ON THE COBORDISM CLASSES OF CODIMENSION ONE FOLIATIONS WHICH ARE ALMOST WITHOUT HOLONY

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(Received 14 September 1981)

INTRODUCTION

Let $M$ be an oriented closed $n$-dimensional $C^\infty$-manifold and $(M, \mathcal{F})$ a transversely oriented codimension one $C^\infty$-foliation of $M$. The purpose of this paper is to study foliated cobordism class of $(M, \mathcal{F})$ assuming that $\mathcal{F}$ is almost without holonomy. In virtue of the works of Haefliger[3, 4], Mather[12, 13] and Thurston[24, 25], foliated cobordism groups can be studied in the following lines. Namely there is a universal space $B\Gamma_{1,\infty}^n$, called the Haefliger’s classifying space for $\Gamma_{1,\infty}$-structures, so that any $(M, \mathcal{F})$ determines an $n$-dimensional homology class of $B\Gamma_{1,\infty}^n$, which turns out to be closely related to the cobordism class of $(M, \mathcal{F})$. On the other hand there is a map $B\text{Diff}_{k,\infty}^n\mathcal{R} \to \Omega B\Gamma_{1,\infty}^n$ from the classifying space of the discrete group $\text{Diff}_{k,\infty}^n\mathcal{R}$ of all $C^\infty$-diffeomorphisms of $\mathcal{R}$ with compact support to the loop space $\Omega B\Gamma_{1,\infty}^n$ of $B\Gamma_{1,\infty}^n$, which induces an isomorphism on integral homology. Thus in some sense the study of homology classes of $B\Gamma_{1,\infty}^n$ can be reduced to the study of those of $\text{Diff}_{k,\infty}^n\mathcal{R}$. In our case, these two fundamental results work very well. Our main result is as follows. Let $(M, \mathcal{F})$ be as before and assume that $\mathcal{F}$ is almost without holonomy. Then it is homologous to a disjoint union of finite number of foliated $S^1$-bundles over $(n - 1)$-dimensional tori. For $n = 3$, in particular, it follows that $(M, \mathcal{F})$ is foliated cobordant to a disjoint union of foliated $S^1$-bundles over $T^2$. The foliated cobordism classes of foliated $S^1$-bundles over $T^2$ were studied by Tsuboi in [26]. Fukui and Oshikiri proved the nullity of the foliated cobordism classes of certain foliated 3-manifolds ([2, 21]). By our method, we can give a wider class of foliated 3-manifolds which are foliated null-cobordant (Corollary to Theorem 2). Also together with results of Wallet[29] and Herman[7], we re-obtain the vanishing of the Godbillon–Vey class of an almost without holonomy foliation $(M, \mathcal{F})$ which we previously proved in [16]. The main tool of this paper is the notion of foliated $J$-bundles which we developed in [16] in order to calculate the Godbillon–Vey class. Associated to each $(M, \mathcal{F})$, there is a foliated $J$-bundle and the original $\mathcal{F}$ can be “embedded” in it as the graph of $\mathcal{F}$. Then we can deform the underlying $\Gamma_{1,\infty}$-structure of $\mathcal{F}$ by simply moving this graph on the total space of the $J$-bundle. This method is originally due to Haefliger. The foliated $J$-bundles associated to $(M, \mathcal{F})$ are determined by the holonomies of the compact leaves and the Novikov transformations (which depend on the non-compact leaves). These two data are essential. In fact, the structure of the foliated $S^1$-bundles over $T^{n-1}$ to which $(M, \mathcal{F})$ is homologous, are determined by these data.

Some of the results in this paper were contained in §5 of our preprint[15] some part of which has been published in [16]. We would like to refer the reader to [16] for the generalities of the foliation which are almost without holonomy and the construction of the associated foliated $J$-bundles and other related notions.

In this paper, all manifolds, foliations and diffeomorphisms are assumed to be smooth ($C^\infty$). Moreover, foliations will mean transversely oriented codimension one foliations.
§1. DEFINITIONS AND STATEMENT OF RESULTS

Let \((M, \mathcal{F})\) be a transversely oriented \(C^\infty\)-foliation of a closed oriented \(n\)-dimensional manifold. It determines an element \([M, \mathcal{F}]\) of \(H_\ast(B\Gamma_1^\infty; \mathbb{Z})\), where \(B\Gamma_1^\infty\) is the Haefliger's classifying space for \(\Gamma_1^\infty\)-structures. We say that two such foliations \((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2)\) are homologous (denoted by \((M_1, \mathcal{F}_1) \sim (M_2, \mathcal{F}_2)\)) if \([M_1, \mathcal{F}_1] = [M_2, \mathcal{F}_2]\). Let \((M_i, \mathcal{F}_i), i = 1, 2, \) be transversely oriented \(C^\infty\)-foliations of oriented manifolds with boundary and suppose that the foliations \(\mathcal{F}_i\) restricted to some neighbourhoods of the boundaries \(\partial M_i\), are isomorphic by a prescribed orientation preserving diffeomorphism \(f\). We say that \((M_1, \mathcal{F}_1)\) and \((M_2, \mathcal{F}_2)\) are homologous relative to the boundary (denoted by \((M_1, \mathcal{F}_1) \sim (M_2, \mathcal{F}_2) \text{ rel. boundary}\)) if the \(\Gamma_1^\infty\)-structure on \(M_i\) induced by \(\mathcal{F}_i\) and \(\mathcal{F}_2\) is homologous to zero in \(B\Gamma_1^\infty\). In this paper, since the diffeomorphism \(f\) is the natural identification obvious from the context, we do not refer to it. We remark here that the above two relations are also defined between the \(\Gamma_1^\infty\)-structures on finite CW-complexes of dimension \(n\), which have (relative) fundamental \(n\)-cycles, such as the suspensions or cones of closed \((n-1)\) dimensional manifolds.

Let \(F\) be an oriented \(q\)-dimensional \(C^\infty\)-manifold and let \(\text{Diff}^\infty(F)\), denote the group of all orientation preserving \(C^\infty\)-diffeomorphisms of \(F\) with discrete topology and let \(\text{Diff}_c^\infty(F)\) denote the subgroup consisting of compactly supported diffeomorphisms. Suppose that we are given a fiber bundle \(\pi : E \rightarrow B\) over a CW-complex \(B\) with fiber \(F\), and suppose further that its structure group is contained in \(\text{Diff}^\infty(F)\). Then we have a \(\Gamma_1^\infty\)-structure \(\mathcal{F}\) on \(E\) whose local level sets are of the form \(\{U \times \{p\}, p \in F\}\) where \(U\) is an open set of \(B\) such that \(E|_U = U \times F\) is a trivial bundle over \(U\). We call the quadruple \((E, B, \mathcal{F}, F)\) a foliated \(F\)-bundle. Of course, if \(E\) and \(B\) are \(C^\infty\)-manifolds, then \(\mathcal{F}\) is a \(C^\infty\)-foliation transverse to fibers. As is well known, a foliated \(F\)-bundle is characterized by its total holonomy homomorphism \(\Phi : \pi_1(B, *) \rightarrow \text{Diff}^\infty(F)\).

We call a foliated \(S^1\)-bundle over \(T_-^n\) linear if its total holonomy (= the image of the total holonomy homomorphism) is smoothly conjugate to a group of rotations and exotic linear if its total holonomy is topologically conjugate to a group of rotations but not smoothly. By a result of Arnold[1], exotic linear foliated \(S^1\)-bundles do exist. A linear foliated \(S^1\)-bundle over \(T_-^n\) is foliated cobordant to zero, because it is defined by a closed \(1\)-form.

Let \((E, K, \mathcal{F}, \mathcal{R})\) be a foliated \(\mathcal{R}\)-bundle over a CW-complex \(K\) which is a \(\text{Diff}^\infty(K)\)-bundle. Choosing an embedding of \(\mathcal{R}\) in \(S^1\), we obtain a foliated \(S^1\)-bundle \((E', K, \mathcal{F}', S')\). We call the foliated \(S^1\)-bundles obtained in this way, layered. In Tsuboi[27], it is proved that if the total holonomy of a layered foliated \(S^1\)-bundle over a closed manifold is contained in a \(1\)-parameter subgroup generated by a \(C^\infty\)-vectorfield, it is homologous to zero in the classifying space \(B\text{Diff}^\infty(S^1)\), consequently, the foliation is homologous to zero in \(B\Gamma_1^\infty\).

We remark that the \(\text{Diff}^\infty(\mathcal{R})\)-bundle \((E, K, \mathcal{F}, \mathcal{R})\) defines a \(\Gamma_1^\infty\)-structure on the suspension \(\Sigma K\) of \(K\), where we consider

\[
\Sigma K = K \times \{RU \pm \infty\}/K \times \{\pm \infty\} = \infty
\]

Clearly, the \(\Gamma_1^\infty\)-structure on \(\Sigma K\) is homologous to that of the layered foliated \(S^1\)-bundle \((E', K, \mathcal{F}', S')\) corresponding to \((E, K, \mathcal{F}, \mathcal{R})\).

For a foliation which is almost without holonomy (this concept was originally introduced by Moussu and studied by several authors[5, 10, 18]), we have a decomposition theorem, Theorem 4 of [16]. We divide the foliations of type III into two classes, namely those of type III₁ and type III₂. This division arose naturally in the proof of [16]. We restate here Theorem 4 of [16] which is due to Hector[5] and Imanishi[9, 10].
THEOREM 1.1 (Hector\[5\], Imanishi\[9, 10\]). Let \((M, \mathcal{F})\) be a codimension one foliation tangent to the boundaries of a compact manifold which is almost without holonomy. Then there is a finite family of foliations \(\{M_k, \mathcal{F}_k\}\) and immersions \(\varphi_k: M_k \to M\) such that

1. \(\varphi_k | \text{Int} M\) is a diffeomorphism onto its image and \(\varphi_k^* \mathcal{F} = \mathcal{F}_k\).
2. \(\varphi_k(\text{Int} M_k)\) are disjoint and \(\cup \varphi_k(\text{Int} M_k) = M\).
3. \((M_k, \mathcal{F}_k)\) is a foliation of one of the following three types.
   - (I) \((M_k, \mathcal{F}_k)\) is a foliation without holonomy of a closed manifold.
   - (II) \((M_k, \mathcal{F}_k)\) is a foliated \(\mathbb{Z}\)-bundle with abelian total holonomy.
   - (III) \((M_k, \mathcal{F}_k)\) is a foliation tangent to the non-empty boundaries such that all leaves other than the boundary leaves have trivial holonomy.

**Remark 1.2.** (i) In the case of III, if the rank of the holonomy (which is free abelian) of all the boundary leaves are not greater than 1, we call it of type III'. Otherwise, we call it of type III''. (ii) The above decomposition is not unique. Some foliated \(I\)-bundles can be decomposed as a union of foliated \(I\)-bundles. Moreover, the classification of the foliations into type II and type III is not exclusive. If a foliation of type III (of type III') is not of type II, we call it of type III' (of type III''). The components of type III' are independent of the choice of the decompositions.

**Remark 1.3.** For any neighborhood \(U\) of the union of all compact leaves of \(\mathcal{F}\), there is a decomposition of Theorem (1.1) such that every component of type II lies in \(U\).

Now we state our theorems.

**Theorem 1.** Let \((M, \mathcal{F})\) be a transversely oriented, smooth, codimension one foliation of a closed \(n\) dimensional manifold which is almost without holonomy. Then \((M, \mathcal{F})\) is homologous to a disjoint union of foliated \(S^1\)-bundles over \(T^{n-1}\) which are either exotic linear or layered. More precisely, if \((M, \mathcal{F})\) is of type I, that is if \((M, \mathcal{F})\) is without holonomy, then all the resulted foliated \(S^1\)-bundles are exotic linear and their total holonomies are determined by the Novikov transformation of \((M, \mathcal{F})\). If \((M, \mathcal{F})\) consists of foliations of type II and of type III, the total holonomies of the exotic linear foliated \(S^1\)-bundles are determined by the Novikov transformations of the components of type III', and those of the layered foliated \(S^1\)-bundles are determined by the total holonomies of the components of type II and the holonomies of the boundary leaves of components (of type II and III); thus the latter is determined by the holonomies of compact leaves.

With some further conditions on \((M, \mathcal{F})\), we have the following Theorem 2. For a component of type II, the total holonomy of foliated \(I\)-bundles and the holonomy of the boundary leaves define a homomorphism of the fundamental group of the compact leaf into the group of germs along \(I\) of the diffeomorphisms of \(\mathbb{R}\) leaving \(I\) invariant. \((f, g \in \text{Diff} \mathbb{R}\) define the same germs along \(I\) if \(f(I) = I, g(I) = I\) and \(f = g\) on some neighborhood of \(I\) in \(\mathbb{R}\).\) We call this homomorphism prolonged holonomy and its image prolonged holonomy group.

**Theorem 2.** Let \((M, \mathcal{F})\) be as in Theorem 1. If the following conditions (A) and (B) are satisfied, \((M, \mathcal{F})\) is homologous to zero.

(A) The rank of the Novikov transformations of components of type I or type III' are not greater than \(n - 1\).
(B) For the components of type II and for each boundary leaf of components of type III, one of the following is satisfied.

(B1) The rank of the (prolonged) holonomy is not greater than \( n - 2 \).

(B2) The rank of the (prolonged) holonomy is \( n - 1 \) and we can find representatives \( f_1, \ldots, f_{n-1} \) of generators of the (prolonged) holonomy group such that

\[
\text{Int}(\text{supp} f_i) \cap \left( \bigcap_{i=2}^{n-1} \text{supp} f_i \right) = \emptyset.
\]

(B3) The (prolonged) holonomy group is contained in a 1-parameter group generated by a smooth vectorfield (germinally).

Remark 1.4. In case \( n \geq 3 \), if none of the holonomy of the compact leaves are infinitely tangent to the germs of the identity, (B3) for holonomy of compact leaves is always satisfied. See Tsuboi[26], Lemmas 3.4 and 3.5.

For manifolds of dimension 3, in particular, we have the following corollary which implies foliations constructed by spinnable structures of 3-manifolds are foliated cobordant to zero (see [2, 21]).

**Corollary.** Let \((M, \mathcal{F})\) be a transversely oriented foliation of a closed oriented 3-manifold. Suppose that \((M, \mathcal{F})\) is almost without holonomy. If one of the following conditions is satisfied, \((M, \mathcal{F})\) is foliated cobordant to zero.

(C) All leaves are proper and there are only finitely many compact leaves.

(D) There exist compact leaves and none of the holonomies of the compact leaves are infinitely tangent to the germ of the identity and none of the holonomies of compact leaves are of rank less than two.

The above Corollary follows from the following observation. If the rank of the Novikov transformation of a component is greater than 1, there exists a dense leaf, therefore if (C) holds, we can take a decomposition of \((M, \mathcal{F})\) in Theorem 1.1 only with the components of type III, and (A) is satisfied. On the other hand, for the boundary compact leaves, by the same reason, its one-sided holonomy is of rank 1. Thus (B1) or (B2) is satisfied. If (D) holds, the number of compact leaves is finite and we can take a decomposition of \((M, \mathcal{F})\) only with the components of type III, thus (A) is automatically satisfied. For compact leaves, by the above remark, we have (B1).

The proof of Theorem 1 consists of construction of various “cobordisms” using foliated J-bundles suitably modified. A foliated J-bundle is a union of foliated \(I_-\), \(I_+\) and \(I_-\)-bundles (§4), where \(I_+\) and \(I_-\) denote the intervals \((-1, 1]\) and \([-1, 1)\) respectively. In the next section we discuss the modification of these bundles which have abelian total holonomy. In §3, we treat cross-sections of foliated R-bundles with abelian total holonomy which concerns the exotic linear foliated S'-bundles arising from the components of types I and III. In §4, we will construct “cobordisms” relative to the boundary for the components of type III. For those of type III, we treat them in §5. We will complete the proof of our theorems in §6.

**§2. Modification of Foliated Bundles**

Let \(I_-\) denote the half open interval \([-1, 1)\) and \(I\) denote the closed interval \([-1, 1]\). Let \((E, L, \mathcal{F}, I_-)\) be a foliated \(I_-\)-bundle over a CW complex \(L\). A modified foliated
1-bundle of $(E, L, \mathcal{F}, I_\perp)$ is a foliated $I$-bundle $(\hat{E}, L, \hat{\mathcal{F}}, I)$ over $L$ with the following properties:

(i) As a topological space, $\hat{E}$ is the total space of the associated $I$-bundle of $E$, i.e. $\hat{E} = I \times L = E \times I$.

(ii) The “foliation” $\hat{\mathcal{F}}$ coincides with $\mathcal{F}$ outside a small neighbourhood of $L \times \{+1\}$.

(iii) The “foliation” $\hat{\mathcal{F}}$ is trivial, i.e. isomorphic to $\{L \times \{\text{const}\}\}$, in a smaller neighbourhood of $L \times \{+1\}$.

**Lemma 2.1.** If the total holonomy of $(E, L, \mathcal{F}, I_\perp)$ is abelian, then there exists a modified foliated 1-bundle $(\hat{E}, L, \hat{\mathcal{F}}, I)$ of $(E, L, \mathcal{F}, I_\perp)$ whose total holonomy is also abelian.

**Proof.** We may assume that $L$ is connected. Let $\Phi: \pi_1(L, \ast) \to \text{Diff}_\mathbb{Z}$ denote the total holonomy homomorphism of $E$. We may also assume that there is an element $f$ in $\text{Im} \Phi$ which is not the identity on a subinterval $(1 - \epsilon, 1)$, where $\epsilon$ is a small positive real. (Otherwise the assertion is trivially true.) Then one can find an interval $[a, b]$ such that $1 - \epsilon < b$, $f([a, b]) = [a, b]$ and $f(x) \neq x$ for $x \in (a, b)$. By a theorem of Kopell[11], the interval $[a, b]$ is invariant under the action of $\pi_1(L, \ast)$ via $\Phi$ (see, e.g. [26]). Moreover, there exists a vectorfield $\xi$, of class $C^1$ on $[a, b]$ and $C^\infty$ on $(a, b)$ such that $\Phi(\pi_1(L, \ast))|_{[a,b]}$ is contained in a 1-parameter subgroup generated by $\xi$ (Sergeraert[23]). Let $\eta$ be a vectorfield of class $C^1$ on $[a + 1]$ and $C^\infty$ on $(a, +1)$ such that

$$\eta = \xi|_{[a, b - \epsilon']} \quad \text{and} \quad \eta = 0 \text{ on } \left[ b - \frac{\epsilon'}{2}, +1 \right]$$

where $\epsilon'$ is a small positive real satisfying $1 - \epsilon < b - \epsilon'$. Let $\hat{\Phi}: \pi_1(L, \ast) \to \text{Diff}_\mathbb{Z}$ be a homomorphism defined by

$$\hat{\Phi}(\pi_1(L, \ast))|_{[a, b]} = \Phi(\pi_1(L, \ast))|_{[a, b]}$$

$$\hat{\Phi}(\pi_1(L, \ast)|_{[a, b + 1]}) = \text{the time } t \text{-map of } \eta \text{ on } [a + 1]$$

$$\text{if } \Phi(\pi_1(L, \ast))|_{[a, b]} \text{ is the time } t \text{-map of } \xi.$$ 

It is easy to see that $\hat{\Phi}$ is a well-defined homomorphism and gives a desired foliated 1-bundle.

For a modification $(\hat{E}, L, \hat{\mathcal{F}}, I)$ of $(E, L, \mathcal{F}, I_\perp)$, let $(CL, \mathcal{F})$ denote the $\Gamma_1^\mathbb{Z}$-structure on the cone $CL = L \times I/L \times \{+1\}$ of $L$ which is induced from $\mathcal{F}$. The following lemma shows that in a case, which we will treat later, the homology class relative to $L \times \{-1\}$ of the $\Gamma_1^\mathbb{Z}$-structure $(CL, \mathcal{F})$ is independent of the choice of the vectorfield $\eta$.

**Lemma 2.2.** Suppose $L$ is an oriented closed manifold and suppose $\Phi(\pi_1(L, \ast))|_{[-1,1]}$ is topologically conjugate to a group of translations of $\mathbb{R}$. Then the homology class of $(CL, \mathcal{F})$ relative to the boundary $L \times \{-1\}$ is independent of the choice of the modification.

**Proof.** By the assumption, the whole interval $I_\perp = [-1, 1]$ serves as the interval $[a, b]$ in the proof of Lemma (2.1). Let $\eta'$ be another vectorfield which plays the same role as $\eta$. We may assume that $\eta$ and $\eta'$ coincide on some interval $[-1, c]$. By a smooth coordinate change, if necessary, $c$ may be supposed to be $1 - \delta$ for a small positive real $\delta$, $0 < \delta < \frac{1}{2}$. Let $\mathcal{F}$ and $\mathcal{F}'$ denote the corresponding $\Gamma_1^\mathbb{Z}$-structures on $CL$, respectively. Let $\eta$ be a
$C^\infty$-vectorfield on $I$ which coincides with $\eta$ on $[-1+\delta, +1]$ and vanishes on $[-1, -1 + (\delta/2)]$ and let $\tilde{\eta}'$ be a $C^\infty$-vectorfield on $I$ which coincides with $\eta'$ on $[-1 + \delta, +1]$ and with $\tilde{\eta}$ on $[-1, -1 + \delta]$. The vectorfields $\tilde{\eta}$ and $\tilde{\eta}'$ define foliated $I$-bundles over $L$ in the same way as before and we write $\mathcal{F}$ and $\mathcal{F}'$ for the resulted $\Gamma_1^\infty$-structures on $CL$, respectively. Since $\mathcal{F}$ and $\mathcal{F}'$ are trivial $\Gamma_1^\infty$-structures near $L \times \{-1\}$, they give $\Gamma_1^\infty$-structures $\mathcal{F}_*, \mathcal{F}'_*$ on the suspension $\Sigma L = CL/L \times \{-1\}$ respectively. We have to show that $(CL, \mathcal{F}) \cup (-CL, \mathcal{F}')$ is homologous to zero. Put $C'L = L \times [0, 1/L \times \{+1\}$. Since $(CL, \mathcal{F})$ coincides with $(CL, \mathcal{F}')$ on $[-1, 0]$, $(CL, \mathcal{F}) \cup (-CL, \mathcal{F}')$ is clearly homologous to $(C'L, \mathcal{F}'_*) \cup (-C'L, \mathcal{F}''_*)$ which is the same as

$$(C'L, \mathcal{F}'_*) \cup (-C'L, \mathcal{F}''_*) = (CL, \mathcal{F}').$$

By the same reason as above, this in turn is homologous to

$$(CL, \mathcal{F}) \cup (-CL, \mathcal{F}').$$

which is again homologous to the disjoint union

$$((\Sigma L, \mathcal{F}_*) \cup (-\Sigma L, \mathcal{F}'_*)).$$

The $\Gamma_1^\infty$-structures $(\Sigma L, \mathcal{F}_*)$ and $(-\Sigma L, \mathcal{F}'_*)$, however, are homologous to zero by a result of [27]. This proves Lemma 2.2.

Of course, we have similar modifications for foliated $I^+$-bundles with abelian total holonomy.

We can also define modifications of foliated $I^+$-bundles $(E, K, \mathcal{F}, I)$ ($(I = (-1, +1)$) if the total holonomy is contained in a 1-parameter subgroup generated by a smooth vectorfield $\xi$ on $I$. In this case, we take a vectorfield $\eta$ which coincides with $\xi$ on $[-1 + \epsilon, 1 - \epsilon]$ and vanishes on $[-1, -1 + (\epsilon/2)] \cup [1 - (\epsilon/2), 1]$ for some small positive $\epsilon$. Then by the construction in the proof of Lemma 2.1, we have a modified foliated $I^+$-bundle $(\tilde{E}, K, \tilde{\mathcal{F}}, I)$ such that

(i) $E \subset \tilde{E}$ and $\tilde{E}$ is the associated $I$-bundle of $E$,

(ii) $\tilde{\mathcal{F}}$ coincides with $\mathcal{F}$ outside a small neighbourhood of $L \times \{-1, 1\}$,

(iii) $\tilde{\mathcal{F}}$ is trivial in a smaller neighbourhood of $L \times \{-1, 1\}$ and

(iv) the total holonomy of $\tilde{E}$ is abelian.

§3. SECTIONS OF FOLIATED $\mathbb{R}$-BUNDLES

In this section, we study $\Gamma_1^\infty$-structures which are defined by foliated $\mathbb{R}$-bundle with abelian total holonomy and prove the following

**Lemma 3.1.** Let $(E, K, \mathcal{F}, \mathcal{R})$ be a foliated $\mathbb{R}$-bundle over an oriented closed $n$-dimensional manifold $K$ and let $s$ be any section; $s: K \to E$. Suppose that the total holonomy of this foliated $\mathbb{R}$-bundle is abelian. Then there is a finite collection $\{(E_i, T^{n-1}, F_i, S_i')\}$, $(i = 1, \ldots, k)$ of foliated $S^1$-bundles over $(n-1)$-dimensional torus $T^{n-1}$ such that $(K, s^*\mathcal{F})$ is homologous to the disjoint union $\bigcup_{i=1}^k (E_i, \mathcal{F}_i)$.

**Proof.** We may suppose $K$ is connected. Let $\Phi: \pi_1(K, *) \to \text{Diff } \mathbb{R}$ be the total holonomy.
homomorphism. Since $\text{Diff} \mathbb{R}$ is a torsion free group, this homomorphism factors through $\mathbb{Z}'$ for some $r \geq 0$:

$$\pi_1(K, \ast) \xrightarrow{\phi} \text{Diff} \mathbb{R}.$$ 

Thus we have a morphism of foliated R-bundles

$$E \xrightarrow{\psi} E' \xrightarrow{s} T'.$$

Since $H_*(T^*, \mathbb{Z})$ is generated by the classes represented by $\langle \ast \rangle$ subtori $\langle (T^r) \rangle$, we have $(B\Psi)_*\langle s(K) \rangle = \sum k_i\langle s_i(T^r) \rangle$ for some $k_i \in \mathbb{Z}$. Of course, this r.h.s. is meant zero unless $r \geq n$.

Let $s : I' \rightarrow E'$ denote a section of $E' \rightarrow I'$. Then the above equality implies

$$(B\Psi)_*\langle s(K) \rangle = \sum k_i\langle s_i(T^r) \rangle.$$

Thus Lemma 3.1 follows from the following Lemma 3.2, which is almost proved in [17].

LEMMA 3.2. Let $(E, T^r, \mathcal{F}, \mathbb{R})$ be a foliated R-bundle over $T^r$ and let $s$ denote a section. Then $(T^r, s^*\mathcal{F})$ is homotopic, as a $\Gamma_{1^r}$-structure, to a foliated $S^1$-bundle over $T^{-1}$. Let $f_i$, $\ldots$, $f_n$ be the elements of $\text{Diff} \mathbb{R}$ which correspond, under the total holonomy homomorphism, to the standard generators of $\pi_1(T^r, \ast)$. The proof of Lemma 3.2 is divided into two cases according to the topological properties of $f_i$'s.

ASSERTION 3.3. If all $f_i$ (i = 1, \ldots, n) have fixed points, $(T^r, s^*\mathcal{F})$ is homotopic to zero in $B\Gamma_{1^r}$.

ASSERTION 3.4. If one of $f_i$'s has no fixed points, $(T^r, s^*\mathcal{F})$ is homotopic to foliated $S^1$-bundle $(T^r, T^{-1}, \mathcal{F}, S^1)$ over $T^{-1}$. If $f_n$ has no fixed points, for example, $(T^r, T^{-1}, \mathcal{F}, S^1)$ is determined by $(n - 1)$ diffeomorphisms $kf^{-1}$ mod. 1 (i = 1, \ldots, n - 1), where $k$ is an orientation preserving diffeomorphism of $\mathbb{R}$ such that $kf^{-1}$ is a translation of $\mathbb{R}$ by +1 or −1.

The above two assertions clearly imply Lemma 3.2 and hence Lemma 3.1.

In the following proof, we fix a bundle trivialization $E \cong T^r \times \mathbb{R}$ under which a compact leaf in $E$ corresponds to a level set $T^r \times \{\ast\}$.

Proof of Assertion 3.3. If all $f_i$ have fixed points, they have a common fixed point $a \in \mathbb{R}$ (see [17] or [26]). We think of $(E, T^r, \mathcal{F}, \mathbb{R})$ with the fiber restricted to [a, +\infty) as a foliated $I^r$-bundle and modify $(E, T^r, \mathcal{F}, \mathbb{R})$ in this part as in Lemma (2.1). Then we obtain a foliated $\mathbb{R}^\ast(1, +\infty)$-bundle $(\bar{E}, \bar{T}^r, \bar{\mathcal{F}}, \mathbb{R}^\ast(1, +\infty))$. Let $\bar{s}_1$ and $\bar{s}_2$ denote the sections of $\bar{E} \rightarrow \bar{T}^r$ and $\bar{E} \rightarrow T^r$ with the image $T^r \times \{x\}$ (x \in \mathbb{R}) respectively. In $(E, T^r, \mathcal{F}, \mathbb{R})$, the $\Gamma_{1^r}$-structure $(T^r, s^*\mathcal{F})$ is homotopic to $(T^r, \bar{s}_1^*\mathcal{F})$; the latter is the same as $(T^r, \bar{s}_2^*\mathcal{F})$. 
Proof of Assertion 3.4. Let \( f \) have no fixed points. Then it is easy to see that there exists an orientation preserving diffeomorphism \( \kappa \) of \( \mathbb{R} \) such that \( \kappa f \kappa^{-1} \) is a translation by \( \pm 1 \) or \( -1 \). Of course, \( \kappa f, \kappa^{-1}, \ldots, \kappa f, \kappa^{-1} \) define a foliated \( \mathbb{R} \)-bundle \((E', T', \mathcal{F}', \mathbb{R})\) which is isomorphic to \((E, T', \mathcal{F}, \mathbb{R})\). On the other hand, \( \kappa f, \kappa^{-1} \) \((i = 1, \ldots, n - 1)\) induce \( n - 1 \) commuting diffeomorphisms of \( S' = \mathbb{R}/\mathbb{Z} \). Assigning these diffeomorphisms to the standard generators of \( \pi_1(T'_{n-1}, *) \), we have a foliated \( S' \)-bundle \((T'', T'_{n-1}, \mathcal{F}', S')\). In [15], we proved that there is a section \( s : T'' \to E'' \) transverse to \( \mathcal{F}'' \) such that \( s^* \mathcal{F}'' = \mathcal{F}' \). Since \((E'', T'', \mathcal{F}'') \) is homotopic to \((E, T', \mathcal{F}, \mathbb{R}) \) in \( \text{BDiff } \mathbb{R} \), the \( \Gamma^1 \)-structure \((T'', s'^* \mathcal{F})\) is homotopic to \((T'', s''^* \mathcal{F}'')\) under the conjugation by \( \kappa \). Since \((T'', s''^* \mathcal{F}'')\) is homotopic to \((T'', s^* \mathcal{F}')\) which is \((T'', \mathcal{F}')\), we have proved (3.4).

As a corollary to Lemma 3.1, we have

**Lemma 3.5.** Let \((E, K, \mathcal{F}, \mathbb{R})\) and \( s \) be as in Lemma 3.1. Suppose that the total holonomy is topologically conjugate to a group of translations of \( \mathbb{R} \). Then \((K, s^* \mathcal{F})\) is homologous to a union of exotic linear foliated \( S' \)-bundles over \( T''_{n-1} \).

**Remark.** Suppose that \((M, \mathcal{F})\) is a foliation which is without holonomy and that the image of the Novikov transformation [see, Lemma 6.1] is of rank \( \leq n - 1 \). Then the above proof of Lemma 3.4 says that \((M, \mathcal{F})\) is homologous to zero.

§4. COMPONENTS OF TYPE III,

First we briefly review the definition of a foliated \( J \)-bundle[16]. Let \( K \) be a CW-complex and \( L^+, L^- \) subcomplexes of \( K \). A family of foliated bundles and foliated bundle isomorphisms \((E; E^+, E^-; b^+, b^-)\) is a foliated \( J \)-bundle over \((K; L^+, L^-)\) if

1. \( E \) is a foliated \( \tilde{I} \)-bundle, \( E^+ \) is a foliated \( I_+ \)-bundle and \( E^- \) is a foliated \( I_- \)-bundle over \( K, L^+ \) and \( L^- \), respectively, and
2. \( b^+ \) and \( b^- \) are foliated bundle isomorphisms;

\[
\begin{align*}
b^+ : \tilde{E}^+ &\to E|_{L^+}, \\
b^- : \tilde{E}^- &\to E|_{L^-},
\end{align*}
\]

where \( \tilde{E}^+ \) (resp. \( \tilde{E}^- \)) is the associated foliated \( \tilde{I} \)-bundle to \( E^+ \) (resp. \( E^- \)).

Let \( \mathcal{E} \) be the space obtained from the disjoint union of \( E, E^+ \) and \( E^- \) by identifying \( \tilde{E}^+ \) (resp. \( \tilde{E}^- \)) with \( E|_{L^+} \) (resp. \( E|_{L^-} \)) by the isomorphism \( b^+ \) (resp. \( b^- \)). \( \mathcal{E} \) is called the total space of the foliated \( J \)-bundle. There is a canonical projection \( \pi : \mathcal{E} \to K \) and \( \mathcal{E} \) has a "foliation" \( \mathcal{F} \) transverse to the fibers of \( \pi \). By abuse of language, \((E, K, \mathcal{F})\) is also called a foliated \( J \)-bundle. A cross section of a foliated \( J \)-bundle \((E, K, \mathcal{F})\) is a continuous map \( s : K \to \mathcal{E} \) such that \( \pi \circ s = \text{id}_K \). Note that \( E, E^+ \) and \( E^- \) are naturally thought as subspaces of \( \mathcal{E} \) and the cross section \( s \) of \( \mathcal{E} \) defines the cross sections of the foliated bundles \( E, E^+ \) and \( E^- \).

In this section and the next, we consider the foliated \( J \)-bundles associated to the components of type III_1 and III_2, respectively.

Let \((E; E^+, E^-, b^+, b^-)\) be a foliated \( J \)-bundle over \((K; L^+, L^-)\), where we suppose \( K \) is a compact manifold of dimension \( n \) and \( L^+ \) and \( L^- \) are disjoint union of components of \( \partial K \). Let \( s \) be a cross section of this foliated \( J \)-bundle such that \( s(L^+) = L^+ \times \{ +1 \} \).
and \( s(L) = L \times \{-1\} \). We assume that the total holonomy of \( E \) is abelian (hence so are those of \( E^+ \) and \( E^- \)).

Let \( ((E^+)^*, L^+, (\mathcal{F}_{|E^+}|^*, I)) \) and \( ((E^-)^*, L^-, (\mathcal{F}_{|E^-}|^*, I)) \) be modified foliated \( I \)-bundles of \( (E^+, L^+, \mathcal{F}_{|E^+}|^*, I) \) and \( (E^-, L^-, \mathcal{F}_{|E^-}|^*, I^-) \), respectively. We will write \( (CL^+, (\mathcal{F}_{|E^+}|^*)) \) and \( (CL^-, (\mathcal{F}_{|E^-}|^*)) \) for the \( \Gamma_1^\times \)-structures induced on the cones of \( L^+ \) and \( L^- \) (see \( \S 2 \)).

The main purpose of this section is to prove the following

**Lemma 4.1.** Let \( (K, L^+, L^-) \), \( (E, E^+, E^-, b^+, b^-) \) and \( s \) be as above and suppose that for any connected component \( L \) of \( L^+ \cup L^- \), the total holonomy of \( (E|_L, \mathcal{F}_{|E|_L}) \) is a free abelian group of rank less than \( n-1 \). Then \( (K, s^* \mathcal{F}) \) is homologous relative to the boundary to a disjoint union of \( (CL^+, (\mathcal{F}_{|E^+}|^*)) \), \( (CL^-, (\mathcal{F}_{|E^-}|^*)) \) and a finite number of foliated \( S^1 \)-bundles over \( T^{n-1} \).

**Proof.** We construct several foliated spaces. Let \( L \) be a connected component of \( L^+ \cup L^- = \partial K \). For the modified foliated \( I \)-bundle \( ((E^+|_L)^*, L^*, (\mathcal{F}_{|E^+|_L}|^*, I)) \) or \( ((E^-|_L)^*, L^*, (\mathcal{F}_{|E^-|_L}|^*, I)) \), we have the total holonomy homomorphism \( q_L : q(L, *) : \text{Diff} I \). Since this factors through \( h' \), where \( r \equiv n - 2 \) from our hypothesis, we have

\[
\pi_1(L, b_L) \overset{q_L}{\longrightarrow} \text{Diff} I
\]

Hence, we have the following commutative diagram in which we may assume all the maps are cellular:

\[
\begin{array}{ccc}
L & \longrightarrow & B\text{Diff} I \\
\downarrow & & \downarrow \text{Diff} I \\
T' & \longrightarrow & T''
\end{array}
\]

Let \( M_L \) be the mapping cylinder of \( f \);

\[
M_L = L \times [0, 1] \cup T'/\langle x, 1 \rangle \sim f(x), \quad x \in L.
\]

The homomorphism

\[
\pi_1(M_L, b_L) \cong \pi_1(T', \star) \overset{\pi_1(f)}{\longrightarrow} \text{Diff} I
\]

defines a foliated \( I \)-bundle \( (E^+_L, M_L, \mathcal{F}_L, I) \) over \( M_L \). Note that \( L \times \{0\} \) in \( M_L \) is the homological boundary of \( M_L \) because \( r \equiv n - 2 \). Since \( \partial E_L = (\partial E_L^+ \cup L \times \{0\}) \) is a modified foliated \( I \)-bundle of \( (E^+_L, L, \mathcal{F}_{|E^+_L}|^*, I^+) \) and \( (E^-_L, L, \mathcal{F}_{|E^-_L}|^*, I^-) \), there is a foliation preserving diffeomorphism from \( \partial E_L \) to \( E_L \) except on a small neighbourhood of \( L \times \{0\} \) (if \( L \subset L^+ \)) or \( L \times \{1\} \) (if \( L \subset L^- \)). If we attach all \( E_L \)'s to \( E \) using these diffeomorphisms, then we obtain a foliated space \( (E', \mathcal{F}') \) and a projection \( E' \longrightarrow K' \), where \( K' \) is the space obtained from the disjoint union \( K \cup (\bigcup_{L \in K} M_L) \) by identifying each \( L \) in \( \partial K \) with \( L \times \{0\} \) in \( M_L \).

As for the foliated \( I \)-bundle \( (E, K, \mathcal{F}_{|E}|^*, I) \) if we restrict it over a component \( L \) of \( \partial K \), its total holonomy homomorphism has the following factorization:
Thus, as before, we obtain foliated $\mathcal{I}$-bundles over $M_L$, and by attaching them to $E$, we get a foliated $\mathcal{I}$-bundle $(E', K', \mathcal{F}', \mathcal{I})$ over $K'$. Topologically (that is, if we forget the foliated bundle structure), $E'$ is homeomorphic to $K' \times (-1, +1)$, thus we have a cross section $s_0'$ of $E' \to K'$ which corresponds to the "zero-section" $K' \to K' \times \{0\} \subset K' \times (-1, +1)$. Since the construction of $E'$ and $E''$ are identical in a neighbourhood of $K' \times \{0\} \subset K' \times (-1, +1)$, the above "zero-section" also gives rise to a section $s_0'$ of $E' \to K'$.

We will define several subspaces of these foliated spaces. For the notational convenience, we will write them as if they were subspaces of $K \times I$ or $K' \times I$. The reader will find this will give rise to no ambiguity because the above foliated spaces are unions of foliated interval bundles each of which is topologically a product space of the base space and an interval.

Put $K_1 = K \times \{0\} \cup L^- \times \{0, 1\} \cup L^+ \times \{0, 1\}$. We have two inclusions

$$i: K_1 \to E \quad \text{and} \quad i': K_1 \to E'. $$

Put

$$K_L = \begin{cases} L \times [-1, 0] \cup M_L \times \{0\} \quad \text{if} \ L \subseteq L^-, \\ L \times [0, +1] \cup M_L \times \{0\} \quad \text{if} \ L \subseteq L^+. \end{cases}$$

We also have inclusions

$$i': K_1 \to E' \quad \text{and} \quad i_L: K_1 \to E'_L. $$

Finally, we put

$$K'_L = \begin{cases} L \times [-1, 1] \cup M_L \times \{+1\} \quad \text{if} \ L \subseteq L^-, \\ L \times [-1, 1] \cup M_L \times \{-1\} \quad \text{if} \ L \subseteq L^+. \end{cases}$$

and we have an inclusion $i_L: K'_L \to E'_L$. Now we are going to prove our lemma. In the foliated $J$-bundle $E$, we have

$$(K, s^*\mathcal{F}) \sim (K_1, i^*\mathcal{F}) \ (\text{rel. boundary}).$$

Since $(E, \mathcal{F})$ and $(E', \mathcal{F}')$ are the same in a neighbourhood of $i(K_1) = i'(K_1)$, we have $(K_1, i^*\mathcal{F}) = (K_1, i'^*\mathcal{F}')$. In $(E', \mathcal{F}')$, one can see that

$$(K_1, i'^*\mathcal{F}') \sim (K', s_1'^*\mathcal{F}') \cup \bigcup_{L \subseteq K} (K_L, i'^*\mathcal{F}') \ (\text{rel. boundary}).$$

Since $(E', \mathcal{F}')$ and $(E'_L, \mathcal{F}_L)$ coincide in a neighbourhood of $i'(K_L) = i_L(K_L)$ for each $L$, we have $(K_L, i'^*\mathcal{F}') = (K_L, i_L^*\mathcal{F}_L)$. Consider everything in $(E_L, \mathcal{F}_L)$. Then it is clear that the latter is homologous relative to the boundary to $(K_L, i_L^*\mathcal{F}_L)$. Since $\mathcal{F}_L$ is trivial near either $M_L \times \{+1\}$ (if $L \subseteq L^-$) or $M_L \times \{-1\}$ (if $L \subseteq L^+$) we have

$$(K'_L, i_L^*\mathcal{F}_L) \sim (C L, (\mathcal{F}_L|_{C L})) \ (\text{rel. boundary}).$$
From the definition, it is clear that

\[
\bigcup_{L \in \mathcal{L}} (CL, (\mathcal{F} |_{L \cup K})) \sim (CL^+, (\mathcal{F} |_{L^+})) \cup (CL^-, (\mathcal{F} |_{L^-})) \text{(rel. boundary).}
\]

On the other hand, since \((E', \mathcal{F}')\) coincides with \((E'', \mathcal{F}'')\) in a neighbourhood of \(s_0(K') = s_0(K)\), we have

\[
(K', s_0^*\mathcal{F}') = (K', s_0\mathcal{F}').
\]

\(K'\) is an \(n\)-dimensional CW complex with a fundamental \(n\)-cycle and Lemma 3.1 applies to \((E'', K', \mathcal{F}'', I)\). Thus \((K', s_0^*\mathcal{F}')\) is homologous to a union of foliated \(S^1\)-bundles over \(T^{n-1}\). This finishes the proof of Lemma 4.1.

The following is an immediate corollary to this lemma and Lemma 3.5.

**Lemma 4.2.** Adding to the assumption of Lemma 4.1, if we further assume that the total holonomy group of \(E\) is topologically conjugate to a group of translations of \(\mathbb{R}\), then \((K, s^*\mathcal{F})\) is homologous relative to the boundary to a disjoint union of \((CL^+, (\mathcal{F} |_{E^+})), (CL^-, (\mathcal{F} |_{E^-}))\) and a finite number of exotic linear foliated \(S^1\)-bundles over \(T^{n-1}\).

The foliated \(J\)-bundle associated to a component of type III, satisfies the assumption of Lemma 4.2 if the dimension of the component is greater than 2 (Lemma 6.2 in §6).

### §5. COMPONENTS OF TYPE III

We use the same notations as in the preceding section and prove the following

**Lemma 5.1.** Suppose that the total holonomy of \(E\) is contained in a 1-parameter subgroup generated by a smooth vectorfield on \(I\). Then \((K, s^\mathcal{F})\) is homologous, relative to the boundary, to the disjoint union of \((CL^+, (\mathcal{F} |_{E^+})), (CL^-, (\mathcal{F} |_{E^-}))\)'s.

**Proof.** We again construct several foliated spaces. Let \((E, K, \mathcal{F}, I)\) be a modified foliated \(I\)-bundle of \((E, K, \mathcal{F}, E, I)\) (§2). We also choose a modified foliated \(I\)-bundle of \((E^+, L^+, \mathcal{F} |_{E^+}, I_+)\) so that it coincides with \((E |_{L^+}, L^+, \mathcal{F} |_{E^+}, I)\) except on a small neighbourhood of \(L^+ \times \{+1\}\) and write \(((E^+)\times, L^+, (\mathcal{F} |_{E^+})\times, I)\) for such a foliated bundle. We choose a similar modified bundle \(((E^-)\times, L^-, (\mathcal{F} |_{E^-})\times, I)\) of \((E^-, L^-, \mathcal{F} |_{E^-}, I_-)\).

Product with the interval \([a, b]\) gives natural foliated \(I\)-bundles;

\[((E^+)\times [a, b], L^+ \times [a, b], (\mathcal{F} |_{E^+})\times [a, b], I)\]

and

\[((E^-)\times [a, b], L^- \times [a, b], (\mathcal{F} |_{E^-})\times [a, b], I)\].

Since, for example, \(((E^+)\times [a], L^+ \times [a], (\mathcal{F} |_{E^+})\times [a], I)\) and \(((E^-)\times [a], L^- \times [a], (\mathcal{F} |_{E^-})\times [a], I)\) are identified by a foliation preserving homeomorphism outside a small neighbourhood of \(I^+ \times \{1\}\), one can attach these two kinds of foliated \(I\)-bundles to \((E, K, \mathcal{F}, I)\) and obtain a foliated space. Let \((E', \mathcal{F}')\) denote it. Naturally, we have a
where $K' = K U (L^+ \times [a, b] U L^- \times [a, b]) / L = L \times \{a\}$ (for all $L \subset L^+ U L^-$).

Let $(E'', \mathcal{F}'')$ denote the foliated space over $K'$ obtained from the original foliated $J$-bundle $(E, \mathcal{F})$ by spreading $I_x$-bundle parts over the collar $L^\pm \times [a, b]$ of $L^\pm$ in $K'$. 

projection, 

$$E : \longrightarrow K'$$
We now define several subspaces of these foliated spaces as in §4. Put
\[ K_i = K \times \{0\} \cup L^- \times [-1, 0] \cup L^+ \times [0, 1]. \]
We have inclusion maps
\[ i: K_i \to E \quad \text{and} \quad i'': K_i \to E''. \]
Put
\[ K_2 = K' \times \{0\} \cup (L^- \times \{b\}) \times [-1, 0] \cup (L^+ \times \{b\}) \times [0, +1]. \]
We have again two inclusion maps

\[ i": K_2 \longrightarrow \mathbb{E}^n \] and \[ i': K_2 \longrightarrow \mathbb{E}'. \]

Finally, consider the subspaces

\[ K \times \{+1\} \cup L^+ \times [-1, +1] \subset \mathbb{E} \]
\[ (L^+ \times [a, b]) \times \{+1\} \cup ((L^+ \times \{b\}) \times [-1, +1]) \subset (E^+) \times \{a, b\} \]
\[ (L^- \times [a, b]) \times \{-1\} \cup ((L^- \times \{b\}) \times [1, +1]) \subset (E^-) \times \{a, b\} \]

and put the union of these three spaces to be \( K_3 \). There is an inclusion map;

\[ i': K_1 \longrightarrow \mathbb{E}'. \]

Now we prove our lemma. In \( \mathbb{E} \), \( s(K) \) and \( i(K) \) are homotopic fixing the boundary, so we have

\[ (K, s^*\mathcal{F}) \sim (K, i^*\mathcal{F}) \quad \text{(rel. boundary)} \]

and it is obvious that \( (K_3, i^*\mathcal{F}) = (K_1, i^*\mathcal{F}) \). Since \( (\mathbb{E}^n, \mathcal{F}') \) restricted over \( L^+ \times \{t\} \cup L^- \times \{t\} \) is a trivial family of foliated \( I \)-bundles, we have

\[ (K_1, i_3^*\mathcal{F}') \sim (K_3, i_3^*\mathcal{F}') \quad \text{(rel. boundary)} \]

Since \( (\mathbb{E}^n, \mathcal{F}') \) coincides with \( (\mathbb{E}', \mathcal{F}') \) in a neighbourhood of \( i'(K_3) = i'(K_1) \), it follows that

\[ (K_3, i_3^*\mathcal{F}') = (K_3, i_3^*\mathcal{F}'). \]

Then in the foliated space \( (\mathbb{E}', \mathcal{F}') \) we have

\[ (K_3, i_3^*\mathcal{F}') \sim (K_3, i_3^*\mathcal{F}'). \]

Since the foliation \( \mathcal{F}' \) of \( \mathbb{E}' \) is trivial in neighbourhoods of \( K \times \{+1\} \) \( \cup (L^- \times [a, b]) \times \{+1\} \) and \( (L^+ \times [a, b]) \times \{-1\} \), \( (K_3, i_3^*\mathcal{F}') \) is homologous (relative to the boundary) to the union of the following three \( \Gamma_1^{\mathcal{R}} \)-structures;

1. the \( \Gamma_1^{\mathcal{R}} \)-structure on the suspension

\[ \Sigma L^+ = L^+ \times [-1, 1]/L^+ \times \{+1\}, L^- \times \{-1\} \]

induced from that on \( (\mathbb{E}, \mathcal{F}) \),

2. the \( \Gamma_1^{\mathcal{R}} \)-structure on the cone

\[ C(L^+ \times \{b\}) - (L^+ \times \{b\}) \times [-1, 1]/(L^+ \times \{b\}) \times \{-1\} \]

induced from that on \( (E^+) \times [a, b], (\mathcal{F} |_{E^+}) \times [a, b] \) and

3. the \( \Gamma_1^{\mathcal{R}} \)-structure on the cone

\[ C(L^- \times \{b\}) = (L^- \times \{b\}) \times [-1, +1]/(L^- \times \{b\}) \times \{+1\} \]

induced from that on \( (E^-) \times [a, b], (\mathcal{F} |_{E^-}) \times [a, b] \).
By a result in [27], we know that the first one is homologous to zero. On the other hand, the second and the third ones are isomorphic to

\[(CL^+, (\mathcal{F}|_{E^+})) \text{ and } (CL^-, (\mathcal{F}|_{E^-}))\]

respectively.

Thus we have proved Lemma 5.1.

The foliated $J$-bundle associated to a component of type III satisfies the assumption of Lemma 5.1 (§6).
§6. THE PROOFS OF THEOREMS

In this section we finish the proof of Theorem 1 and give the proof of Theorem 2.

Let \((M, \mathcal{F})\) be a transversely oriented codimension one foliation of a closed oriented manifold which is almost without holonomy. We may assume that \(M\) is connected. We need to show that \((M, \mathcal{F})\) is homologous to a union of foliated \(S^1\)-bundles over tori. Using Theorem (1.1), we can decompose \((M, \mathcal{F})\) into immersed images of foliations. If \((M, \mathcal{F})\) is a component of type I, then Theorem 1 follows from Lemma (3.5) and the following lemma which is proved in [17].

**Lemma 6.1.** Let \((M, \mathcal{F})\), be a codimension one foliation without holonomy of a closed connected manifold. Then there is a locally trivial foliated \(\mathbb{R}\)-bundle \((E, M, \mathcal{F}, \mathbb{R})\) such that

1. there is a cross section \(s: M \to E\) transverse to \(\mathcal{F}\) and the induced foliation \(s^*\mathcal{F}\) coincides with \(\mathcal{F}\), and
2. the image of the total holonomy homomorphism \(q: \pi_1(M, h) \to \text{Diff} \mathbb{R}\) is topologically conjugate to a group of translations of \(\mathbb{R}\). (Following Imanishi[9], we call \(q\) the Novikov transformation.)

In the rest of this section we suppose \((M, \mathcal{F})\) is decomposed as a union of components of types II and III.
For a component \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) of type III, we proved the following lemma in [16].

**Lemma 6.2.** Let \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) be a component of type III, that is, let \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) be a foliation tangent to the non-empty boundary such that all leaves other than the boundary leaves have trivial holonomy. Then there exist a foliated \(J\)-bundle \((E, M_{	ext{III}}, \mathcal{F}) = (E; E^*, E^-)\) over \((M_{	ext{III}}; L^*, L^-)\) where \(\partial M_{	ext{III}} = L^* \cup L^-\), and a section \(s\) which is transverse to \(\mathcal{F}\) such that \(s^* \mathcal{F}\) coincides with \(\mathcal{F}_{	ext{III}}\). Moreover the holonomy group of \(E\) is topologically conjugate to a group of translations of \(\mathbb{R}\). If \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) is of type III\(_2\), or if \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) is a foliated \(I\)-bundle, the holonomy group of \(E\) is contained in a one parameter subgroup generated by a smooth vectorfield on \(I\), that is, this group is differentiably conjugate to a group of translations of \(\mathbb{R}\).

We will take a convenient decomposition of \((M, \mathcal{F})\) to our purpose.

If all leaves are compact, by the Reeb stability theorem, \(M\) is a fiber bundle over \(S^1\) and \((M, \mathcal{F})\) is the associated bundle foliation, so \((M, \mathcal{F})\) is also of type I. \((M, \mathcal{F})\) is easily shown to be homologous to zero. For, \((M, \mathcal{F})\) is the boundary of the \(\Gamma_1\)-structure on the mapping cylinder of the bundle projection \(M \rightarrow S^1\) which is again a fiber bundle over \(S^1\), if \(\dim M \geq 2\).

Now suppose that there are both compact and non-compact leaves. Then, by Remark 1.3, we can assume that all components of type II are contained in a sufficiently small neighborhood \(U\) of the union of compact leaves. If there are two adjacent components of type II we regard the union of them as a new component of type II. After this process, our decomposition has the property that each component of type II is contained in \(U\) and sandwiched by components of type III.

We now return to the proof of Theorem 1. First we suppose that the dimension of \(M\) is greater than 2. By Lemma 6.2, there is associated a foliated \(J\)-bundle to each component of the decomposition of \((M, \mathcal{F})\) and the foliated \(J\)-bundles associated to the components of type III\(_1\) satisfy the assumption of Lemma 4.2 and those associated to the components of type III\(_2\) satisfy the assumption of Lemma 5.1. For a component \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) of type II of \((M, \mathcal{F})\), let \(M'_{	ext{III}}, M''_{	ext{III}}\) be the two components of type III adjacent to the component \((M_{	ext{III}}, \mathcal{F}_{	ext{III}})\) \((M'_{	ext{III}}\) and \(M''_{	ext{III}}\) can be the same components). Put

\[ L' = \varphi_{\text{III}}(M_{	ext{III}}) \cap \varphi_{\text{III}}(M''_{	ext{III}}) \]

and

\[ L'' = \varphi_{\text{III}}(M_{	ext{III}}) \cap \varphi_{\text{III}}(M'_{	ext{III}}). \]

Then applying Lemmas 4.2 and 5.1 to the foliated \(J\)-bundles associated to \(M'_{	ext{III}}\) and \(M''_{	ext{III}}\), we have \(\Gamma_1\)-structures \((CL', (\mathcal{F}_{	ext{III}}'))\) and \((CL'', (\mathcal{F}_{	ext{III}}''))\) on cones of \(L'\) and \(L''\). Consequently we obtain an induced \(\Gamma_1\)-structure

\[ (\mathcal{F}_{	ext{III}}') \cup (\mathcal{F}_{	ext{III}}') \cup (\mathcal{F}_{	ext{III}}'') \]

on \(CL' \cup M_{	ext{III}} \cup CL''\).

For a compact leaf \(L\) which is the common boundary of the components \(M'_{	ext{III}}, M''_{	ext{III}}\) of type III, we obtain an induced \(\Gamma_1\)-structure

\[ (\mathcal{F}_{	ext{III}}') \cup (\mathcal{F}_{	ext{III}}') \]

on \(CL' \cup CL''\) in a similar way \((L' = L'' = L)\).

By applying Lemmas 4.2 and 5.1 to all the components of type III\(_1\), and III\(_2\), respectively,
we can see that \((M, \mathcal{F})\) is homologous to a union of exotic linear foliated \(S^1\)-bundles over \(T^{* - 1}\) which arise from the components of type III, and the \(\Gamma_1^\infty\)-structures of the form
\[
(CL' \cup M_\| \cup CL'', (\mathcal{F}_{\|} \| \cup (\mathcal{F}_{\|}))
\]
or
\[
(CL' \cup CL'', (\mathcal{F}_{\|} \| \cup (\mathcal{F}_{\|})).
\]
Since the latter \(\Gamma_1^\infty\)-structures are clearly homologous to the ones defined by layered foliated \(S^1\)-bundles with abelian holonomy, they are homologous to a union of layered foliated \(S^1\)-bundles over \(T^{* - 1}\).

The rest of Theorem 1 follows from the construction of foliated \(J\)-bundles and Lemma 2.2.

For the foliations of a manifold of dimension two, our theorem is trivial because \(B\Gamma_1^\infty\) is known to be 2-connected[12]. In this case, however, our method gives a good understanding of a fact due to Thurston that all transversely oriented foliations of 2-manifolds are foliated cobordant to zero. For, every foliation of a 2-manifold is almost without holonomy. If it has components of type III they are foliations of annuli. So the Novikov transformations of them are of rank 1. And consequently, the foliated \(J\)-bundles associated to them satisfy the assumption of Lemma 5.1. Thus proceeding as in the case of dimension greater than 2, we have that every foliation on a closed 2-manifold is homologous to a union of foliated \(S^1\)-bundles over the circle. Then the fact that \(\text{Diff}^c(S^1)\) is a perfect group implies that our foliation is homologous to zero[6]. Since the homology group with integral coefficient is isomorphic to the oriented bordism group in dimension 2, this proves that every foliation on a closed 2-manifold is cobordant to zero. This finishes the proof of Theorem 1.

**Proof of Theorem 2.** Theorem 2 is a rather easy corollary to Theorem 1 and what we have already proved. If \((M, \mathcal{F})\) satisfies condition (A), the exotic linear foliated \(S^1\)-bundles over \(T^{* - 1}\) in Theorem 1 are homologous to zero in \(B\Gamma_1^\infty\). For, it represents a degenerate \(n\) complex in \(B\Gamma_1^\infty\). If \((M, \mathcal{F})\) satisfies (B), layered foliated \(S^1\)-bundles in Theorem 1 are all homologous to zero by the same argument as in §4 of [26]. As for the condition of (B), see also [27].

As a final remark, we mention that the nullity of the Godbillon-Vey class of \((M, \mathcal{F})\) is obtained by using Theorem 1, which we proved in [16]. If \(\dim M = 3\), by Theorem 1, \((M, \mathcal{F})\) is homologous to a union of foliated \(S^1\)-bundles over \(T^2\) whose Godbillon-Vey classes are zero by Herman[7] and Wallet[29]. In the case when \(\dim M > 3\), it is enough to show the nullity of the Godbillon-Vey class evaluated on elements in \(H_3(M; \mathbb{R})\) which are represented by closed 3-manifolds. Let \(N\) represent such an element of \(H_3(M; \mathbb{R})\). We may suppose \(N\) is transverse to the boundary leaves of the decomposition of \((M, \mathcal{F})\). Consider the induced decomposition and the pull back of the foliated \(J\)-bundle to \(N\). Proceeding as in the proof of Theorem 1, we see that the \(\Gamma_1^\infty\)-structure on \(N\) induced by \((M, \mathcal{F})\) is homologous to a union of foliated \(S^1\)-bundles over \(T^2\) and we have done.

**REFERENCES**


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