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The Roelcke compactification of groups of homeomorphisms

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Abstract

Let X be a zero-dimensional compact space such that all non-empty clopen subsets of X are homeomorphic to each other, and let $\operatorname{Aut} X$ be the group of all self-homeomorphisms of X, equipped with the compact-open topology. We prove that the Roelcke compactification of $\operatorname{Aut} X$ can be identified with the semigroup of all closed relations on X whose domain and range are equal to X. We use this to prove that the group $\operatorname{Aut} X$ is topologically simple and minimal. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let G be a topological group. There are at least four natural uniform structures on G which are compatible with the topology [4]: the left uniformity $\mathcal{L}$, the right uniformity $\mathcal{R}$, their least upper bound $\mathcal{L} \vee \mathcal{R}$ and their greatest lower bound $\mathcal{L} \wedge \mathcal{R}$. In [4] the uniformity $\mathcal{L} \wedge \mathcal{R}$ is called the *lower uniformity* on G; we shall call it the *Roelcke uniformity*, as in [6]. Let $\mathcal{N}(G)$ be the filter of neighborhoods of unity in G. When U runs over $\mathcal{N}(G)$, the covers of the form $\{xU\colon x \in G\}$, $\{Ux\colon x \in G\}$, $\{xU \cap Ux\colon x \in G\}$ and $\{UxU\colon x \in G\}$ are uniform for $\mathcal{L}$, $\mathcal{R}$, $\mathcal{L} \vee \mathcal{R}$ and $\mathcal{L} \wedge \mathcal{R}$, respectively, and generate the corresponding uniformity.

All topological groups are assumed to be Hausdorff. A uniform space X is *precompact* if its completion is compact or, equivalently, if every uniform cover of X has a finite subcover. For any topological group G the following are equivalent:

(1) G is $\mathcal{L}$-precompact;
(2) G is $\mathcal{R}$-precompact;

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(3) G is $\mathcal{L} \vee \mathcal{R}$-precompact;

(4) G is a topological subgroup of a compact group.

If these conditions are satisfied, G is said to be *precompact*. Let us say that G is *Roelcke-precompact* if G is precompact with respect to the Roelcke uniformity. A group G is precompact if and only if for every $U \in \mathcal{N}(G)$ there exists a finite set $F \subset G$ such that $UF = FU = G$. A group G is Roelcke-precompact if and only if for every $U \in \mathcal{N}(G)$ there exists a finite $F \subset G$ such that $UFU = G$. Every precompact group is Roelcke-precompact, but not vice versa. For example, the unitary group of a Hilbert space or the group $\mathrm{Symm}(E)$ of all permutations of a discrete set E, both with the pointwise convergence topology, are Roelcke-precompact but not precompact [6,4]. Unlike the usual precompactness, the property of being Roelcke-precompact is not inherited by subgroups. (If H is a subgroup of G, in general the Roelcke uniformity of H is finer than the uniformity induced on H by the Roelcke uniformity of G.) Moreover, every topological group is a subgroup of a Roelcke-precompact group [7].

The *Roelcke completion* of a topological group G is the completion of the uniform space $(G, \mathcal{L} \wedge \mathcal{R})$. If G is Roelcke-precompact, the Roelcke completion of G will be called the *Roelcke compactification* of G.

A topological group is *minimal* if it does not admit a strictly coarser Hausdorff group topology. Let us say that a group G is *topologically simple* if G has no closed normal subgroups besides G and $\{1\}$. It was shown in [6,7] that the Roelcke compactification of some important topological groups has a natural structure of an ordered semigroup with an involution, and that the study of this structure can be used to prove that a given group is minimal and topologically simple. In the present paper we apply this method to some groups of homeomorphisms.

A *semigroup* is a set with an associative binary operation. Let S be a semigroup with the multiplication $(x, y) \mapsto xy$. We say that a self-map $x \mapsto x^*$ of S is an *involution* if $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Every group has a natural involution $x \mapsto x^{-1}$. An element $x \in S$ is *symmetrical* if $x^* = x$, and a subset $A \subset S$ is *symmetrical* if $A^* = A$. An *ordered semigroup* is a semigroup with a partial order $\leqslant$ such that the conditions $x \leqslant x'$ and $y \leqslant y'$ imply $xy \leqslant x'y'$. An element $x \in S$ is *idempotent* if $x^2 = x$.

Let K be a compact space. A *closed relation* on K is a closed subset of K^2. Let $E(K)$ be the compact space of all closed relations on K, equipped with the Vietoris topology. If $R, S \in E(K)$, then the *composition* of R and S is the relation $RS = \{(x, y)\colon \exists z((x, z) \in S \text{ and } (z, y) \in R)\}$. The relation RS is closed, since it is the image of the closed subset $\{(x, z, y)\colon (x, z) \in S,\ (z, y) \in R\}$ of K^3 under the projection $K^3 \to K^2$ which is a closed map. If $R \in E(K)$, then the *inverse relation* $\{(x, y)\colon (y, x) \in R\}$ will be denoted by R^* or by R^{-1}; we prefer the first notation, since we are interested in the algebraic structure on $E(K)$, and in general R^{-1} is not an inverse of R in the algebraic sense. The set $E(K)$ has a natural partial order. Thus $E(K)$ is an ordered semigroup with an involution. In general the map $(R, S) \mapsto RS$ from $E(K)^2$ to $E(K)$ is not (even separately) continuous.

For $R \in E(K)$ let $\mathrm{Dom}\, R = \{x\colon \exists y((x, y) \in R)\}$ and $\mathrm{Ran}\, R = \{y\colon \exists x((x, y) \in R)\}$. Put $E_0(K) = \{R \in E(K)\colon \mathrm{Dom}\, R = \mathrm{Ran}\, R = K\}$. The set $E_0(K)$ is a closed symmetrical subsemigroup of $E(K)$.

Denote by $\mathrm{Aut}(K)$ the group of all self-homeomorphisms of K, equipped with the compact-open topology. For every $h \in \mathrm{Aut}(K)$ let $\Gamma(h) = \{(x, h(x)) \colon x \in K\}$ be the graph of h. The map $h \mapsto \Gamma(h)$ from $\mathrm{Aut}(K)$ to $E_0(K)$ is a homeomorphic embedding and a morphism of semigroups with an involution. Identifying every self-homeomorphism of K with its graph, we consider the group $\mathrm{Aut}(K)$ as a subspace of $E_0(K)$.

We say that a compact space X is *h-homogeneous* if X is zero-dimensional and all non-empty clopen subsets of X are homeomorphic to each other.

Main Theorem 1.1. *Let X be an h-homogeneous compact space, and let $G = \mathrm{Aut}(X)$ be the topological group of all self-homeomorphisms of X. Then*:

(1) *G is Roelcke-precompact*; *the Roelcke compactification of G can be identified with the semigroup $E_0(X)$ of all closed relations R on X such that* $\mathrm{Dom}\, R = \mathrm{Ran}\, R = X$;

(2) *G is minimal and topologically simple.*

In the case when $X = 2^\omega$ is the Cantor set, the minimality of $\mathrm{Aut}(X)$ was proved by Gamarnik [3].

Let us explain how to deduce the second part of Theorem 1.1 from the first. Let $G = \mathrm{Aut}(X)$ be such as in Theorem 1.1, and let $f : G \to G'$ be a continuous onto homomorphism. We must prove that either f is a topological isomorphism or $|G'| = 1$. Let $\Theta = E_0(X)$. The first part of Theorem 1.1 implies that f can be extended to a map $F : \Theta \to \Theta'$, where Θ' is the Roelcke compactification of G'. Let e' be the unity of G', and let $S = F^{-1}(e')$. Then S is a closed symmetrical subsemigroup of Θ. Let Δ be the diagonal in X^2. The set $\{r \in S \colon \Delta \subset r\}$ has a largest element. Denote this element by p. Then p is a symmetrical idempotent in Θ and hence an equivalence relation on X. The semigroup S is invariant under inner automorphisms of Θ, and so is the relation p. But there are only two G-invariant closed equivalence relations on X, namely Δ and X^2. If $p = \Delta$, then $S \subset G$, $G = F^{-1}(G')$ and f is perfect. Since G has no non-trivial compact normal subgroups, we conclude that f is a homeomorphism. If $p = X^2$, then $S = \Theta$ and $G' = \{e'\}$.

A similar argument was used in [7] to prove that every topological group is a subgroup of a Roelcke-precompact topologically simple minimal group, and in [6] to yield an alternative proof of Stoyanov's theorem asserting that the unitary group of a Hilbert space is minimal [5,2]. For more information on minimal groups, see the recent survey by Dikranjan [1].

We prove the first part of Theorem 1.1 in Section 2, and the second part in Section 4.

2. Proof of Main Theorem, part 1

Let X be an h-homogeneous compact space, and let $G = \mathrm{Aut}(X)$. Let $\Theta = E_0(X)$ be the semigroup of all closed relations R on X such that $\mathrm{Dom}\, R = \mathrm{Ran}\, R = X$. We identify G with the set of all invertible elements of Θ. We prove in this section that Θ can be identified with the Roelcke compactification of G.

The space Θ, being compact, has a unique compatible uniformity. Let $\mathcal{U}$ be the uniformity that G has as a subspace of Θ. The first part of Theorem 1.1 is equivalent to the following:

Theorem 2.1. *Let X be an h-homogeneous compact space, $\Theta = E_0(X)$, and $G = \mathrm{Aut}(X)$. Identify G with the set of all invertible elements of Θ. Then*:
(1) *G is dense in Θ*;
(2) *the uniformity $\mathcal{U}$ induced by the embedding of G into Θ coincides with the Roelcke uniformity $\mathcal{L} \wedge \mathcal{R}$ on G.*

Let us first introduce some notation. Let $\gamma = \{U_\alpha\colon \alpha \in A\}$ be a finite clopen partition of X. A *γ-rectangle* is a set of the form $U_\alpha \times U_\beta$, $\alpha, \beta \in A$. Given a relation $R \in \Theta$, denote by $M(\gamma, R)$ the set of all pairs $(\alpha, \beta) \in A \times A$ such that R meets the rectangle $U_\alpha \times U_\beta$. Let $\mathcal{V}(\gamma, R)$ be the family $\{U_\alpha \times U_\beta\colon (\alpha, \beta) \in M(\gamma, R)\}$ of all γ-rectangles which meet R. If r is a subset of $A \times A$, put

$$O_{\gamma,r} = \{R \in \Theta\colon M(\gamma, R) = r\}.$$

The sets of the form $O_{\gamma,r}$ constitute a base of Θ. Denote by $E_0(A)$ the set of all relations r on A such that $\mathrm{Dom}\, r = \mathrm{Ran}\, r = A$. A set $O_{\gamma,r}$ is non-empty if and only if $r \in E_0(A)$.

Let $O_\gamma(R)$ be the set of all relations $S \in \Theta$ which meet the same γ-rectangles as R. We have $O_\gamma(R) = O_{\gamma,r}$, where $r = M(\gamma, R)$. The sets of the form $O_\gamma(R)$ constitute a base at R. If λ is another clopen partition of X which refines γ, then $O_\lambda(R) \subset O_\gamma(R)$.

Proof of Theorem 2.1. Our proof proceeds in three parts.

(a) We prove that G is dense in Θ.

Let $\gamma = \{U_\alpha\colon \alpha \in A\}$ be a finite clopen partition of X and $r \in E_0(A)$. We must prove that $O_{\gamma,r}$ meets G. Decomposing each U_α into a suitable number of clopen pieces, we can find a clopen partition $\{W_{\alpha,\beta}\colon (\alpha, \beta) \in r\}$ of X such that $U_\alpha = \bigcup\{W_{\alpha,\beta}\colon (\alpha, \beta) \in r\}$ for every $\alpha \in A$. Similarly, there exists a clopen partition $\{W'_{\alpha,\beta}\colon (\alpha, \beta) \in r\}$ of X such that $U_\beta = \bigcup\{W'_{\alpha,\beta}\colon (\alpha, \beta) \in r\}$ for every $\beta \in A$. Let $f \in G$ be a self-homeomorphism of X such that $f(W_{\alpha,\beta}) = W'_{\alpha,\beta}$ for every $(\alpha, \beta) \in r$. The graph of f meets each rectangle of the form $W_{\alpha,\beta} \times W'_{\alpha,\beta}$, $(\alpha, \beta) \in r$, and is contained in the union of such rectangles. It follows that $M(\gamma, f) = r$ and $f \in G \cap O_{\gamma,r} \neq \emptyset$.

(b) We prove that the uniformity $\mathcal{U}$ is coarser than $\mathcal{L} \wedge \mathcal{R}$.

This is a special case of the following:

Lemma 2.2. *For every compact space K the map $h \mapsto \Gamma(h)$ from $\mathrm{Aut}(K)$ to $E_0(K)$ is $\mathcal{L} \wedge \mathcal{R}$-uniformly continuous.*

Proof. It suffices to prove that the map under consideration is $\mathcal{L}$-uniformly continuous and $\mathcal{R}$-uniformly continuous. Let d be a continuous pseudometric on K. Let d_2 be the pseudometric on K^2 defined by $d_2((x, y), (x', y')) = d(x, x') + d(y, y')$, and let d_H be the corresponding Hausdorff pseudometric on $E_0(K)$. If $R, S \in E_0(K)$ and $a > 0$, then

$d_H(R, S) \leqslant a$ if and only if each of the relations R and S is contained in the closed a-neighbourhood of the other with respect to d_2. The pseudometrics of the form d_H generate the uniformity of $E_0(K)$.

Let d_s be the right-invariant pseudometric on $\mathrm{Aut}(K)$ defined by $d_s(f, g) = \sup\{d(f(x), g(x))\colon x \in K\}$. The pseudometrics of the form d_s generate the right uniformity $\mathcal{R}$ on $\mathrm{Aut}(K)$. Since $d_H(\Gamma(f), \Gamma(g)) \leqslant d_s(f, g)$, the map $\Gamma : \mathrm{Aut}(K) \to E_0(K)$ is $\mathcal{R}$-uniformly continuous. For the left uniformity $\mathcal{L}$ we can either use a similar argument, or note that the involution on $\mathrm{Aut}(K)$ is an isomorphism between $\mathcal{L}$ and $\mathcal{R}$, and use the formula $\Gamma(f) = \Gamma(f^{-1})^*$ to reduce the case of $\mathcal{L}$ to the case of $\mathcal{R}$. □

(c) We prove that $\mathcal{U}$ is finer than $\mathcal{L} \wedge \mathcal{R}$.

Let $\gamma = \{U_\alpha\colon \alpha \in A\}$ be a finite clopen partition of X. Put $V_\gamma = \{f \in G\colon f(U_\alpha) = U_\alpha$ for every $\alpha \in A\}$. The open subgroups of the form V_γ constitute a base at unity of G. We must show that if $f, g \in G$ are close enough in Θ, then $f \in V_\gamma g V_\gamma$.

The set of all pairs $(R, S) \in \Theta^2$ such that $M(\gamma, R) = M(\gamma, S)$ is a neighbourhood of the diagonal in Θ^2 and therefore an entourage for the unique compatible uniformity on Θ. It suffices to prove that for every $f, g \in G$ the condition $M(\gamma, f) = M(\gamma, g)$ implies that $f \in V_\gamma g V_\gamma$. Suppose that $M(\gamma, f) = M(\gamma, g) = r$. The following conditions are equivalent for every $\alpha, \beta \in A$:

(1) $f(U_\alpha) \cap U_\beta \neq \emptyset$;
(2) $g(U_\alpha) \cap U_\beta \neq \emptyset$;
(3) $(\alpha, \beta) \in r$.

Pick $u \in G$ such that $u(f(U_\alpha) \cap U_\beta) = g(U_\alpha) \cap U_\beta$ for every $(\alpha, \beta) \in r$. Such a self-homeomorphism u of X exists, since all non-empty clopen subsets of X are homeomorphic. Since for a fixed $\beta \in A$ the sets $f(U_\alpha) \cap U_\beta$ cover U_β, we have $u(U_\beta) = U_\beta$. Thus $u \in V_\gamma$. It follows that $uf(U_\alpha) \cap U_\beta = u(f(U_\alpha) \cap U_\beta) = g(U_\alpha) \cap U_\beta$ for all $\alpha, \beta \in A$ and hence $uf(U_\alpha) = g(U_\alpha)$ for every $\alpha \in A$. Put $v = g^{-1}uf$. Since $uf(U_\alpha) = g(U_\alpha)$, we have $v(U_\alpha) = U_\alpha$ for every $\alpha \in A$. Thus $v \in V_\gamma$ and $f = u^{-1}gv \in V_\gamma g V_\gamma$. □

3. Continuity-like properties of composition

We preserve all the notation of the previous section. In particular, X is an h-homogeneous compact space, $G = \mathrm{Aut}(X)$, $\Theta = E_0(X)$.

Recall that is a non-empty collection $\mathcal{F}$ of non-empty subsets of a set Y is a *filter base* on Y if for every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $C \subset A \cap B$. If Y is a topological space, $\mathcal{F}$ is a filter base on Y and $x \in Y$, then x is a *cluster point* of $\mathcal{F}$ if every neighbourhood of x meets every member of $\mathcal{F}$, and $\mathcal{F}$ *converges* to x if every neighbourhood of x contains a member of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are two filter bases on G, let $\mathcal{F}\mathcal{G} = \{AB\colon A \in \mathcal{F},\ B \in \mathcal{G}\}$.

For every $R \in \Theta$ let

$$\mathcal{F}_R = \{G \cap V\colon V \text{ is a neighbourhood of } R \text{ in } \Theta\}.$$

In other words, $\mathcal{F}_R$ is the trace on G of the filter of neighborhoods of R in Θ. We have noted that the multiplication on Θ is not continuous. If $R, S \in \Theta$, it is not true in general that $\mathcal{F}_R\mathcal{F}_S$ converges to RS. However, RS is a cluster point of $\mathcal{F}_R\mathcal{F}_S$. This fact will be used in the next section.

Proposition 3.1. *If $R, S \in \Theta$, then RS is a cluster point of the filter base $\mathcal{F}_R\mathcal{F}_S$.*

We need some lemmas. First we note that for any compact space K the composition of relations is upper-semicontinuous on $E(K)$ in the following sense:

Lemma 3.2. *Let K be a compact space, $R, S \in E(K)$. Let O be an open set in K^2 such that $RS \subset O$. Then there exist open sets V_1, V_2 in K^2 such that $R \subset V_1$, $S \subset V_2$, and for every $R', S' \in E(K)$ such that $R' \subset V_1$, $S' \subset V_2$ we have $R'S' \subset O$.*

Proof. Consider the following three closed sets in K^3:

$$F_1 = \big\{(x, z, y)\colon (z, y) \in R\big\},$$
$$F_2 = \big\{(x, z, y)\colon (x, z) \in S\big\},$$
$$F_3 = \big\{(x, z, y)\colon (x, y) \notin O\big\}.$$

The intersection of these three sets is empty. There exist neighborhoods of these sets with empty intersection. We may assume that the neighborhoods of F_1 and F_2 are of the form $\{(x, z, y)\colon (z, y) \in V_1\}$ and $\{(x, z, y)\colon (x, z) \in V_2\}$, respectively, where V_1 and V_2 are open in K^2. The sets V_1 and V_2 are as required. □

Lemma 3.3. *Let $\gamma = \{U_\alpha\colon \alpha \in A\}$ be a finite clopen partition of X. For every $R, S \in \Theta$ we have $M(\gamma, RS) \subset M(\gamma, R)M(\gamma, S)$ (the product on the right means the composition of relations on A).*

Proof. Let $(\alpha, \beta) \in M(\gamma, RS)$. Then RS meets the rectangle $U_\alpha \times U_\beta$. Pick $(x, y) \in RS \cap (U_\alpha \times U_\beta)$. There exists $z \in X$ such that $(x, z) \in S$ and $(z, y) \in R$. Pick $\delta \in A$ such that $z \in U_\delta$. Then $(x, z) \in S \cap (U_\alpha \times U_\delta)$, $(z, y) \in R \cap (U_\delta \times U_\beta)$, hence $(\alpha, \delta) \in M(\gamma, S)$ and $(\delta, \beta) \in M(\gamma, R)$. It follows that $(\alpha, \beta) \in M(\gamma, R)M(\gamma, S)$. □

Lemma 3.4. *Let $\lambda = \{U_\alpha\colon \alpha \in A\}$ be a finite clopen partition of X, and let $r, s \in E_0(A)$. There exist $f, g \in G$ such that $M(\lambda, f) = r$, $M(\lambda, g) = s$ and $M(\lambda, fg) = rs$.*

Proof. We modify the proof of Theorem 2.1. For every $\gamma \in A$ take a clopen partition $\{V_{\alpha,\gamma,\beta}\colon (\alpha, \gamma) \in s,\ (\gamma, \beta) \in r\}$ of U_γ. For every $(\gamma, \beta) \in r$ put $W_{\gamma,\beta} = \bigcup\{V_{\alpha,\gamma,\beta}\colon (\alpha, \gamma) \in s\}$. For every $(\alpha, \gamma) \in s$ put $Y'_{\alpha,\gamma} = \bigcup\{V_{\alpha,\gamma,\beta}\colon (\gamma, \beta) \in r\}$. Take a clopen partition $\{W'_{\gamma,\beta}\colon (\gamma, \beta) \in r\}$ of X such that for every $\beta \in A$ we have $U_\beta = \bigcup\{W'_{\gamma,\beta}\colon (\gamma, \beta) \in r\}$. Take a clopen partition $\{Y_{\alpha,\gamma}\colon (\alpha, \gamma) \in s\}$ of X such that for every $\alpha \in A$ we have $U_\alpha = \bigcup\{Y_{\alpha,\gamma}\colon (\alpha, \gamma) \in s\}$. There exist $f \in G$ such that $f(W_{\gamma,\beta}) = W'_{\gamma,\beta}$ for every $(\gamma, \beta) \in r$. There exists $g \in G$ such that $g(Y_{\alpha,\gamma}) = Y'_{\alpha,\gamma}$ for every $(\alpha, \gamma) \in s$. The graph of f meets every rectangle $W_{\gamma,\beta} \times W'_{\gamma,\beta}$, $(\gamma, \beta) \in r$, and is contained in the union of such rectangles. Since $W_{\gamma,\beta} \times W'_{\gamma,\beta} \subset U_\gamma \times U_\beta$, it follows that $M(\lambda, f) = r$. Similarly,

$M(\lambda, g) = s$. We claim that $M(\lambda, fg) = rs$. Let $(\alpha, \beta) \in rs$. There exists $\gamma \in A$ such that $(\alpha, \gamma) \in s$ and $(\gamma, \beta) \in r$. We have $g(U_\alpha) \supset g(Y_{\alpha,\gamma}) = Y'_{\alpha,\gamma} \supset V_{\alpha,\gamma,\beta}$ and $f^{-1}(U_\beta) \supset f^{-1}(W'_{\gamma,\beta}) = W_{\gamma,\beta} \supset V_{\alpha,\gamma,\beta}$. Thus $V_{\alpha,\gamma,\beta} \subset g(U_\alpha) \cap f^{-1}(U_\beta) \neq \emptyset$. It follows that the graph of fg meets the rectangle $U_\alpha \times U_\beta$. This means that $(a, \beta) \in M(\lambda, fg)$. We have proved that $rs \subset M(\lambda, fg)$. The reverse inclusion follows from Lemma 3.3. □

Proof of Proposition 3.1. Let U_1, U_2, U_3 be neighborhoods in Θ of R, S and RS, respectively. We must show that U_3 meets the set $(U_1 \cap G)(U_2 \cap G)$.

Fix a clopen partition λ of X such that $O_\lambda(RS) \subset U_3$. Lemma 3.2 implies that there exists a clopen partition γ of X such that for every $R' \in O_\gamma(R)$ and $S' \in O_\gamma(S)$ we have $R'S' \subset \bigcup \mathcal{V}(\lambda, RS)$ (recall that $\mathcal{V}(\lambda, RS)$ is the family of all λ-rectangles that meet RS). We may assume that γ refines λ and that $O_\gamma(R) \subset U_1$, $O_\gamma(S) \subset U_2$. Put $r = M(\gamma, R)$, $s = M(\gamma, S)$. According to Lemma 3.4, there exist $f, g \in G$ such that $M(\gamma, f) = r$, $M(\gamma, g) = s$ and $M(\gamma, fg) = rs$. Then $f \in G \cap O_\gamma(R)$ and $g \in G \cap O_\gamma(S)$. Lemma 3.3 implies that $M(\gamma, RS) \subset rs = M(\gamma, fg)$. This means that (the graph of) fg meets every member of the family $\mathcal{V}(\gamma, RS)$. Then every member of $\mathcal{V}(\lambda, RS)$ meets fg, since every member of $\mathcal{V}(\lambda, RS)$ contains a member of $\mathcal{V}(\gamma, RS)$. On the other hand, by the choice of γ we have $fg \subset \bigcup \mathcal{V}(\lambda, RS)$. It follows that $M(\lambda, fg) = M(\lambda, RS)$. Thus $fg \in O_\lambda(RS) \subset U_3$ and hence $fg \in (U_1 \cap G)(U_2 \cap G) \cap U_3 \neq \emptyset$. □

4. Proof of Main Theorem, part 2

Let X, as before, be a compact h-homogeneous space, $G = \mathrm{Aut}(X)$, $\Theta = E_0(X)$. We saw that G is Roelcke-precompact and that Θ can be identified with the Roelcke compactification of G. In this section we prove that G is minimal and topologically simple.

If H is a group and $g \in H$, we denote by l_g (respectively, r_g) the left shift of H defined by $l_g(h) = gh$ (respectively, the right shift defined by $r_g(h) = hg$).

Proposition 4.1. *Let H be a topological group, and let K be the Roelcke completion of H. Let $g \in H$. Each of the following self-maps of H extends to a self-homeomorphism of K:*

(1) *the left shift l_g;*
(2) *the right shift r_g;*
(3) *the inversion $g \mapsto g^{-1}$.*

Proof. Let $\mathcal{L}$ and $\mathcal{R}$ be the left and the right uniformity on H, respectively. In each of the cases (1)–(3) the map $f : H \to H$ under consideration is an automorphism of the uniform space $(H, \mathcal{L} \wedge \mathcal{R})$. This is obvious for the case (3). For the cases (1) and (2), observe that the uniformities $\mathcal{L}$ and $\mathcal{R}$ are invariant under left and right shifts, hence the same is true for their greatest lower bound $\mathcal{L} \wedge \mathcal{R}$. It follows that in all cases f extends to an automorphism of the completion K of the uniform space $(H, \mathcal{L} \wedge \mathcal{R})$. □

For $g \in G$ define self-maps $L_g : \Theta \to \Theta$ and $R_g : \Theta \to \Theta$ by $L_g(R) = gR$ and $R_g(R) = Rg$.

Proposition 4.2. *For every $g \in G$ the maps $L_g : \Theta \to \Theta$ and $R_g : \Theta \to \Theta$ are continuous.*

Proof. We have $gR = \{(x, g(y)): (x, y) \in R\}$. Let $\lambda = \{U_\alpha: \alpha \in A\}$ be a clopen partition of X. Let $r = M(\lambda, gR)$, and let $O_\lambda(gR) = \{S \in \Theta: M(\lambda, S) = r\}$ be a basic neighbourhood of gR. Let U be the set of all $T \in \Theta$ such that T meets every member of the family $\{U_\alpha \times g^{-1}(U_\beta): (\alpha, \beta) \in r\}$ and is contained in the union of this family. Then U is a neighbourhood of R and $L_g(U) = O_\lambda(gR)$. Thus L_g is continuous. The argument for R_g is similar. □

Let Δ be the diagonal in X^2.

Proposition 4.3. *Let S be a closed subsemigroup of Θ, and let T be the set of all $p \in S$ such that $p \supset \Delta$. If $T \neq \emptyset$, then T has a greatest element p, and p is an idempotent.*

Proof. We claim that every non-empty closed subset of Θ has a maximal element. Indeed, if C is a non-empty linearly ordered subset of Θ, then C has a least upper bound $b = \overline{\bigcup C}$ in Θ, and b belongs to the closure of C in Θ. Thus our claim follows from Zorn's lemma.

The set T is a closed subsemigroup of Θ. Let p be a maximal element of T. For every $q \in T$ we have $pq \supset p\Delta = p$, whence $pq = p$. It follows that p is an idempotent and that $p = pq \supset \Delta q = q$. Thus p is the greatest element of T. □

An *inner automorphism* of Θ is a map of the form $p \mapsto gpg^{-1}$, $g \in G$.

Proposition 4.4. *There are precisely two elements in Θ which are invariant under all inner automorphisms of Θ, namely Δ and X^2.*

Proof. A relation $R \in \Theta$ is invariant under all inner automorphisms if and only if the following holds: if $x, y \in X$ and $(x, y) \in R$, then $(f(x), f(y)) \in R$ for every $f \in G$. Suppose that $R \in \Theta$ has this property and $\Delta \neq R$. Pick $(x, y) \in R$ such that $x \neq y$. We claim that the set $B = \{(f(x), f(y)): f \in G\}$ is dense in X^2. Indeed, pick disjoint clopen neighborhoods U_1 and U_2 of x and y, respectively, such that X is not covered by U_1 and U_2. Given disjoint clopen non-empty sets V_1 and V_2, by h-homogeneity of X we can find an $f \in G$ such that $f(U_i) \subset V_i$, $i = 1, 2$. It follows that $V_1 \times V_2$ meets B, hence B is dense in X^2. Since $B \subset R$, it follows that $R = X^2$. □

Proposition 4.5. *The group G has no compact normal subgroups other than* $\{1\}$.

We shall prove later that actually G has no non-trivial closed normal subgroups.

Proof. Let $H \neq \{1\}$ be a normal subgroup of G. We show that H is not compact.

Let Y be the collection of all non-empty clopen sets in X. Consider Y as a discrete topological space. The group G has a natural continuous action on Y. Pick $f \in H$, $f \neq 1$. Pick $U \in Y$ such that $f(U) \cap U = \emptyset$ and $X \setminus (f(U) \cup U) \neq \emptyset$. Let Y_1 be the set of all $V \in Y$ such that V is a proper subset of $X \setminus U$. If $V \in Y_1$, there exists $h \in G$ such that $h(U) = U$ and $h(f(U)) = V$. Put $g = hfh^{-1}$. Then $g(U) = V$. Since H is a normal subgroup of G, we have $g \in H$. It follows that the H-orbit of U contains Y_1. Since Y_1 is infinite, H cannot be compact. □

Proposition 4.6. *For every topological group H the following conditions are equivalent*:

(1) *H is minimal and topologically simple*;
(2) *if $f : H \to H'$ is a continuous onto homomorphism of topological groups, then either f is a homeomorphism, or $|H'| = 1$.* □

We are now ready to prove Theorem 1.1, part 2:

For every compact h-homogeneous space X the topological group $G = \mathrm{Aut}(X)$ is minimal and topologically simple.

Proof. Let $f : G \to G'$ be a continuous onto homomorphism. According to Proposition 4.6, we must prove that either f is a homeomorphism or $|G'| = 1$.

Since G is Roelcke-precompact, so is G'. Let Θ' be the Roelcke compactification of G'. The homomorphism f extends to a continuous map $F : \Theta \to \Theta'$. Let e' be the unity of G', and let $S = F^{-1}(e') \subset \Theta$.

Claim 1. *S is a subsemigroup of Θ.*

Let $p, q \in S$. In virtue of Proposition 3.1, there exist filter bases $\mathcal{F}_p$ and $\mathcal{F}_q$ on G such that $\mathcal{F}_p$ converges to p (in Θ), $\mathcal{F}_q$ converges to q and pq is a cluster point of the filter base $\mathcal{F}_p\mathcal{F}_q$. The filter bases $\mathcal{F}'_p = F(\mathcal{F}_p)$ and $\mathcal{F}'_q = F(\mathcal{F}_q)$ on G' converge to $F(p) = F(q) = e'$, hence the same is true for the filter base $\mathcal{F}'_p\mathcal{F}'_q = F(\mathcal{F}_p\mathcal{F}_q)$. Since pq is a cluster point of $\mathcal{F}_p\mathcal{F}_q$, $F(pq)$ is a cluster point of the convergent filter base $F(\mathcal{F}_p\mathcal{F}_q)$. A convergent filter on a Hausdorff space has only one cluster point, namely the limit. Thus $F(pq) = e'$ and hence $pq \in S$.

Claim 2. *The semigroup S is closed under involution.*

In virtue of Proposition 4.1, the inversion on G' extends to an involution $x \mapsto x^*$ of Θ'. Since $F(p^*) = F(p)^*$ for every $p \in G$, the same holds for every $p \in \Theta$. Let $p \in S$. Then $F(p^*) = F(p)^* = e'$ and hence $p^* \in S$.

Claim 3. *If $g \in G$ and $g' = f(g)$, then $F^{-1}(g') = gS = Sg$.*

We saw that the left shift $h \mapsto gh$ of G extends to a continuous self-map $L = L_g$ of Θ defined by $L(p) = gp$ (Proposition 4.2). According to Proposition 4.1, the self-map $x \mapsto$

$g'x$ of G' extends to self-homeomorphism L' of Θ'. The maps FL and $L'F$ from Θ to Θ' coincide on G and hence everywhere. Replacing g by g^{-1}, we see that $FL^{-1} = (L')^{-1}F$. Thus $F^{-1}(g') = F^{-1}L'(e') = LF^{-1}(e') = gS$. Using right shifts instead of left shifts, we similarly conclude that $F^{-1}(g') = Sg$.

Claim 4. *S is invariant under inner automorphisms of Θ.*

We have just seen that $gS = Sg$ for every $g \in G$, hence $gSg^{-1} = S$.

Let $T = \{r \in S\colon r \supset \Delta\}$. According to Proposition 4.3, there is a greatest element p in T. Claim 4 implies that p is invariant under inner automorphisms. In virtue of Proposition 4.4, either $p = \Delta$ or $p = X^2$. We shall show that either f is a homeomorphism or $|G'| = 1$, according to which of the cases $p = \Delta$ or $p = X^2$ holds.

First assume that $p = \Delta$.

Claim 5 ($p = \Delta$). *All elements of S are invertible in Θ.*

Let $r \in S$. Then $r^*r \in S$ and $rr^* \in S$, since S is a symmetrical semigroup. Since $\operatorname{Dom} r = \operatorname{Ran} r = X$, we have $r^*r \supset \Delta$ and $rr^* \supset \Delta$. The assumption $p = \Delta$ implies that every relation $s \in S$ such that $s \supset \Delta$ must be equal to Δ. Thus $rr^* = r^*r = \Delta$ and r is invertible.

Claim 6 ($p = \Delta$). $|S| = 1$.

Claim 5 implies that S is a subgroup of G. This subgroup is normal (Claim 4) and compact, since S is closed in Θ. Proposition 4.5 implies that $|S| = 1$.

Claim 7 ($p = \Delta$). *$f : G \to G'$ is a homeomorphism.*

Claims 6 and 3 imply that $G = F^{-1}(G')$ and that the map $f : G \to G'$ is bijective. Since F is a map between compact spaces, it is perfect, and hence so is the map $f : G = F^{-1}(G') \to G'$. Thus f, being a perfect bijection, is a homeomorphism.

Now consider the case $p = X^2$.

Claim 8. *If $p = X^2 \in S$, then $G' = \{e'\}$.*

Let $g \in G$ and $g' = f(g)$. We have $gp = p \in S$. On the other hand, Claim 3 implies that $gp \in gS = F^{-1}(g')$. Thus $g' = F(gp) = F(p) = e'$. □

5. Remarks

The group $\operatorname{Aut}(K)$ is Roelcke-precompact also for some compact spaces K which are not zero-dimensional. For example, let $I = [0, 1]$ and $G = \operatorname{Aut}(I)$. Identify G with a subspace of $E(I)$, as above. The Roelcke compactification of G can be identified with the closure of G in $E(I)$. Let G_0 be the subgroup of all $f \in G$ which leave the end-points of the interval I fixed. The closure of G_0 in $E(I)$ is the set of all curves c in the square I^2 such that c connects the points $(0,0)$ and $(1,1)$ and has the following property: there are no points $(x, y) \in c$ and $(x', y') \in c$ such that $x < x'$ and $y > y'$. This can be used to yield an alternative proof of D. Gamarnik's theorem saying that G is minimal [3].

Let $K = I^\omega$ be the Hilbert cube and $G = \mathrm{Aut}(K)$. I do not know if G is minimal or Roelcke-precompact in this case.

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