alpha)]}{G(\alpha \rightarrow X\alpha)} (\wedge E2)}{\frac{G\alpha}{(\alpha \wedge G(\alpha \rightarrow X\alpha)) \rightarrow G\alpha} (\rightarrow I)}$$

**Definition 34.** Let  $\alpha$  be a formula occurring in a proof  $D$  in NPB[l]. Then  $\alpha$  is called a *maximum formula* in  $D$  iff  $\alpha$  satisfies the following conditions: (1)  $\alpha$  is the conclusion of an introduction rule,  $(\vee E)$  or (FE), and (2)  $\alpha$  is the major premise of an elimination rule. A proof is said to be *normal* iff it contains no maximum formula.

In order to define a reduction relation  $\triangleright$  on the set of proofs, we assume the usual definition of substitution of proofs (for assumptions). The set of proofs is closed under substitution.

**Definition 35.** Let  $\gamma$  be a maximum formula in a proof which is the conclusion of an inference rule  $R$ . The reduction relation  $\triangleright$  at  $\gamma$  is defined as follows.

1.  $R$  is  $(\rightarrow I)$ , and  $\gamma$  is  $X^i(\alpha \rightarrow \beta)$ :

$$\frac{\frac{\begin{array}{c} [X^i \alpha] \\ \vdots \\ D \\ X^i \beta \end{array}}{X^i(\alpha \rightarrow \beta)} R \quad \begin{array}{c} \vdots \\ E \\ X^i \alpha \\ \vdots \\ D \\ X^i \beta \end{array}}{X^i \beta} \triangleright \begin{array}{c} \vdots \\ E \\ X^i \alpha \\ \vdots \\ D \\ X^i \beta \end{array}$$

2.  $R$  is  $(\wedge I)$ , and  $\gamma$  is  $X^i(\alpha_1 \wedge \alpha_2)$ :

$$\frac{\begin{array}{c} \vdots \\ D_1 \\ X^i \alpha_1 \end{array} \quad \begin{array}{c} \vdots \\ D_2 \\ X^i \alpha_2 \end{array}}{X^i(\alpha_1 \wedge \alpha_2)} R \quad \begin{array}{c} \vdots \\ D_n \\ X^i \alpha_n \end{array}}{X^i \alpha_n} \triangleright \begin{array}{c} \vdots \\ D_n \\ X^i \alpha_n \end{array}$$

where  $n$  is 1 or 2.

3.  $R$  is  $(\vee I1)$  or  $(\vee I2)$ , and  $\gamma$  is  $X^i(\alpha_1 \vee \alpha_2)$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i\alpha_n \end{array} R \quad \frac{\begin{array}{c} [X^i\alpha_1] \\ \vdots E_1 \end{array} \quad \begin{array}{c} [X^i\alpha_2] \\ \vdots E_2 \end{array}}{X^i\beta} \quad \begin{array}{c} \vdots D \\ X^i\alpha_n \\ \vdots E_n \end{array}}{X^i\beta} \triangleright \begin{array}{c} \vdots D \\ X^i\beta \end{array}$$

where  $n$  is 1 or 2.

4.  $R$  is  $(\vee E)$ :

$$\frac{\begin{array}{c} [X^i\alpha_1] \quad [X^i\alpha_2] \\ \vdots D_1 \quad \vdots D_2 \quad \vdots D_3 \\ X^i(\alpha_1 \vee \alpha_2) \quad X^i\gamma \quad X^i\gamma \end{array} R \quad \begin{array}{c} E_1 \quad \dots \quad E_l \end{array} R'}{X^i\beta} \triangleright \frac{\begin{array}{c} [X^i\alpha_1] \quad [X^i\alpha_2] \\ \vdots D_2 \quad \vdots D_3 \\ X^i(\alpha_1 \vee \alpha_2) \quad X^i\gamma \quad E_1 \quad \dots \quad E_l \end{array} R' \quad \begin{array}{c} X^i\gamma \quad E_1 \quad \dots \quad E_l \\ X^i\beta \end{array} R'}{X^i\beta} R$$

where  $R'$  is an arbitrary inference rule, and both  $E_1, \dots, E_l$  are proofs of the minor premises of  $R'$  if they exist.

5.  $R$  is  $(XI)$ , and  $\gamma$  is  $X^{i+m}\alpha$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i\alpha \end{array} R \quad \begin{array}{c} \vdots D \\ X^{i+m}\alpha \end{array}}{X^i\alpha} \triangleright \begin{array}{c} \vdots D \\ X^i\alpha \end{array}$$

6.  $R$  is  $(GI)$ , and  $\gamma$  is  $X^iG\alpha$ :

$$\frac{\begin{array}{c} \vdots D_j \\ \{X^{i+j}\alpha\}_{j \in \omega_1} \end{array} R \quad \begin{array}{c} \vdots D_k \\ X^{i+k}\alpha \end{array}}{X^iG\alpha} \triangleright \begin{array}{c} \vdots D_k \\ X^{i+k}\alpha \end{array}$$

7.  $R$  is  $(FI)$ , and  $\gamma$  is  $X^iF\alpha$ :

$$\frac{\begin{array}{c} \vdots D_k \quad \{X^{i+j}\alpha\}_{j \in \omega_1} \\ X^{i+k}\alpha \quad \vdots E_j \end{array} R \quad \begin{array}{c} \vdots D_k \\ X^{i+k}\alpha \\ \vdots E_k \end{array}}{X^iF\alpha} \triangleright \begin{array}{c} \vdots D_k \\ X^i\beta \end{array}$$

8.  $R$  is  $(FE)$ :

$$\frac{\begin{array}{c} \{X^{i+j}\alpha\}_{j \in \omega_1} \\ \vdots D \quad \vdots D_j \\ X^iF\alpha \quad X^i\gamma \end{array} R \quad \begin{array}{c} E_1 \quad \dots \quad E_l \end{array} R'}{\frac{X^i\gamma}{X^i\beta}} \triangleright \frac{\begin{array}{c} \{X^{i+j}\alpha\}_{j \in \omega_1} \\ \vdots D_j \\ \vdots D \quad X^i\gamma \quad E_1 \quad \dots \quad E_l \end{array} R' \quad \begin{array}{c} X^i\gamma \quad E_1 \quad \dots \quad E_l \\ X^i\beta \end{array} R}{X^i\beta} R$$

where  $R'$  is an arbitrary inference rule, and both  $E_1, \dots, E_l$  are proofs of the minor premises of  $R'$  if they exist.



9. R is ( $\sim I$ ), and  $\gamma$  is  $X^i \sim \sim \alpha$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i \alpha \end{array} R}{\frac{X^i \sim \sim \alpha}{X^i \alpha}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i \alpha \end{array}}$$

10. R is ( $\sim \rightarrow I$ ), and  $\gamma$  is  $X^i \sim (\alpha \rightarrow \beta)$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i (\alpha \wedge \sim \beta) \end{array} R}{\frac{X^i \sim (\alpha \rightarrow \beta)}{X^i (\alpha \wedge \sim \beta)}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i (\alpha \wedge \sim \beta) \end{array}}$$

11. R is ( $\sim \wedge I$ ), and  $\gamma$  is  $X^i \sim (\alpha \wedge \beta)$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i (\sim \alpha \vee \sim \beta) \end{array} R}{\frac{X^i \sim (\alpha \wedge \beta)}{X^i (\sim \alpha \vee \sim \beta)}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i (\sim \alpha \vee \sim \beta) \end{array}}$$

12. R is ( $\sim \vee I$ ), and  $\gamma$  is  $X^i \sim (\alpha \vee \beta)$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i (\sim \alpha \wedge \sim \beta) \end{array} R}{\frac{X^i \sim (\alpha \vee \beta)}{X^i (\sim \alpha \wedge \sim \beta)}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i (\sim \alpha \wedge \sim \beta) \end{array}}$$

13. R is ( $\sim GI$ ), and  $\gamma$  is  $X^i \sim G\alpha$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i F \sim \alpha \end{array} R}{\frac{X^i \sim G\alpha}{X^i F \sim \alpha}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i F \sim \alpha \end{array}}$$

14. R is ( $\sim FI$ ), and  $\gamma$  is  $X^i \sim F\alpha$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i G \sim \alpha \end{array} R}{\frac{X^i \sim F\alpha}{X^i G \sim \alpha}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i G \sim \alpha \end{array}}$$

15. R is ( $\sim XI$ ), and  $\gamma$  is  $\sim X^i \alpha$ :

$$\frac{\begin{array}{c} \vdots D \\ X^i \sim \alpha \end{array} R}{\frac{\sim X^i \alpha}{X^i \sim \alpha}} \triangleright \frac{\begin{array}{c} \vdots D \\ X^i \sim \alpha \end{array}}$$

16. Let  $D, D', E, F, D_j$  ( $j \in \omega_l$ ) be proofs. If  $D \triangleright D'$ , then

$$\begin{array}{l} \frac{D}{\alpha} (I) \triangleright \frac{D'}{\alpha} (I) \\ \frac{D \ E}{\alpha} (R) \triangleright \frac{D' \ E}{\alpha} (R) \\ \frac{E \ D}{\alpha} (R) \triangleright \frac{E \ D'}{\alpha} (R) \\ \frac{D \ E \ F}{\alpha} (\vee E) \triangleright \frac{D' \ E \ F}{\alpha} (\vee E) \end{array}$$

$$\frac{E \quad D \quad F}{\alpha} (\vee E) \triangleright \frac{E \quad D' \quad F}{\alpha} (\vee E)$$

$$\frac{E \quad F \quad D}{\alpha} (\vee E) \triangleright \frac{E \quad F \quad D'}{\alpha} (\vee E)$$

$$\frac{D_0 \cdots D \cdots D_l}{\alpha} (GI) \triangleright \frac{D_0 \cdots D' \cdots D_l}{\alpha} (GI)$$

$$\frac{D_0 \cdots D \cdots D_{l+1}}{\alpha} (FE) \triangleright \frac{D_0 \cdots D' \cdots D_{l+1}}{\alpha} (FE)$$

where  $I \in \{\rightarrow I, \wedge E1, \wedge E2, \vee I1, \vee I2, XI, XI, GE, FI \sim I, \sim E, \sim \rightarrow I, \sim \rightarrow E, \sim \wedge I, \sim \wedge E, \sim \vee I, \sim \vee E, \sim GI, \sim GE, \sim FI, \sim FE, \sim XI, \sim XI\}$ ,  $R \in \{\rightarrow E, \wedge I\}$  and  $D_0, \dots, D_{l+1}$  can be  $D$  or  $D'$ .

**Definition 36.** If  $D'$  is obtained from  $D$  by the reduction at  $\gamma$ , then such a fact is denoted by  $D \triangleright D'$ . A sequence  $D_0, D_1, \dots$  of proofs is called a *reduction sequence* iff it satisfies the following conditions: (1)  $D_i \triangleright D_{i+1}$  for all  $i \geq 0$  and (2) the last proof in the sequence is normal iff the sequence is finite. A proof  $D$  is called *normalizable* iff there is a finite reduction sequence starting from  $D$ .

Let  $P$  be a proof. The expression  $oa(P)$  denotes the set of open assumptions of  $P$ , and the expression  $end(P)$  denotes the end-formula of  $P$ .

From the following theorem, we can obtain the fact that a formula  $\alpha$  is provable in  $NPB[I]$  if and only if the sequent  $\Rightarrow \alpha$  is provable in  $PB[I]$ .

**Theorem 37** (Equivalence between  $NPB[I]$  and  $PB[I]$ ). *We have the following.*

1. If  $P$  is a proof in  $NPB[I]$  such that  $oa(P) = \Gamma$  and  $end(P) = \{\beta\}$ , then the sequent  $\Gamma \Rightarrow \beta$  is provable in  $PB[I]$ .
2. If a sequent  $\Gamma \Rightarrow \beta$  is provable in  $PB[I]$ -(cut), then there is a proof  $Q$  in  $NPB[I]$  which satisfies the following conditions: (a)  $oa(Q) = \Gamma$ , (b)  $end(Q) = \{\beta\}$ , and (c)  $Q$  is normal.

**Proof.** First, we show (1) by induction on a proof  $P$  of  $PB[I]$  such that  $oa(P) = \Gamma$  and  $end(P) = \{\beta\}$ . We distinguish the cases according to the last inference of  $P$ . We show some cases.

Case (GI):  $P$  is of the form:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ P' \\ \{X^{i+j}\alpha\}_{j \in \omega_i} \end{array}}{X^i G \alpha} (GI)$$

where  $oa(P) = \Gamma$  and  $end(P) = \{X^i G \alpha\}$ . By the hypothesis of induction, the sequents  $\Gamma_j \Rightarrow X^{i+j}\alpha$  for any  $j \in \omega_i$  where  $\Gamma_j$  is a subset of  $\Gamma$  are provable in  $PB[I]$ . Then the sequent  $\Gamma \Rightarrow X^i G \alpha$  is provable in  $PB[I]$  by using (we-left) and (Gright).

Case (FE):  $P$  is of the form:

$$\frac{\begin{array}{cccc} \Gamma_{l+1} & \Gamma_0[X^i\alpha] & \Gamma_1[X^{i+1}\alpha] & \Gamma_l[X^{i+l}\alpha] \\ \vdots & \vdots & \vdots & \vdots \\ X^i F \alpha & \gamma & \gamma & \cdots \quad \gamma \end{array}}{\gamma} (FE)$$

where  $oa(P) = \Gamma = \bigcup_{j \in \omega_{l+1}} \Gamma_j$  and  $end(P) = \{\gamma\}$ . By the hypothesis of induction, the following sequents are provable in  $PB[I]$ :  $(\Gamma_{l+1} \Rightarrow X^i F \alpha)$ ,  $(X^i \alpha, \Gamma_0 \Rightarrow \gamma)$ ,  $(X^{i+1} \alpha, \Gamma_1 \Rightarrow \gamma)$ ,  $\dots$ ,  $(X^{i+l} \alpha, \Gamma_l \Rightarrow \gamma)$ . Then we obtain the required fact:

$$\frac{\begin{array}{cccc} \Gamma_{l+1} \Rightarrow X^i F \alpha & X^i \alpha, \Gamma_0 \Rightarrow \gamma & X^{i+1} \alpha, \Gamma_1 \Rightarrow \gamma & X^{i+l} \alpha, \Gamma_l \Rightarrow \gamma \\ \vdots & \vdots \text{ (we-left)} & \vdots \text{ (we-left)} & \vdots \text{ (we-left)} \\ \vdots & X^i \alpha, \Gamma \Rightarrow \gamma & X^{i+1} \alpha, \Gamma \Rightarrow \gamma & \cdots \quad X^{i+l} \alpha, \Gamma \Rightarrow \gamma \end{array}}{\Gamma \Rightarrow X^i F \alpha} \text{ (Fleft)}$$

$$\frac{\Gamma \Rightarrow X^i F \alpha \quad X^i F \alpha, \Gamma \Rightarrow \gamma}{\Gamma, \Gamma \Rightarrow \gamma} \text{ (cut)}$$

$$\frac{\Gamma, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma} \text{ (co)}$$

Case ( $\sim \rightarrow I$ ):  $P$  is of the form:

$$\frac{\begin{array}{c} \Gamma \\ \vdots \\ P' \\ X^i(\alpha \wedge \sim\gamma) \end{array}}{X^i\sim(\alpha \rightarrow \gamma)} (\sim \rightarrow I)$$

where  $oa(P) = \Gamma$  and  $end(P) = \{X^i\sim(\alpha \rightarrow \gamma)\}$ . By the hypothesis of induction, the sequent  $\Gamma \Rightarrow X^i(\alpha \wedge \sim\gamma)$  is provable in  $PB[I]$ . Then the sequent  $\Gamma \Rightarrow X^i\sim(\alpha \rightarrow \gamma)$  is provable in  $PB[I]$  as follows.

$$\frac{\frac{\frac{X^i\alpha \Rightarrow X^i\alpha}{X^i\alpha, X^i\sim\gamma \Rightarrow X^i\alpha} \text{ (we-left)} \quad \frac{X^i\sim\gamma \Rightarrow X^i\sim\gamma}{X^i\alpha, X^i\sim\gamma \Rightarrow X^i\sim\gamma} \text{ (we-left)}}{X^i\alpha, X^i\sim\gamma \Rightarrow X^i\sim(\alpha \rightarrow \gamma)} (\sim \rightarrow \text{right})}{\vdots} (\wedge \text{left1, 2), (co)}$$

$$\frac{\Gamma \Rightarrow X^i(\alpha \wedge \sim\gamma) \quad X^i(\alpha \wedge \sim\gamma) \Rightarrow X^i\sim(\alpha \rightarrow \gamma)}{\Gamma \Rightarrow X^i\sim(\alpha \rightarrow \gamma)} \text{ (cut)}$$

Second, we prove (2) by induction on a cut-free proof  $P$  of  $\Gamma \Rightarrow \beta$  in  $PB[I]$ -(cut). We distinguish the cases according to the inference of  $P$ . We show some cases.

Case (Gleft):  $P$  is of the form:

$$\frac{X^{i+k}\alpha, \Gamma \Rightarrow \gamma}{X^iG\alpha, \Gamma \Rightarrow \gamma} \text{ (Gleft)}$$

By the hypothesis of induction, there is a normal proof  $Q'$  in  $NPB[I]$  of the form:

$$\begin{array}{c} \Gamma, X^{i+k}\alpha \\ \vdots \\ Q' \\ \gamma \end{array}$$

where  $oa(Q') = \Gamma \cup \{X^{i+k}\alpha\}$  and  $end(Q') = \{\gamma\}$ . Then we obtain a normal proof  $Q$  as follows.

$$\frac{\frac{X^iG\alpha}{\Gamma \quad X^{i+k}\alpha} \text{ (GE)}}{\vdots} \frac{Q'}{\gamma}$$

where  $oa(Q) = \Gamma \cup \{X^iG\alpha\}$  and  $end(Q) = \{\gamma\}$ .

Case (Fleft):  $P$  is of the form:

$$\frac{\{X^{i+j}\alpha, \Gamma \Rightarrow \gamma\}_{j \in \omega_1}}{X^iF\alpha, \Gamma \Rightarrow \gamma} \text{ (Fleft)}$$

By the hypothesis of induction, there are normal proofs  $\{Q_j\}_{j \in \omega_1}$  in  $NPB[I]$  of the form: for any  $j \in \omega_1$ ,

$$\begin{array}{c} \Gamma, X^{i+j}\alpha \\ \vdots \\ Q_j \\ \gamma \end{array}$$

where  $oa(Q_j) = \Gamma \cup \{X^{i+j}\alpha\}$  and  $end(Q_j) = \{\gamma\}$ . Then we obtain a normal proof  $Q$  as follows:

$$\frac{\frac{\Gamma[X^i\alpha] \quad \Gamma[X^{i+1}\alpha]}{\vdots} \frac{Q_0 \quad \dots \quad Q_1}{\gamma} \text{ (FE)}}{X^iF\alpha \quad \gamma} \gamma$$

where  $oa(Q) = \Gamma \cup \{X^iF\alpha\}$  and  $end(Q) = \{\gamma\}$ .

Case ( $\sim \rightarrow$ right):  $P$  is of the form:

$$\frac{\begin{array}{c} \vdots \\ P_1 \end{array} \quad \begin{array}{c} \vdots \\ P_2 \end{array}}{\Gamma \Rightarrow X^i\alpha \quad \Gamma \Rightarrow X^i\sim\gamma} (\sim \rightarrow \text{right})$$

$$\frac{\Gamma \Rightarrow X^i\alpha \quad \Gamma \Rightarrow X^i\sim\gamma}{\Gamma \Rightarrow X^i\sim(\alpha \rightarrow \gamma)}$$

By the hypothesis of induction, there are normal proofs  $Q_1$  and  $Q_2$  in  $\text{NPB}[I]$  of the form:

$$\begin{array}{c} \Gamma \\ \vdots \\ Q_1 \\ \vdots \\ X^i\alpha, \end{array} \quad \begin{array}{c} \Gamma \\ \vdots \\ Q_2 \\ \vdots \\ X^i\sim\gamma \end{array}$$

where  $\text{oa}(Q_1) = \text{oa}(Q_2) = \Gamma$ ,  $\text{end}(Q_1) = \{X^i\alpha\}$  and  $\text{end}(Q_2) = \{X^i\sim\gamma\}$ . Then we obtain a normal proof  $Q$  as follows.

$$\frac{\frac{\frac{\Gamma}{\vdots} Q_1 \quad \frac{\Gamma}{\vdots} Q_2}{X^i\alpha \quad X^i\sim\gamma} (\wedge I)}{\frac{X^i(\alpha \wedge \sim\gamma)}{X^i\sim(\alpha \rightarrow \gamma)} (\sim \rightarrow I)}$$

where  $\text{oa}(Q) = \Gamma$  and  $\text{end}(Q) = \{X^i\sim(\alpha \rightarrow \gamma)\}$ .  $\square$

**Theorem 38** (Normalization for  $\text{NPB}[I]$ ). All proofs in  $\text{NPB}[I]$  are normalizable. More precisely, if a proof  $P$  in  $\text{NPB}[I]$  is given, then there is a normal proof  $Q$  such that  $\text{oa}(Q) = \text{oa}(P)$  and  $\text{end}(Q) = \text{end}(P)$ .

**Proof.** Suppose  $\text{oa}(P) = \Gamma$  and  $\text{end}(P) = \{\beta\}$ . By Theorem 37(1), the sequent  $\Gamma \Rightarrow \beta$  is provable in  $\text{PB}[I]$ . By Theorem 23, the sequent  $\Gamma \Rightarrow \beta$  is also provable in  $\text{PB}[I]$ -(cut). Then, by Theorem 37(2), there is a normal proof  $Q$  in  $\text{NPB}[I]$  such that  $\text{oa}(Q) = \text{oa}(P)$  and  $\text{end}(Q) = \text{end}(P)$ .  $\square$

Next, we discuss  $\text{NIB}[I]$ . The proof of the equivalence and normalization theorems are omitted.

**Definition 39** ( $\text{NIB}[I]$ ).  $\text{NIB}[I]$  is obtained from  $\text{NPB}[I]$  by deleting the inference rules concerning  $\sim$  and adding the inference rule of the form:

$$\frac{X^i\perp}{\alpha} (\perp E)$$

The notions of proof, reduction, etc. are defined similarly as for  $\text{NPB}[I]$ .

**Theorem 40** (Equivalence between  $\text{NIB}[I]$  and  $\text{IB}[I]$ ). We have the following.

1. If  $P$  is a proof in  $\text{NIB}[I]$  such that  $\text{oa}(P) = \Gamma$  and  $\text{end}(P) = \{\beta\}$ , then the sequent  $\Gamma \Rightarrow \beta$  is provable in  $\text{IB}[I]$ .
2. If a sequent  $\Gamma \Rightarrow \beta$  is provable in  $\text{IB}[I]$ -(cut), then there is a proof  $Q$  in  $\text{NIB}[I]$  which satisfies the following conditions: (a)  $\text{oa}(Q) = \Gamma$ , (b)  $\text{end}(Q) = \{\beta\}$ , and (c)  $Q$  is normal.

**Theorem 41** (Normalization for  $\text{NIB}[I]$ ). All proofs in  $\text{NIB}[I]$  are normalizable. More precisely, if a proof  $P$  in  $\text{NIB}[I]$  is given, then there is a normal proof  $Q$  such that  $\text{oa}(Q) = \text{oa}(P)$  and  $\text{end}(Q) = \text{end}(P)$ .

## 5. Display calculi

### 5.1. A display calculus for $\text{IB}[I]$

In this subsection, we present a display sequent calculus  $\delta\text{IB}[I]$  for  $\text{IB}[I]$ . In comparison to the sequent calculus from Section 2,  $\delta\text{IB}[I]$  has some advantages from a philosophical point of view, see also [4,11,35]. In particular, if the introduction rules of a sequent calculus are viewed as meaning assignments, then the sequent calculus from Section 2 is *holistic* in the sense that it assigns a meaning to the operators  $X^i$  only in combination with *each* of the other object language connectives. By suitably generalizing the notion of a sequent and exploiting the fact that (i)  $\wedge$  and  $\rightarrow$ , (ii)  $G$  and  $P$  (“sometimes in the past”), (iii)  $H$  (“always in the past”) and  $F$ , and (iv)  $X^i$  and  $E^i$  (“ $i$  steps earlier”) form residuated pairs, it is possible to state introduction rules for the connectives in such a way that every operation is introduced as the main connective of a single-antecedent (single-succedent) conclusion sequent. Moreover, the right and left introduction rules exhibit only one occurrence of the operation and no occurrence of another connective from the object language. Furthermore, the interpretation of some *structural* connectives in the display calculus as backward-looking temporal operators in either antecedent or succedent position allows one to add introduction rules with the just mentioned property also for the backward-looking modalities. Certain properties of the assumed temporal order such as the boundedness of the time domain can then be expressed by purely structural sequent rules not exhibiting any operations of the logical object language.

In ordinary sequent calculi, the comma, ‘,’ may be seen as a context-sensitive structural connective. It is to be understood as conjunction in antecedent position and as disjunction in succedent position of a sequent. In  $\delta\text{IB}[I]$  we shall use one binary

operation and certain unary operations as structural connectives. A sequent is an expression of the shape  $\Delta \Rightarrow \Gamma$ , where  $\Delta$  and  $\Gamma$  are structures (or ‘Gentzen terms’). We assume the empty structure **I**, and the set of structures is inductively defined from a set *Atom* of atomic formulas as follows:

formulas:  $\alpha \in \text{Form}(\text{Atom})$   
 structures  $\Delta \in \text{Struc}(\text{Form})$   
 $\Delta ::= A \mid \mathbf{I} \mid (\Delta; \Delta) \mid \triangleleft \Delta \mid \triangleright \Delta \mid \times \Delta \mid \times^i \Delta$

In antecedent position, ; is to be interpreted as conjunction and in succedent position as implication. In antecedent position,  $\triangleleft$  is to be read as P and in succedent position as G, whereas  $\triangleright$  is to be understood as F in antecedent position and as H in succedent position. The structure  $\times^i \Delta$  for any  $i \in \omega$  is inductively defined by  $(\times^0 \Delta := \Delta)$  and  $(\times^{n+1} \Delta := \times^n \times \Delta)$ . Similarly,  $(\times^0 \Delta := \Delta)$  and  $(\times^{n+1} \Delta := \times^n \times \Delta)$ . In succedent position  $\times^i$  means  $E^i$  and in antecedent position it means  $X^i$ . In succedent position  $\times^i$  means  $X^i$  and in antecedent position it means  $E^i$ .

The suggested interpretation of the structural connectives justifies a number of ‘display postulates’ (*dp*) (we omit outer brackets)<sup>3</sup>:

$$\frac{\Gamma \Rightarrow \Delta; \Sigma}{\Delta; \Gamma \Rightarrow \Sigma} \quad \frac{\Delta \Rightarrow \Gamma; \Sigma}{\Delta; \Gamma \Rightarrow \Sigma}$$

$$\frac{\Delta \Rightarrow \Gamma; \Sigma}{\Gamma \Rightarrow \Delta; \Sigma}$$

$$\frac{\triangleleft \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \triangleleft \Gamma} \quad \frac{\Delta \Rightarrow \triangleleft \Gamma}{\triangleleft \Delta \Rightarrow \Gamma} \quad \frac{\triangleright \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \triangleright \Gamma} \quad \frac{\Delta \Rightarrow \triangleright \Gamma}{\triangleright \Delta \Rightarrow \Gamma}$$

$$\frac{\times^i \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \times^i \Gamma} \quad \frac{\Delta \Rightarrow \times^i \Gamma}{\times^i \Delta \Rightarrow \Gamma} \quad \frac{\times^i \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \times^i \Gamma} \quad \frac{\Delta \Rightarrow \times^i \Gamma}{\times^i \Delta \Rightarrow \Gamma}$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \times^i \times^i \Gamma} \quad \frac{\Delta \Rightarrow \times^i \times^i \Gamma}{\Delta \Rightarrow \Gamma} \quad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \times^i \times^i \Gamma} \quad \frac{\Delta \Rightarrow \times^i \times^i \Gamma}{\Delta \Rightarrow \Gamma}$$

Moreover, we assume initial sequents  $p \Rightarrow p$ , a cut-rule:

$$\frac{\Delta \Rightarrow \alpha \quad \alpha \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma} \text{ (cut)}$$

rules which govern the empty structure:

$$\frac{\Delta; \mathbf{I} \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma} \quad \frac{\mathbf{I}; \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma}$$

$$\frac{\Delta \Rightarrow \Gamma}{\mathbf{I}; \Delta \Rightarrow \Gamma} \quad \frac{\Delta \Rightarrow \Gamma}{\Delta; \mathbf{I} \Rightarrow \Gamma}$$

and versions of the standard left structural rules from ordinary Gentzen calculi, weakening, exchange, and contraction, together with associativity:

$$\frac{\Delta \Rightarrow \Gamma}{\Delta; \Sigma \Rightarrow \Gamma} \text{ (lm)} \quad \frac{\Delta; \Sigma \Rightarrow \Gamma}{\Sigma; \Delta \Rightarrow \Gamma} \text{ (le)} \quad \frac{\Delta; \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma} \text{ (lc)} \quad \frac{(\Delta; \Gamma); \Sigma \Rightarrow \Theta}{\Delta; (\Gamma; \Sigma) \Rightarrow \Theta} \text{ (la)}$$

We also assume further structural rules, for any  $k \in \omega$ , to express the boundedness of the temporal order (rules (b), (b’)), to capture the interaction between the temporal operators (rules (lg)–(rf)), and to capture part of the interaction between  $X^i$  and  $\rightarrow$  (rule  $\star$ ):

$$\frac{\Delta \Rightarrow \times^i \Gamma; \times^i \Sigma}{\Delta \Rightarrow \times^i (\Gamma; \Sigma)} \star \quad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \times^{l+m} \times^l \Gamma} \text{ (b)} \quad \frac{\Delta \Rightarrow \Gamma}{\times^{l+m} \times^l \Delta \Rightarrow \Gamma} \text{ (b')}$$

$$\frac{\Delta \Rightarrow \triangleleft \Gamma}{\Delta \Rightarrow \times^i \times^{i+k} \Gamma} \text{ (lg)} \quad \frac{\{\Delta \Rightarrow \times^{i+j} \Gamma\} j \in \omega_l}{\triangleleft \times^i \Delta \Rightarrow \Gamma} \text{ (rg)}$$

$$\frac{\{\times^{i+j} \Delta \Rightarrow \Gamma\} j \in \omega_l}{\Delta \Rightarrow \triangleright \times^i \Gamma} \text{ (lf)} \quad \frac{\triangleright \Delta \Rightarrow \Gamma}{\times^i \times^{i+k} \Delta \Rightarrow \Gamma} \text{ (rf)}$$

**Definition 42.** The display sequent calculus  $\delta\text{IB}[I]$  consists of the above sequent rules together with the following right and left introduction rules<sup>4</sup>:

<sup>3</sup> Note that the following display postulates are derivable:

$$\frac{\Delta \Rightarrow \Gamma}{\times^i \times^i \Delta \Rightarrow \Gamma} \quad \frac{\times^i \times^i \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma} \quad \frac{\Delta \Rightarrow \Gamma}{\times^i \times^i \Delta \Rightarrow \Gamma} \quad \frac{\times^i \times^i \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma}$$

<sup>4</sup> We use the same names for the logical inference rules as in the standard-style  $\text{IB}[I]$ .

$$\begin{array}{c}
\frac{}{\perp \Rightarrow \Delta} \text{ (\perp left)} \\
\frac{\Delta \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Delta; \Gamma \Rightarrow (\alpha \wedge \beta)} \text{ (\wedge right)} \quad \frac{\alpha; \beta \Rightarrow \Delta}{(\alpha \wedge \beta) \Rightarrow \Delta} \text{ (\wedge left)} \\
\frac{\Delta \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Delta \Rightarrow (\alpha \vee \beta)} \text{ (\vee right)} \quad \frac{\alpha \Rightarrow \Delta \quad \beta \Rightarrow \Delta}{(\alpha \vee \beta) \Rightarrow \Delta} \text{ (\vee left)} \\
\frac{\Delta \Rightarrow \alpha; \beta}{\Delta \Rightarrow (\alpha \rightarrow \beta)} \text{ (\rightarrow right)} \quad \frac{\Delta \Rightarrow \alpha \quad \beta \Rightarrow \Gamma}{(\alpha \rightarrow \beta) \Rightarrow \Delta; \Gamma} \text{ (\rightarrow left)} \\
\frac{\triangleleft \Delta \Rightarrow \alpha}{\Delta \Rightarrow G\alpha} \text{ (Gright)} \quad \frac{\alpha \Rightarrow \Delta}{G\alpha \Rightarrow \triangleleft \Delta} \text{ (Gleft)} \\
\frac{\Delta \Rightarrow \alpha}{\triangleright \Delta \Rightarrow F\alpha} \text{ (Fright)} \quad \frac{\alpha \Rightarrow \triangleright \Delta}{F\alpha \Rightarrow \Delta} \text{ (Fleft)} \\
\frac{\times^i \Delta \Rightarrow \alpha}{\Delta \Rightarrow X^i \alpha} \text{ (X^i right)} \quad \frac{\alpha \Rightarrow \times^i \Delta}{X^i \alpha \Rightarrow \Delta} \text{ (X^i left)}
\end{array}$$

**Proposition 43.** In  $\delta\text{IB}[l]$ ,  $\alpha \Rightarrow \alpha$  is provable for every formula  $\alpha$ .

**Proof.** By induction on  $\alpha$ .  $\square$

We take up the earlier example of a natural deduction proof and present a proof in  $\delta\text{IB}[l]$  of the temporal induction axiom for the case  $l = 1$ . We first present a proof  $\Pi_1$  of  $G(\alpha \rightarrow X\alpha) \Rightarrow X(\alpha \rightarrow X\alpha)$ .

$$\frac{\frac{\frac{\alpha \rightarrow X\alpha \Rightarrow \alpha \rightarrow X\alpha}{G(\alpha \rightarrow X\alpha) \Rightarrow \triangleleft(\alpha \rightarrow X\alpha)}}{G(\alpha \rightarrow X\alpha) \Rightarrow \times(\alpha \rightarrow X\alpha)} \text{ (lg)}}{\times G(\alpha \rightarrow X\alpha) \Rightarrow \alpha \rightarrow X\alpha} \\
\frac{}{G(\alpha \rightarrow X\alpha) \Rightarrow X(\alpha \rightarrow X\alpha)}$$

Next, we give a proof  $\Pi_2$  of  $\alpha; X(\alpha \rightarrow X\alpha) \Rightarrow X\alpha$ .

$$\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha \quad X\alpha \Rightarrow X\alpha}{\alpha \rightarrow X\alpha \Rightarrow \alpha; X\alpha}}{\alpha \rightarrow X\alpha \Rightarrow \times \times (\alpha; X\alpha)} \text{ (b)}}{X(\alpha \rightarrow X\alpha) \Rightarrow \times \times (\alpha; X\alpha)}}{X(\alpha \rightarrow X\alpha) \Rightarrow \alpha; X\alpha} \\
\frac{}{\alpha; X(\alpha \rightarrow X\alpha) \Rightarrow X\alpha}$$

We combine  $\Pi_1$  and  $\Pi_2$  to obtain a proof  $\Pi_3$  of  $\alpha \wedge G(\alpha \rightarrow X\alpha) \Rightarrow X\alpha$ .

$$\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha \quad X(\alpha \rightarrow X\alpha) \Rightarrow X(\alpha \rightarrow X\alpha)}{\alpha; X(\alpha \rightarrow X\alpha) \Rightarrow \alpha \wedge X(\alpha \rightarrow X\alpha)} \quad \frac{\Pi_2}{\alpha \wedge X(\alpha \rightarrow X\alpha) \Rightarrow X\alpha}}{\alpha; X(\alpha \rightarrow X\alpha) \Rightarrow X\alpha}}{\Pi_1 \quad X(\alpha \rightarrow X\alpha) \Rightarrow \alpha; X\alpha} \\
\frac{G(\alpha \rightarrow X\alpha) \Rightarrow \alpha; X\alpha}{\alpha; G(\alpha \rightarrow X\alpha) \Rightarrow X\alpha} \\
\frac{}{\alpha \wedge G(\alpha \rightarrow X\alpha) \Rightarrow X\alpha}$$

We can now use  $\Pi_3$  in a proof of the induction axiom (for the case  $l = 1$ ).

$$\frac{\frac{\frac{\alpha \Rightarrow \alpha}{\alpha; G(\alpha \rightarrow X\alpha) \Rightarrow \alpha}}{\alpha \wedge G(\alpha \rightarrow X\alpha) \Rightarrow \alpha} \quad \frac{\frac{\frac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \times \times \alpha}}{\Pi_3 \quad X\alpha \Rightarrow \times \alpha}}{\alpha \wedge G(\alpha \rightarrow X\alpha) \Rightarrow \times \alpha} \text{ (rg)}}{\frac{\triangleleft(\alpha \wedge G(\alpha \rightarrow X\alpha)) \Rightarrow \alpha}{\alpha \wedge G(\alpha \rightarrow X\alpha) \Rightarrow G\alpha}}$$

If two sequents are interderivable by means of the display postulates, the sequents are said to be *display equivalent*. In display logic, any substructure of a given sequent  $s$  may be displayed as either the entire antecedent or the entire succedent

of a display equivalent sequent  $s'$ . In order to state this claim precisely, we need the notions of antecedent parts and succedent parts of a sequent. A succedent part of a sequent  $\Delta \Rightarrow \Gamma$  is a certain occurrence of a substructure of  $\Gamma$ . Suppose  $\Sigma$  occurs as a substructure of  $\Gamma$ . Then this occurrence of  $\Sigma$  is said to be a succedent part of the sequent  $\Delta \Rightarrow \Gamma$  iff

1.  $\Gamma \equiv \Sigma$ , or
2.  $\Gamma \equiv \Gamma_1; \Gamma_2$  and  $\Sigma$  is a succedent part of  $\Delta; \Gamma_1 \Rightarrow \Gamma_2$ , or
3.  $\Gamma \equiv \sharp\Gamma_1$ , and  $\Sigma$  is a succedent part of  $\sharp\Delta \Rightarrow \Gamma_1$ ,  $\sharp \in \{\triangleright, \triangleleft, \times^i, \times^i\}$ .

An antecedent part of  $s \equiv \Delta \Rightarrow \Gamma$  is either an occurrence of a substructure of  $\Delta$  or an occurrence of a substructure of  $\Gamma$  that is not a succedent part of  $s$ .

**Theorem 44** (Display property for  $\delta\text{IB}[I]$ ). (See Belnap [4].) For every sequent  $s$  and every antecedent part (succedent part)  $\Delta$  of  $s$  there exists a sequent  $s'$  display equivalent to  $s$  such that  $\Delta$  is the entire antecedent (succedent) of  $s'$ .<sup>5</sup>

**Theorem 45** (Cut-elimination for  $\delta\text{IB}[I]$ ). (See Belnap [4].) Every proof of a sequent  $\Delta \Rightarrow \Gamma$  in  $\delta\text{IB}[I]$ , can be converted into a cut-free proof of  $\Delta \Rightarrow \Gamma$  in  $\delta\text{IB}[I]$ .

**Proof.** This follows from Belnap's cut-elimination theorem for properly displayable logics. The calculus  $\delta\text{IB}[I]$  satisfies Belnap's conditions (C1)–(C8) and hence is properly displayable.  $\square$

The context-sensitive reading of the structural connectives is made explicit by the following translation from sequents into formulas, where  $\top$  is defined as  $\alpha \rightarrow \alpha$  for some fixed atomic formula  $\alpha$ .

**Definition 46.** If  $\Delta \Rightarrow \Gamma$  is a sequent then its translation  $\tau(\Delta \Rightarrow \Gamma)$  is the formula  $\tau_1(\Delta) \rightarrow \tau_2(\Gamma)$ , where the translations  $\tau_1$  and  $\tau_2$  from structures into formulas are inductively defines as follows:

- (1) If  $\Sigma$  is a formula  $\alpha$ , then  $\tau_1(\Sigma) \equiv \tau_2(\Sigma) := \alpha$ .
- (2) If  $\Sigma \equiv \mathbf{I}$ , then  $\tau_1(\Sigma) := \top$ ;  $\tau_2(\Sigma) := \perp$ .
- (3)  $\tau_1(\Sigma; \Theta) := \tau_1(\Sigma) \wedge \tau_1(\Theta)$ ;  $\tau_2(\Sigma; \Theta) := \tau_1(\Sigma) \rightarrow \tau_2(\Theta)$ .
- (4)  $\tau_1(\triangleleft\Sigma) := \text{P}\tau_1(\Sigma)$ ;  $\tau_2(\triangleleft\Sigma) := \text{G}\tau_2(\Sigma)$ .
- (5)  $\tau_1(\triangleright\Sigma) := \text{H}\tau_1(\Sigma)$ ;  $\tau_2(\triangleright\Sigma) := \text{F}\tau_2(\Sigma)$ .
- (6)  $\tau_1(\times^i\Sigma) := \text{X}^i\tau_1(\Sigma)$ ;  $\tau_2(\times^i\Sigma) := \text{E}^i\tau_2(\Sigma)$ .
- (7)  $\tau_1(\times^i\Sigma) := \text{E}^i\tau_1(\Sigma)$ ;  $\tau_2(\times^i\Sigma) := \text{X}^i\tau_2(\Sigma)$ .

**Theorem 47** (Soundness of  $\delta\text{IB}[I]$ ). If  $\delta\text{IB}[I] \vdash \Delta \Rightarrow \Gamma$ , then  $\tau(\Delta \Rightarrow \Gamma)$  is valid in the class of all Kripke frames.

**Proof.** The evaluation conditions for formulas  $\text{E}\alpha$  are:  $(x, i) \models \text{E}\alpha$  iff  $(x, i-1) \models \alpha$  if  $i > 0$  and otherwise  $(x, 0) \models \alpha$ . For formulas  $\text{P}\alpha$  we have:  $(x, i) \models \text{P}\alpha$  iff  $\exists j \in N_l [j \leq i \text{ and } (x, j) \models \alpha]$  if  $i > 0$  and otherwise  $(x, 0) \models \alpha$ . By induction on proofs in  $\delta\text{IB}[I]$ , it can be shown that if  $\delta\text{IB}[I] \vdash \Delta \Rightarrow \Gamma$ , then for every Kripke frame and valuation  $\models$ ,  $\tau(\Delta \Rightarrow \Gamma)$  is true at every state  $x$  and every moment  $i$ . Thus,  $\forall x (x, 0) \models \tau(\Delta \Rightarrow \Gamma)$ .  $\square$

If  $\Delta \equiv \langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle$  ( $1 \leq n$ ), let  $\Delta^\wedge$  and  $\Delta^\vee$  stand for  $\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n$  and  $\alpha_1 \vee \alpha_2 \vee \dots \vee \alpha_n$ , respectively. If  $\Delta$  is the empty sequence, let  $\Delta^\wedge$  and  $\Delta^\vee$  stand for  $\top$  and  $\perp$ , respectively.

**Theorem 48** (Completeness of  $\delta\text{IB}[I]$ ). If the sequent  $\Delta \Rightarrow \Gamma$  is provable in the sequent calculus for  $\text{IB}[I]$  from Section 2, then  $\Delta^\wedge \Rightarrow \Gamma^\vee$  is provable in  $\delta\text{IB}[I]$ .

**Proof.** By induction on proofs in the standard sequent system for  $\text{IB}[I]$ .

Initial sequents: By Proposition 43 and

$$\frac{\perp \Rightarrow \times^i \perp}{\text{X}^i \perp \Rightarrow \perp}$$

<sup>5</sup> An elegant method for proving the display property can be found in [28].

(cut); the cases of the other structural rules are obvious:

$$\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha \quad \Sigma^\wedge \Rightarrow \Sigma^\wedge}{\alpha; \Sigma^\wedge \Rightarrow \alpha \wedge \Sigma^\wedge} \quad \alpha \wedge \Sigma^\wedge \Rightarrow \Delta^\vee}{\alpha; \Sigma^\wedge \Rightarrow \Delta^\vee}}{\alpha \Rightarrow \Sigma^\wedge; \Delta^\vee}}{\frac{\Gamma^\wedge \Rightarrow \alpha \quad \alpha \Rightarrow \Sigma^\wedge \rightarrow \Delta^\vee}{\Gamma^\wedge \Rightarrow \Sigma^\wedge \rightarrow \Delta^\vee} \quad \frac{\Sigma^\wedge \Rightarrow \Sigma \wedge \quad \Delta^\vee \Rightarrow \Delta^\vee}{\Sigma \wedge \rightarrow \Delta^\vee \Rightarrow \Sigma \wedge; \Delta^\vee}}{\frac{\Gamma^\wedge \Rightarrow \Sigma^\wedge; \Delta^\vee}{\Gamma^\wedge; \Sigma^\wedge \Rightarrow \Delta^\vee}} \quad \frac{\Gamma^\wedge \wedge \Sigma^\wedge \Rightarrow \Delta^\vee}$$

( $\rightarrow$ left): By the induction hypothesis, the sequents  $\Gamma^\wedge \Rightarrow X^i\alpha$  and  $X^i\beta \wedge \Sigma^\wedge \Rightarrow \Delta^\vee$  are provable in  $\delta\text{IB}[I]$ . If  $X^i(\alpha \rightarrow \beta) \Rightarrow X^i\alpha \rightarrow X^i\beta$  is provable in  $\delta\text{IB}[I]$ , we can proceed as follows:

$$\frac{\frac{\frac{\frac{X^i\beta \Rightarrow X^i\beta \quad \Sigma^\wedge \Rightarrow \Sigma^\wedge}{X^i\beta; \Sigma^\wedge \Rightarrow X^i\beta \wedge \Sigma^\wedge} \quad X^i\beta \wedge \Sigma^\wedge \Rightarrow \Delta^\vee}{X^i\beta; \Sigma^\wedge \Rightarrow \Delta^\vee}}{\frac{\Gamma^\wedge \Rightarrow X^i\alpha \quad X^i\beta \Rightarrow \Sigma^\wedge; \Delta^\vee}{X^i\alpha \rightarrow X^i\beta \Rightarrow \Gamma^\wedge; (\Sigma^\wedge; \Delta^\vee)}}{\frac{X^i(\alpha \rightarrow \beta) \Rightarrow X^i\alpha \rightarrow X^i\beta \quad X^i\alpha \rightarrow X^i\beta \Rightarrow \Gamma^\wedge; (\Sigma^\wedge; \Delta^\vee)}{X^i(\alpha \rightarrow \beta) \Rightarrow \Gamma^\wedge; (\Sigma^\wedge; \Delta^\vee)}} \quad \vdots \quad \frac{X^i(\alpha \rightarrow \beta) \wedge \Gamma^\wedge \wedge \Sigma^\wedge \Rightarrow \Delta^\vee}$$

Thus, it remains to show that  $X^i(\alpha \rightarrow \beta) \Rightarrow X^i\alpha \rightarrow X^i\beta$  is provable in  $\delta\text{IB}[I]$ . To this end we will combine the following four proofs,  $\Pi_1$ – $\Pi_4$ .

$\Pi_1$ :

$$\frac{\frac{\frac{\alpha \rightarrow \beta \Rightarrow \alpha \rightarrow \beta}{\alpha \rightarrow \beta \Rightarrow \times^i \times^i (\alpha \rightarrow \beta)}}{X^i(\alpha \rightarrow \beta) \Rightarrow \times^i (\alpha \rightarrow \beta)}}{X^i(\alpha \rightarrow \beta); X^i\alpha \Rightarrow \times^i (\alpha \rightarrow \beta)} \quad \frac{\times^i (X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow (\alpha \rightarrow \beta)}$$

$\Pi_2$ :

$$\frac{\frac{\frac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \times^i \times^i \alpha}}{X^i\alpha \Rightarrow \times^i (\alpha \rightarrow \beta)}}{X^i(\alpha \rightarrow \beta); X^i\alpha \Rightarrow \times^i \alpha} \quad \frac{\times^i (X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow \alpha}$$

$\Pi_3$ :

$$\frac{\frac{\frac{\Pi_1 \quad \Pi_2}{\times^i (X^i(\alpha \rightarrow \beta); X^i\alpha); \times^i (X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow (\alpha \rightarrow \beta) \wedge \alpha}}{\times^i (X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow (\alpha \rightarrow \beta) \wedge \alpha}}{X^i(\alpha \rightarrow \beta); X^i\alpha \Rightarrow \times^i ((\alpha \rightarrow \beta) \wedge \alpha)}$$

$\Pi_4$ :

$$\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\alpha \rightarrow \beta \Rightarrow \alpha; \beta}}{(\alpha \rightarrow \beta); \alpha \Rightarrow \beta}}{(\alpha \rightarrow \beta) \wedge \alpha \Rightarrow \beta}}{(\alpha \rightarrow \beta) \wedge \alpha \Rightarrow \times^i \times^i \beta} \quad \frac{\times^i ((\alpha \rightarrow \beta) \wedge \alpha) \Rightarrow \times^i \beta}$$



We obtain the following proof:

$$\frac{\frac{\frac{\frac{\Pi_3 \quad \Pi_4}{(X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow \times^i\beta}}{\times^i(X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow \beta}}{(X^i(\alpha \rightarrow \beta); X^i\alpha) \Rightarrow X^i\beta}}{X^i(\alpha \rightarrow \beta) \Rightarrow X^i\alpha; X^i\beta}}{X^i(\alpha \rightarrow \beta) \Rightarrow X^i\alpha \rightarrow X^i\beta}$$

( $\rightarrow$ right): Here the essential step is to show that  $X^i\alpha \rightarrow X^i\beta \Rightarrow X^i(\alpha \rightarrow \beta)$  is provable in  $\delta IB[l]$ . We use the structural sequent rule  $\star$ :

$$\frac{\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha}{\times^i \times^i \alpha \rightarrow \alpha} \quad \frac{\beta \Rightarrow \beta}{\beta \Rightarrow \times^i \times^i \beta}}{\times^i \alpha \rightarrow X^i\alpha} \quad X^i\beta \Rightarrow \times^i \alpha}}{X^i\alpha \rightarrow X^i\beta \Rightarrow \times^i\alpha; \times^i\beta} \star}{X^i\alpha \rightarrow X^i\beta \Rightarrow \times^i(\alpha; \beta)} \star}{\frac{\times^i(X^i\alpha \rightarrow X^i\beta) \Rightarrow \alpha; \beta}{\times^i(X^i\alpha \rightarrow X^i\beta) \Rightarrow \alpha \rightarrow \beta}}{X^i\alpha \rightarrow X^i\beta \Rightarrow X^i(\alpha \rightarrow \beta)}$$

( $\wedge$ left1, 2): The proof uses (cut), (*lm*), and the provability of  $\times^i\gamma \Rightarrow X^i\gamma$ .

( $\wedge$ right): The proof uses (cut), (*lc*), and the provability of  $X^i\gamma \Rightarrow \times^i\gamma$ .

( $\vee$ left): The proof uses (cut) and the provability of  $\times^i\gamma \Rightarrow X^i\gamma$ .

( $\vee$ right1, 2): The proof makes use of (cut) and the provability of  $X^i\gamma \Rightarrow \times^i\gamma$ .

(Xleft): The proof makes use of (cut) and the provability of  $X^{l+m}\gamma \Rightarrow X^l\gamma$  by means of the boundedness rule (*b*).

(Xright): We may use (cut) and the provability of  $X^l\gamma \Rightarrow X^{l+m}\gamma$  by means of the boundedness rule (*b'*).

(Gleft): The proof makes use of (cut) and the following subproof:

$$\frac{\frac{\frac{\alpha \Rightarrow \alpha}{G\alpha \Rightarrow \alpha}}{G\alpha \Rightarrow \times^i \times^{i+k} \alpha} (lg)}{\vdots} (dp)$$

$$\frac{G\alpha \Rightarrow \times^i X^{i+k} \alpha}{X^i G\alpha \Rightarrow X^{i+k} \alpha}$$

(Gright):

$$\frac{\frac{\{ \Gamma^\wedge \Rightarrow \times^{i+j} \alpha \} j \in \omega_l}{\triangleleft \times^i \Gamma^\wedge \Rightarrow \alpha} (rg)}{\frac{\times^i \Gamma^\wedge \Rightarrow G\alpha}{\Gamma^\wedge \Rightarrow X^i G\alpha}}$$

(Fleft):

$$\frac{\frac{\{ \times^{i+j} \alpha \Rightarrow \Gamma^\wedge; \Delta^\vee \} j \in \omega_l}{\alpha \Rightarrow \triangleright \times^i (\Gamma^\wedge; \Delta^\vee)} (lf)}{\frac{F\alpha \Rightarrow \times^i (\Gamma^\wedge; \Delta^\vee)}{X^i F\alpha \Rightarrow \Gamma^\wedge; \Delta^\vee}} \frac{X^i F\alpha; \Gamma^\wedge \Rightarrow \Delta^\vee}{X^i F\alpha \wedge \Gamma^\wedge \Rightarrow \Delta^\vee}$$

(Fright):

$$\frac{\frac{\frac{\frac{\alpha \Rightarrow \alpha}{\triangleright \alpha \Rightarrow \mathbf{F}\alpha}}{\times^i \times^{i+k} \alpha \Rightarrow \mathbf{F}\alpha}}{\times^{i+k} \alpha \Rightarrow \mathbf{X}^i \mathbf{F}\alpha}}{\alpha \Rightarrow \times^{i+k} \mathbf{X}^i \mathbf{F}\alpha}}{\Gamma^\wedge \Rightarrow \mathbf{X}^{i+k} \alpha \quad \mathbf{X}^{i+k} \alpha \Rightarrow \mathbf{X}^i \mathbf{F}\alpha}}{\Gamma^\wedge \Rightarrow \mathbf{X}^i \mathbf{F}\alpha} \quad \square$$

Due to the presence of both the forward-looking *and* the backward-looking structural connectives, introduction rules for the backward-looking counterparts of IB[I]'s temporal operators are easily available:

$$\begin{array}{ll} \frac{\triangleright \Delta \Rightarrow \alpha}{\Delta \Rightarrow \mathbf{H}\alpha} \text{ (Hright)} & \frac{\alpha \Rightarrow \Delta}{\mathbf{H}\alpha \Rightarrow \triangleright \Delta} \text{ (Hleft)} \\ \frac{\Delta \Rightarrow \alpha}{\triangleleft \Delta \Rightarrow \mathbf{P}\alpha} \text{ (Pright)} & \frac{\alpha \Rightarrow \triangleleft \Delta}{\mathbf{P}\alpha \Rightarrow \Delta} \text{ (Pleft)} \\ \frac{\times^i \Delta \Rightarrow \alpha}{\Delta \Rightarrow \mathbf{E}^i \alpha} \text{ (E}^i \text{right)} & \frac{\alpha \Rightarrow \times^i \Delta}{\mathbf{E}^i \alpha \Rightarrow \Delta} \text{ (E}^i \text{left)} \end{array}$$

## 5.2. A display calculus for PB[I]

A sound and complete display calculus  $\delta\text{PB}[I]$  for PB[I] can be obtained in a natural and straightforward way, see also the display calculi presented in [36] for extensions of Heyting–Brouwer logic by strong negation. Again, the inferential understanding of  $X^i$  as laid down by the introduction rules is (basically) non-holistic. It is only the meaning of  $\sim$  that is specified in combination with each of the other object language connectives.

**Definition 49.** The display calculus  $\delta\text{PB}[I]$  is obtained from  $\delta\text{IB}[I]$  by removing ( $\perp$ left) and adding initial sequents  $\sim p \Rightarrow \sim p$  together with the following sequent rules:

$$\begin{array}{ll} \frac{\Delta \Rightarrow \sim \alpha}{\Delta \Rightarrow \sim(\alpha \wedge \beta)} \quad \frac{\Delta \Rightarrow \sim \beta}{\Delta \Rightarrow \sim(\alpha \wedge \beta)} \text{ } (\sim \wedge \text{right}) & \frac{\sim \alpha \Rightarrow \Delta \quad \sim \beta \Rightarrow \Delta}{\sim(\alpha \wedge \beta) \Rightarrow \Delta} \text{ } (\sim \wedge \text{left}) \\ \frac{\Delta \Rightarrow \sim \alpha \quad \Gamma \Rightarrow \sim \beta}{\Delta; \Gamma \Rightarrow \sim(\alpha \vee \beta)} \text{ } (\sim \vee \text{right}) & \frac{\sim \alpha; \sim \beta \Rightarrow \Delta}{\sim(\alpha \vee \beta) \Rightarrow \Delta} \text{ } (\sim \vee \text{left}) \\ \frac{\Delta \Rightarrow \alpha \quad \Gamma \Rightarrow \sim \beta}{\Delta; \Gamma \Rightarrow \sim(\alpha \rightarrow \beta)} \text{ } (\sim \rightarrow \text{right}) & \frac{\alpha; \sim \beta \Rightarrow \Delta}{\sim(\alpha \rightarrow \beta) \Rightarrow \Delta} \text{ } (\sim \rightarrow \text{left}) \\ \frac{\Delta \Rightarrow \sim \alpha}{\triangleright \Delta \Rightarrow \sim \mathbf{G}\alpha} \text{ } (\sim \mathbf{G} \text{right}) & \frac{\sim \alpha \Rightarrow \triangleright \Delta}{\sim \mathbf{G}\alpha \Rightarrow \Delta} \text{ } (\sim \mathbf{G} \text{left}) \\ \frac{\triangleleft \Delta \Rightarrow \sim \alpha}{\Delta \Rightarrow \sim \mathbf{F}\alpha} \text{ } (\sim \mathbf{F} \text{right}) & \frac{\sim \alpha \Rightarrow \Delta}{\sim \mathbf{F}\alpha \Rightarrow \triangleleft \Delta} \text{ } (\sim \mathbf{F} \text{left}) \\ \frac{\Delta \Rightarrow \times^i \sim \alpha}{\Delta \Rightarrow \sim \mathbf{X}^i \alpha} \text{ } (\sim \mathbf{X}^i \text{right}) & \frac{\times^i \sim \alpha \Rightarrow \Delta}{\sim \mathbf{X}^i \alpha \Rightarrow \Delta} \text{ } (\sim \mathbf{X}^i \text{left}) \end{array}$$

Note that for every formula  $\alpha$ , the sequent  $\alpha \Rightarrow \alpha$  is provable in  $\delta\text{PB}[I]$ . Moreover, the modifications leading from  $\delta\text{IB}[I]$  to  $\delta\text{PB}[I]$  clearly do not spoil the display property.

**Theorem 50** (Display property for  $\delta\text{PB}[I]$ ). *For every sequent  $s$  and every antecedent part (succedent part)  $\Delta$  of  $s$ , there exists a sequent  $s'$  display equivalent to  $s$  such that  $\Delta$  is the entire antecedent (succedent) of  $s'$ .*

**Theorem 51** (Soundness of  $\delta\text{PB}[I]$ ). *If  $\delta\text{PB}[I] \vdash \Delta \Rightarrow \Gamma$ , then  $\tau(\Delta \Rightarrow \Gamma)$  is  $p$ -valid in the class of all Kripke models for  $\delta\text{PB}[I]$ .*

**Proof.** The verification conditions for formulas  $\mathbf{P}\alpha$  and  $\mathbf{E}\alpha$  are analogous to the intuitionistic case. The refutation conditions are:  $(x, i) \models^- \mathbf{E}\alpha$  iff  $(x, i-1) \models^- \alpha$  if  $i > 0$  and otherwise  $(x, 0) \models^- \alpha$ ;  $(x, i) \models^- \mathbf{P}\alpha$  iff  $\exists j \in N_1 [j \leq i \text{ and } (x, j) \models^- \alpha]$  if  $i > 0$  and otherwise  $(x, 0) \models^- \alpha$ . By induction on proofs in  $\delta\text{PB}[I]$ , it can be shown that if  $\delta\text{PB}[I] \vdash \Delta \Rightarrow \Gamma$ , then for every Kripke frame and valuations  $\models^+$  and  $\models^-$ ,  $(x, i) \models^+ \tau(\Delta \Rightarrow \Gamma)$  for every state  $x$  and moment  $i$ .  $\square$

**Theorem 52** (Completeness of  $\delta\text{PB}[I]$ ). *If the sequent  $\Delta \Rightarrow \gamma$  is provable in the sequent calculus for PB[I] from Section 3, then  $\Delta^\wedge \Rightarrow \gamma$  is provable in  $\delta\text{PB}[I]$ .*



$$\begin{array}{l}
\frac{\Delta \Rightarrow \sim\alpha}{\triangleleft\Delta \Rightarrow \sim H\alpha} \text{ (~Hright)} \quad \frac{\sim\alpha \Rightarrow \triangleright\Delta}{\sim H\alpha \Rightarrow \Delta} \text{ (~Hleft)} \\
\frac{\triangleright\Delta \Rightarrow \sim\alpha}{\Delta \Rightarrow \sim P\alpha} \text{ (~Pright)} \quad \frac{\sim\alpha \Rightarrow \Delta}{\sim P\alpha \Rightarrow \triangleright\Delta} \text{ (~Pleft)} \\
\frac{\times^i \Delta \Rightarrow \sim\alpha}{\Delta \Rightarrow \sim E^i \alpha} \text{ (~E}^i \text{right)} \quad \frac{\sim\alpha \Rightarrow \times^i \Delta}{\sim E^i \alpha \Rightarrow \Delta} \text{ (~E}^i \text{left)}
\end{array}$$

## 6. Remarks on the infinite linear time domain

In this section, we suggest to construct some infinite time domain versions of the proposed systems. Infinite (unbounded) versions of IB[I] and PB[I] can naturally be considered. Let IB[ $\omega$ ] and PB[ $\omega$ ] be obtained from IB[I] and PB[I], respectively, by deleting (Xleft) and (Xright), and replacing  $\omega_l$  by  $\omega$ . Embedding theorems of IB[ $\omega$ ] and PB[ $\omega$ ] into *infinitary* intuitionistic logic (see [13]) and *infinitary* Nelson's paraconsistent logic, respectively, can be obtained, and then the cut-elimination theorems for IB[ $\omega$ ] and PB[ $\omega$ ] may also be obtained using these embedding theorems. Also, normalizable natural deduction systems NIB[ $\omega$ ] and NPB[ $\omega$ ] can be obtained. Unbounded display calculi  $\delta$ IB[ $\omega$ ] and  $\delta$ PB[ $\omega$ ] can be defined from  $\delta$ IB[I] and  $\delta$ PB[I], respectively, by deleting the boundedness rules (b) and (b'), and replacing  $\omega_l$  by  $\omega$ . However, the Kripke completeness theorems for these systems may not be shown for the natural Kripke semantics with the following valuation conditions:

$$\begin{array}{l}
(x, i) \models X\alpha \text{ iff } (x, i+1) \models \alpha, \\
(x, i) \models G\alpha \text{ iff } \forall j \in N[i \leq j \text{ implies } (x, j) \models \alpha], \\
(x, i) \models F\alpha \text{ iff } \exists j \in N[i \leq j \text{ and } (x, j) \models \alpha] \\
\text{(and analogously for } \models^+ \text{),} \\
(x, i) \models^- X\alpha \text{ iff } (x, i+1) \models^- \alpha, \\
(x, i) \models^- G\alpha \text{ iff } \exists j \in N_l[i \leq j \text{ and } (x, j) \models^- \alpha], \\
(x, i) \models^- F\alpha \text{ iff } \forall j \in N_l[i \leq j \text{ implies } (x, j) \models^- \alpha].
\end{array}$$

The reason of the failure of the completeness proof is that in the proof of Lemmas 17 and 32, the cases for  $\alpha \equiv G\beta$  and  $\alpha \equiv F\beta$ , respectively, require the following facts:

$$\vdash X^i \beta, X^{i+1} \beta, X^{i+2} \beta, \dots, \infty \Rightarrow X^i G\beta$$

where the antecedent is an infinite sequence of formulas, and

$$\vdash X^i F\beta \Rightarrow X^i \beta \vee X^{i+1} \beta \vee X^{i+2} \beta \vee \dots \infty$$

where the succedent is an infinite disjunction of formulas. Thus, in order to obtain completeness, the notion of a sequent should be extended to encompass infinite sequents that permit us to have infinite antecedents. Moreover, infinite disjunctions should be allowed, and the treatment of unbounded display calculi would require infinite conjunctions as well. By imposing these modifications, the completeness theorems can be obtained, but the corresponding logics may differ from IB[ $\omega$ ], PB[ $\omega$ ],  $\delta$ IB[ $\omega$ ], and  $\delta$ PB[ $\omega$ ]. Moreover, the cut-elimination theorems for the corresponding sequent calculi with these modifications cannot be shown. The reason of the failure of the cut-elimination proof is that the infinite antecedents require an infinite version of the cut rule of the form:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \Sigma}{\Gamma, \Delta^* \Rightarrow \Sigma} \text{ } (\omega\text{-cut})$$

where  $\Delta$  contains  $\alpha$  (it can appear infinitely many times), and  $\Delta^*$  is obtained from  $\Delta$  by deleting all occurrences of  $\alpha$ .

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